DEVELOPMENTAL MATH (NROC)

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Developmental Math (NROC)

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SECTION OVERVIEW

1.1: Introduction to Whole Numbers

Topic hierarchy

1.1.1: Place Value and Names for Whole Numbers

1.1.2: Rounding Whole Numbers

1.1.3: Comparing Whole Numbers

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1.1.1: Place Value and Names for Whole Numbers

Learning Objectives

- Find the place value of a digit in a whole number.
- Write a whole number in words and in standard form.
- Write a whole number in expanded form.

Introduction

Mathematics involves solving problems that involve numbers. We will work with **whole numbers**, which are any of the numbers 0, 1, 2, 3 and so on. We first need to have a thorough understanding of the number system we use. Suppose the scientists preparing a lunar command module know it has to travel 382, 564 kilometers to get to the moon. How well would they do if they didn't understand this number? Do you think it would make more of a difference if the 8 was off by 1 or if the 4 was off by 1?

In this section, you will take a look at digits and place value. You will also learn how to write whole numbers in words, standard form, and expanded form based on the place values of their digits.

The Number System

A **digit** is one of the symbols is one of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. All numbers are made up of one or more digits. Numbers such as 2 have one digit, whereas numbers such as 89 have two digits. To understand what a number really means, you need to understand what the digits represent in a given number.

The position of each digit in a number tells its value, or **place value**. We can use a **place-value chart** like the one below to easily see the place value for each digit. The place values for the digits in 1, 456 are shown in this chart.

Place-Value Chart

Trillions			Billions			Millions			Thousands			Ones		
											1	4	5	6
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

In the number 1, 456 the digit 1 is in the thousands place. The digit 4 is in the hundreds place. The digit 5 is in the tens place, and the digit 6 is in the ones place.

As you see above, you can tell a digit's value by looking at its position. Look at the number of digits to the right of the digit, or write your number into a place-value chart, with the last digit in the ones column. Both these methods are shown in the example below.

Example

The development of a city over the past twenty years cost $ 962, 234, 532, 274, 312. **What is the value of the digit 6 in this number?**

Solution

Place-Value Chart

Trillions			Billions			Millions			Thousands			Ones		
9	6	2	2	3	4	5	3	2	2	7	4	3	1	2
Hundreds	**Tens**	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

Write the number in the place-value chart. Read the value of the 6 from the chart.

$ 962, 234, 532, 274, 312

60, 000, 000, 000, 000

Answer: The value of the digit 6 is 60 trillion.

Exercise

In a far away galaxy, there are 2, 968, 351, 472 stars. What does the digit 3 represent in this problem?

a. three hundred thousands
b. three hundreds
c. three hundred trillions
d. three hundred millions

Answer

a. Correct. The digit 3 is in the hundred thousands place.
b. Incorrect. The digit 4 is in the hundreds place. The correct answer is three hundred thousands.
c. Incorrect. The number is less than a trillion, so this digit does not exist here. The correct answer is three hundred thousands.
d. Incorrect. The digit 9 is in the hundred millions place. The correct answer is three hundred thousands.

Periods and Standard Form

The **standard form** of a number refers to a type of notation in which digits are separated into groups of three by commas. These groups of three digits are known as **periods**. For example, 893, 450, 243 has three periods with three digits in each period, as shown below.

Place-Value Chart

Trillions			Billions			Millions			Thousands			Ones		
						8	9	3	4	5	0	2	4	3
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

Let’s examine the number of digits and periods in a greater number. The number of body cells in an average adult human is about one hundred trillion. This number is written as 100, 000, 000, 000, 000. Notice that there are 15 digits and 5 periods. Here is how the number would look in a place-value chart.

Place-Value Chart

Trillions			Billions			Millions			Thousands			Ones		
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

You are now familiar with the place values of greater numbers, so let’s examine a problem that involves converting from standard form to a word name.

Converting Standard Form to Word Names

We often use word names to write numbers. A word name for 42 is “forty-two.” The total number of weeks in a year, 52, is written as “fifty-two.”

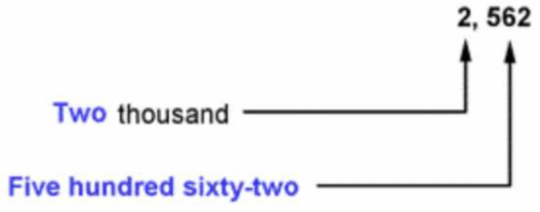
For whole numbers with three digits, use the word “hundred” to describe how many hundreds there are in the number. For example, for the number of days in a normal year, 365, the digit 3 is in the hundreds place. The word name for the number is “three hundred sixty-five.”

For whole numbers with four digits, begin the name with the number of thousands, followed by the period name, as in the example below.

✓ Example

A man owes $2, 562 on a car. Write the word name for this.

Solution


2, 562
Two thousand
Five hundred sixty-two

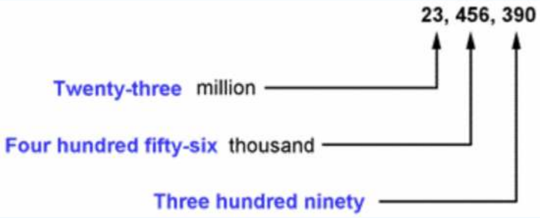
The word name is two thousand, five hundred sixty-two.

For word names of greater numbers, begin at the left with the greatest period. For each period, write the one- to three-digit number in the period, and then the period name. See the example below.

✓ Example

The construction of a new athletic center cost $23, 456, 390. Write the word name for this number.

Solution


23, 456, 390
Twenty-three million
Four hundred fifty-six thousand
Three hundred ninety

The word name is twenty-three million, four hundred fifty-six thousand, three hundred ninety.

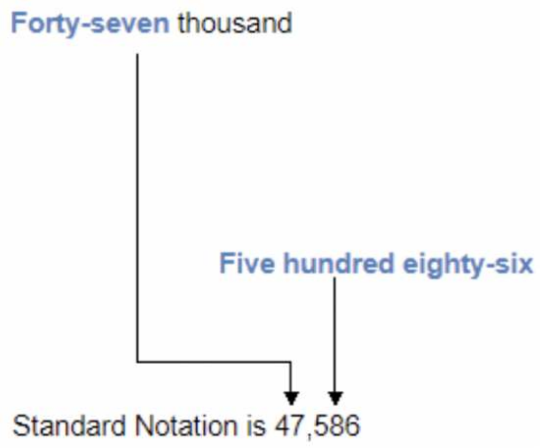
Converting Word Names to Standard Form

When converting word names to standard form, the word “thousand” tells you which period the digits are in. See the example below.

✓ Example

Forty-seven thousand, five hundred eighty-six blueberries are produced on a farm over the course of three years. Write this number in standard form.

Solution


Forty-seven thousand
Five hundred eighty-six
Standard Notation is 47,586

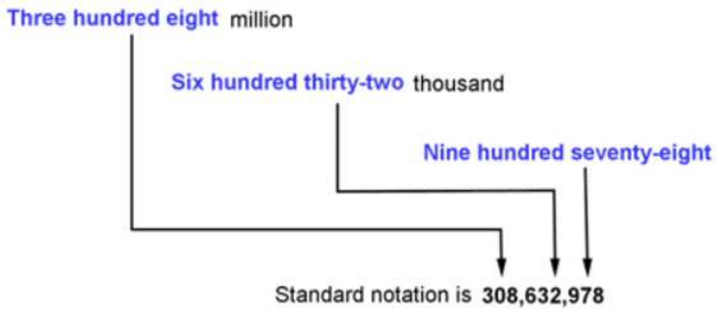
The number in standard form is 47, 586.

Below is an example with a number containing more digits. The words “million” and “thousand” tell you which periods the digits are in. The periods are separated by commas.

✓ Example

There are three hundred eight million, six hundred thirty-two thousand, nine hundred seventy-eight bacteria in a sample of soil. Write this number in standard form.

Solution


Three hundred eight million
Six hundred thirty-two thousand
Nine hundred seventy-eight
Standard notation is 308,632,978

The number in standard form is 308, 632, 978

Some numbers in word form may not mention a specific period. For example, three million, one hundred twelve written in standard form is 3, 000, 112. Because the thousands period is not mentioned, you would write three zeros in the thousands period. You can use a place-value chart to make it easier to see the values of the digits. See the example below.

✓ Example

A company had a new office building constructed. The final cost was seventy-four million, three hundred sixty-two dollars. Write this number in standard form.

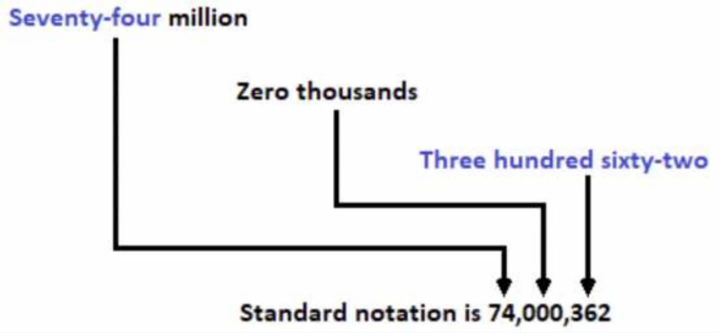
Solution

Place-Value Chart

Trillions			Billions			Millions			Thousands			Ones		
							7	4	0	0	0	3	6	2
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

Placing this number in a place- value chart shows that the thousands period is zero.

Remember to separate each period with a comma.



The number written in standard form is $74, 000, 362.

Writing Numbers in Expanded Form

Sometimes it is useful to write numbers in **expanded form**. In expanded form, the number is written as a sum of the value of each digit.

Example

During the week, Mike drives a total of 264 miles. Write 264 in expanded form.

Solution

First, identify the value of each digit.

In numerical form:	
The 2 in 264	200
The 6 in 264	60
The 4 in 264	4
In word form:	
The 2 in 264	2 **hundreds**
The 6 in 264	6 **tens**
The 4 in 264	4 **ones**
Then, write the numbers as a sum.	

264 written in expanded form is

200 + 60 + 4 or

2 hundreds + 6 tens + 4 or

$(2 \cdot 100) + (6 \cdot 10) + (4 \cdot 1)$

You can also use a place-value chart to help write a number in expanded form. Suppose the number of cars and pick-up trucks in the U.S. at this very moment is 251,834,697. Place this number in a place-value chart.

Place-Value Chart

Trillions			Billions			Millions			Thousands			Ones		
						2	5	1	8	3	4	6	9	7

Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

2 hundred millions	200,000,000
+5 ten millions	+50,000,000
+1 million	+1,000,000
+8 hundred thousands	+800,000
+3 ten thousands	+30,000
+4 thousands	+4,000
+6 hundreds	+600
+9 tens	+90
+7 ones	+7

Summary

Whole numbers that are greater than 9 consist of multiple digits. Each digit in a given number has a place value. To better understand place value, numbers can be put in a place-value chart so that the value of each digit can be identified. Numbers with more than three digits can be separated into groups of three digits, known as periods. Any whole number can be expressed in standard form, expanded form, or as a word name.

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1.1.2: Rounding Whole Numbers

Learning Objectives

- Learn the rules for rounding.
- Round whole numbers to specific place values, including tens, hundreds, and thousands.

Introduction

In some situations, you don't need an exact answer. In these cases, **rounding** the number to a specific **place value** is possible. For example, if you travelled 973 miles, you might want to round the distance to 1,000 miles, which is easier to think about. Rounding also comes in handy to see if a calculation is reasonable.

Rounding Whole Numbers

These are the rules for rounding **whole numbers**:

First, identify the digit with the place value to which you are rounding. You might circle or highlight the digit so you can focus on it better.

Then, determine the possible numbers that you would obtain by rounding. These possible numbers are close to the number that you're rounding to, but have zeros in the digits to the right.

If you are rounding 186 to the nearest ten, then 180 and 190 are the two possible numbers to round to, as 186 is between 180 and 190. But how do you know whether to round to 180 and 190?

> Usually, round a number to the number that is closest to the original number.
>
> When a number is halfway between the two possible numbers, round up to the greater number.

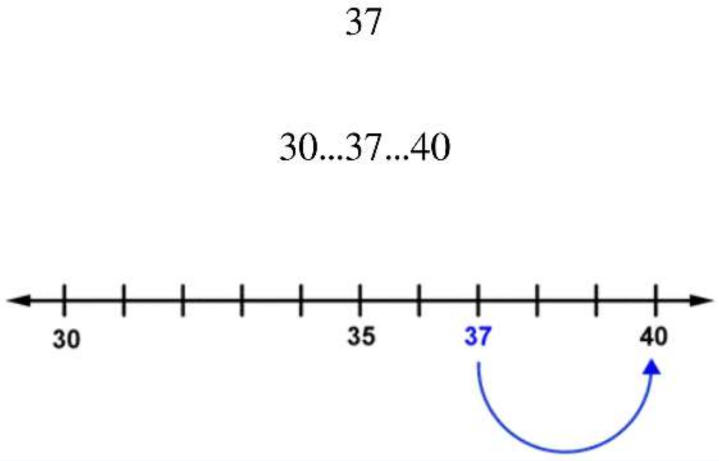
Since 186 is between 180 and 190, and 186 is closer to 190, you round up to 190.

You can use a number line to help you round numbers.

Example

A camera is dropped out of a boat, and sinks to the bottom of a pond that is 37 feet deep. Round 37 to the nearest ten.

Solution



The digit you're rounding to is the tens digit, 3.

37 is between 30 and 40.

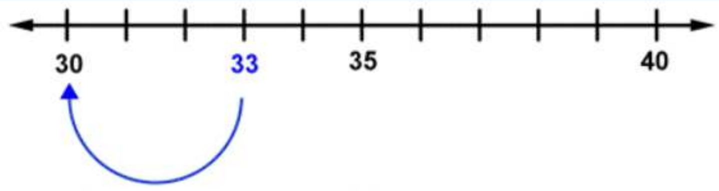
37 is only 3 away from 40, but it's 7 away from 30. So, 37 is closer to 40.

To the nearest ten, 37 rounds to 40.

✓ Example

Round 33 to the nearest ten.

Solution



33 rounds to 30 because 33 is closer to 30.

To the nearest ten, 33 rounds to 30.

You can determine where to round without using a number line by looking at the digit to the right of the one you're rounding to. If that digit is less than 5, round down. If it's 5 or greater, round up. In the example above, you can see without a number line that 33 is rounded to 30 because the ones digit, 3, is less than 5.

✓ Example

Round 77 to the nearest ten.

Solution

77 rounds to 80 because the ones digit, 7, is 5 or greater.

77 rounded to the nearest ten is 80.

✓ Example

There are 576 jellybeans in a jar. Round this number to the nearest ten.

Solution

576 rounds to 580 because the ones digit, 6, is 5 or greater.

576 rounded to the nearest ten is 580.

In the previous examples, you rounded to the tens place. The rounded numbers had a 0 in the ones place. If you round to the nearest hundred, the rounded number will have zeros in the tens and ones places. The rounded number will resemble 100, 500, or 1, 200.

✓ Example

A runner ran 1,539 meters, but describes the distance he ran with a rounded number. Round 1,539 to the nearest hundred.

Solution

1,539 rounds to 1,500 because the next digit is less than 5.

1,539 rounded to the nearest hundred is 1,500.

If you round to the nearest thousand, the rounded number will have zeros in the hundreds, tens, and ones places. The rounded number will resemble 1,000 , 2,000 , or 14,000.

✓ Example

A plane's altitude increased by 2,721 feet. Round this number to the nearest thousand.

Solution

2,721 rounds to 3,000 because the next digit, 7, is 5 or greater.

2,721 rounded to the nearest thousand is 3,000.

Now that you know how to round to the nearest ten, hundred, and thousand, try rounding to the nearest ten thousand.

✓ Example

Round 326,749 to the nearest ten thousand.

Solution

326,749 rounds to 330,000 because the next digit, 6, is 5 or greater.

326,749 rounded to the nearest ten thousand is 330,000.

? Exercise

A record number of 23,386 people voted in a city election. Round this number to the nearest hundred.

A. 23,300
B. 23,400
C. 23,000
D. 23,390

Answer

A. Incorrect. The two possible numbers are 23,300 and 23,400, but 23,386 is closer to 23,400. The tens digit, 8, is 5 or greater, so you should round up. The correct answer is 23,400.
B. Correct. The two possible numbers are 23,300 and 23,400, and 23,386 is closer to 23,400. The tens digit, 8, is 5 or greater, so you should round up.
C. Incorrect. This number is rounded to the nearest thousand, not the nearest hundred. The correct answer is 23,400.
D. Incorrect. This number is rounded to the nearest ten, not the nearest hundred. The correct answer is 23,400.

Summary

In situations when you don't need an exact answer, you can round numbers. When you round numbers, you are always rounding to a particular place value, such as the nearest thousand or the nearest ten. Whether you round up or round down usually depends on which number is closest to your original number. When a number is halfway between the two possible numbers, round up to the larger number.

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1.1.3: Comparing Whole Numbers

Learning Objectives

- Use > or < to compare whole numbers.

Introduction

There will be times when it's helpful to compare two numbers and determine which number is greater, and which one is less. This is a useful way to compare quantities such as travel time, income, or expenses. The symbols < and > are used to indicate which number is greater, and which is less than the other.

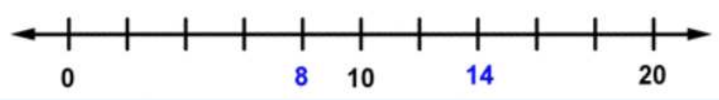
Comparing Whole Numbers

When comparing the values of two numbers, you can use a number line to determine which number is greater. The number on the right is always greater than the number on the left. In the example below, you can tell that 14 is greater than 8 because 14 is to the right of 8 on the number line.

Example

Which number is greater, 8 or 14?

Solution



14 is to the right of 8, so 14 is greater than 8.

14 is greater than 8.

In the example below, you can determine which number is greater by comparing the digits in the ones place value.

Example

Which number is greater, 33 or 38?

Solution

In both 33 and 38, the digit in the tens place is 3.

Because they have the same number in the tens place, you can determine which one is greater by comparing the digits in the ones place.

In the number 38, the digit in the ones place is 8.

In the number 33, the digit in the ones place is 3.

Because 8 is greater than 3,38 is greater than 33.

38 is greater than 33. This answer was obtained from comparing their digits in the ones place value, which are 8 and 3, respectively.

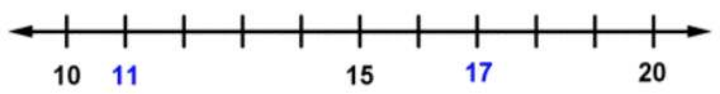
Exercise

Which number is greater, 17 or 11?

A. 17
B. 11

Answer

A. Correct. The number 17 is 6 units to the right of 11 on the number line.
B. Incorrect. The number 11 is to the left of 17 on the number line, so 17 is greater. The correct answer is 17.



If one number is significantly greater than another number, it may be difficult to compare the numbers effectively with a number line. In general, **whole numbers** with more digits are greater than whole numbers with fewer digits. For example, 542 is greater than 84 because 542 has the digit 5 in the hundreds place. There are no hundreds in 84.

? Exercise

Which number is greater, 71 or 710?

A. 71
B. 710

Answer

A. Incorrect. You can see that there is no digit in the hundreds place, which means that 71 is less than 710. The correct answer is 710.
B. Correct. The number 710 has 7 hundreds, but 71 has no hundreds.

Inequalities

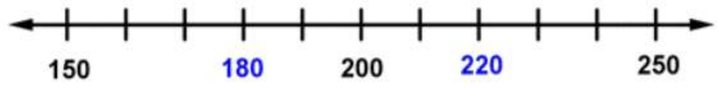
An **inequality** is a mathematical sentence that compares two numbers that aren’t equal. Instead of an equal sign (=), inequalities use greater than (>) or less than (<) symbols. The important thing to remember about these symbols is that the small end points towards the lesser number, and the larger (open) end is always on the side of the greater number.

There are other ways to remember this. For example, the wider part of the symbol represents the jaws of an alligator, which “gobbles up” the greater number. So "35 is greater than 28" can be written as 35 > 28, and "52 is less than 109" can be written as 52 < 109.

✓ Example

Replace the question mark with < or > to make a true sentence: 180 ? 220.

Solution



180 is to the left of 220, so 180 < 220. The symbol points at 180, which is the lesser number.

180 < 220

? Exercise

Which expression correctly compares the numbers 85 and 19?

A. 85 < 19
B. 19 = 85
C. 85 > 19
D. 19 > 85

Answer

A. Incorrect. The symbol should point at the lesser number, 19. On a number line, 85 is to the right of 19, so 85 is greater than 19. The correct answer is 85 > 19

B. Incorrect. This symbol says that 85 is equal to 19, which is false. On a number line, 85 is to the right of 19, so 85 is greater than 19. The correct answer is 85 > 19.
C. Correct. The open part of the symbol faces the larger number, 85, and the symbol points at the smaller number, 19.
D. Incorrect. The symbol should point at the smaller number, 19. On a number line, 85 is to the right of 19, so 85 is greater than 19. The correct answer is 85 > 19.

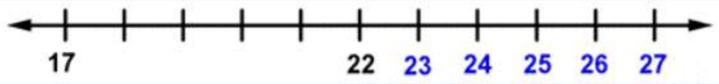
Many times, an answer needs to be a range of values rather than just a single value. For example, you want to make more than $22 an hour. This can be expressed as all numbers greater than 22. See the example below.

✓ Example

? > 22. What whole number(s) will make this statement true?

Solution

The symbol points at 22, so the numbers you want to replace the question mark with are *greater* than 22. There are many numbers that work.



23, 24, 25, 26 and any additional whole numbers that are greater than 26 make this statement true.

? Exercise

A farmer has produced 230 pumpkins for the autumn harvest. Last year, he produced 198. Write an expression that compares these two numbers.

A. 230 > 198
B. 230 < 198
C. 198 = 230
D. 198 > 230

Answer

A. Correct. 230 is greater than 198, and this is reflected in the symbol because the open part of the symbol faces 230.
B. Incorrect. 230 is greater than 198, and the symbol is pointing in the wrong direction, with the open part facing the lesser number. The correct answer is 230 > 198.
C. Incorrect. This statement says that 198 is equal to 230, which is incorrect. The correct answer is 230 > 198.
D. Incorrect. 230 is greater than 198, and the symbol is pointing in the wrong direction, with the open part facing the lesser number. The correct answer is 230 > 198.

Summary

To compare two values that are not the same, you can write an inequality. You can use a number line or place value to determine which number is greater than another number. Inequalities can be expressed using greater than (>) or less than (<) symbols.

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SECTION OVERVIEW

1.2: Adding and Subtracting Whole Numbers

Topic hierarchy

1.2.1: Adding Whole Numbers and Applications

1.2.2: Subtracting Whole Numbers and Applications

1.2.3: Estimation

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1.2.1: Adding Whole Numbers and Applications

Learning Objectives

- Add whole numbers without regrouping.
- Add whole numbers with regrouping.
- Find the perimeter of a polygon.
- Solve application problems using addition.

Introduction

Adding is used to find the total number of two or more quantities. The total is called the **sum**, or the number that results from the addition. You use addition to find the total distance that you travel if the first distance is 1,240 miles and the second distance is 530 miles. The two numbers to be added, 1,240 and 530, are called the **addends**. The total distance, 1,770 miles, is the sum.

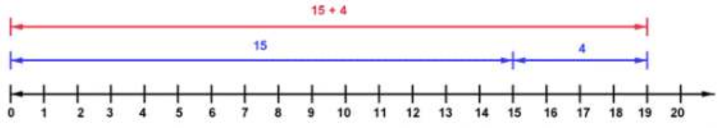
Adding Whole Numbers without Regrouping

Adding numbers with more than one digit requires an understanding of **place value**. The place value of a digit is the value based on its position within the number. In the number 492, the 4 is in the hundreds place, the 9 is in the tens place, and the 2 is in the ones place. You can use a number line to add. In the example below, the blue lines represent the two quantities, 15 and 4, that are being added together. The red line represents the resulting quantity.

Example

15 + 4 = ?

Solution



On the number line, the blue line segment stretches across 15 units, representing the number 15. The second blue segment shows that if you add 4 more units, the resulting number is 19.

15 + 4 = 19

You can solve the same problem without a number line, by adding vertically. When adding numbers with more than 1 digit, it is important to line up your numbers by place value, as in the example below. You must add ones to ones, tens to tens, hundreds to hundreds, and so on.

Example

15 + 4 = ?

Solution

$\begin{array}{r} 15 \\ +\ \ 4 \\ \hline \end{array}$	Because 5 and 4 have the same place value, make sure they are aligned when you add.
$\begin{array}{r} 15 \\ +\ \ 4 \\ \hline 9 \end{array}$	First, add the ones digits (the numbers on the right). The result goes in the ones place for the answer.
$\begin{array}{r} 15 \\ +\ \ 4 \\ \hline 19 \end{array}$	Then, add the tens digits and put the result in the tens place of the answer. In this case, there is no tens digit in the second number, so the result is the same as the tens digit of the first number (1).

15 + 4 = 19

This strategy of lining up the numbers is effective for adding a series of numbers as well.

✓ Example

1 + 2 + 3 + 2 = ?

Solution

$$\begin{array}{r} 1 \\ 2 \\ 3 \\ +2 \\ \hline 8 \end{array}$$

1 + 2 + 3 + 2 = 8

Adding Whole Numbers with Regrouping

When adding whole numbers, a place-value position can have only one digit in it. If the sum of digits in a place value position is more than 10, you have to **regroup** the number of tens to the next greater place value position.

When you add, make sure you line up the digits according to their place values, as in the example below. As you regroup, place the regrouped digit above the appropriate digit in the next higher place value position and add it to the numbers below it.

✓ Example

45 + 15 = ?

Solution

$\begin{array}{r} 1 \\ 45 \\ +15 \\ \hline 0 \end{array}$	Add the ones. Regroup as needed. The sum of 5 and 5 is 10. This is 1 ten and 0 ones. Write the number of ones (0) in the ones place and the 1 ten in the tens place above the 4.
$\begin{array}{r} 1 \\ 45 \\ +\ \ 15 \\ \hline 60 \end{array}$	Add the tens, 1 + 4 + 1 is 6 tens. The final sum is 60.

45 + 15 = 60

You must add digits in the ones place first, the digits in the tens place next, and so on. Go from right to left.

✓ Example

4,576 + 698 = ?

Solution

$\begin{array}{r} 4,576 \\ +\ \ 698 \\ \hline \end{array}$	First, write the problem with one addend on top of the other. Be sure you line up the place values!
$\begin{array}{r} 1 \\ 4,576 \\ +\ \ 698 \\ \hline 4 \end{array}$	Add the numbers, 6 and 8, in the ones place. Since the sum is 14, write the ones value (4) in the ones place of the answer. Write the 1 ten in the tens place above the 7.

$\begin{array}{r} 11 \\ 4,576 \\ +\quad 698 \\ \hline 74 \end{array}$	Add the numbers in the tens place. Since the sum is 17 tens, regroup 17 tens as 1 hundred, 7 in the tens place in the answer and write the 1 hundred in the hundreds place above the 5.
$\begin{array}{r} 1\ \ 11 \\ 4,576 \\ +\quad 698 \\ \hline 274 \end{array}$	Add the numbers in the hundreds place, including the 1. Again, the sum is more than one digit. Rename 12 hundreds as 2 hundreds and 1 thousand. Write the 2 in the hundreds place and the 1 above the 4 in the thousands place.
$\begin{array}{r} 1\ \ 11 \\ 4,576 \\ +\quad 698 \\ \hline 5,274 \end{array}$	Add the numbers in the thousands place, including the 1. The final sum is 5,274.

4,576 + 698 = 5,274

Adding Numbers Using the Partial Sums Method

Another way to add is the partial sums method. In the example below, the sum of 23 + 46 is found using the partial sums method. In this method, you add together all the numbers with the same place value and record their *values* (not just a single digit). Once you have done this for each place value, add their sums together.

Example

23 + 46 = ?

Solution

Step 1: Add Tens	
$\begin{array}{rr} 23 & 20 \\ 46 & +40 \\ \hline & 60 \end{array}$	Let's begin by adding the values in the tens position, the 2 and 4. The values are written as 20 and 40.
Step 2: Add Ones	
$\begin{array}{rr} 23 & 3 \\ 46 & +6 \\ \hline & 9 \end{array}$	Add the values in the ones place, the 3 and 6.
Step 3: Add Parts	
$\begin{array}{r} 60 \\ +\quad 9 \\ \hline 69 \end{array}$	Finally, add the two sums, 60 and 9, together.

23 + 46 = 69

The next example adds a series of three numbers. Notice that hundreds is the greatest place value now, so hundreds are added before the tens. (You can add in any order that you prefer.) Also notice that in Step 3, the value in the ones column for 350 is zero, but you still add that in to make sure everything is accounted for.

Example

225 + 169 + 350 = ?

Solution

Step 1: Add Hundreds	

$\begin{array}{rr} 225 & 200 \\ 169 & 100 \\ 350 & +300 \\ \hline & 600 \end{array}$	Add the values represented by the digits 2, 1, and 3 in the hundreds place first. This gives a sum of 600.
Step 2: Add Tens	
$\begin{array}{rr} 225 & 20 \\ 169 & 60 \\ 350 & +50 \\ \hline & 130 \end{array}$	Next, add the values from the digits in the tens place, the 2, 6, and 9. The sum is 130.
Step 3: Add Ones	
\(\ \begin{array}{r} 22\color{blue}5 & \color{blue}5\ 16\color{blue}9& \color{blue} 9\ 35\color{blue}0&\color{blue}+0\ \hline &\color{blue}14 \end{array}\)	Add the values from the digits in the ones place, the 5, 9, and 0. The sum is 14.
Step 4: Add Parts	
$\begin{array}{r} 600 \\ 130 \\ +\ \ 14 \\ \hline 744 \end{array}$	At this point, you have a sum for each place value. Add together these three sums, which gives a final value of 744.

225 + 169 + 350 = 744

Exercise

A local company built a playground at a park. It took the company 124 hours to plan out the playground, 243 hours to prepare the site, and 575 hours to build the playground. Find the total number of hours the company spent on the project.

A. 937 hours
B. 812 hours
C. 742 hours
D. 942 hours

Answer

A. Incorrect. You probably did not add the ones correctly. The correct answer is 942 hours.
B. Incorrect. You probably did not add the tens correctly. The correct answer is 942 hours.
C. Incorrect. You probably did not add the hundreds correctly. The correct answer is 942 hours.
D. Correct. You carried out the partial sums process effectively. The parts should be 800 + 130 + 12.

When adding multi-digit numbers, use the partial sums method or any method that works best for you.

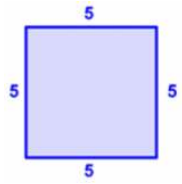
Finding the Perimeter of a Polygon

A **polygon** is a many-sided closed figure with sides that are straight line segments. Triangles, rectangles, and pentagons (five-sided figures) are polygons, but a circle or semicircle is not. The **perimeter** of a polygon is the distance around the polygon. To find the perimeter of a polygon, add the lengths of its sides, as in the example below.

Example

One side of a square has a length of 5cm. Find the perimeter.

Solution

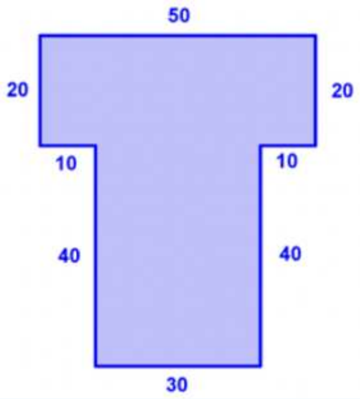
	Draw the polygon and label the lengths of the sides. Since the side lengths of a square are equal, each side is 5cm.
$\begin{array}{r} 5\\ 5\\ 5\\ +\quad 5\\ \hline 20 \end{array}$	Add the lengths of each side, 5 + 5 + 5 + 5.

The perimeter is 20cm.

The key part of completing a polygon problem is correctly identifying the side lengths. Once you know the side lengths, you add them as you would in any other addition problem.

Example

A company is planning to construct a building. Below is a diagram illustrating the shape of the building's floor plan. The length of each side is given in the diagram. Measurements for each side are in feet. Find the perimeter of the building.



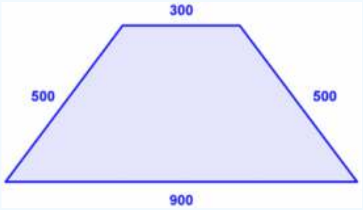
Solution

$\begin{array}{r} 50\\ 20\\ 20\\ 10\\ 10\\ 40\\ 40\\ +30\\ \hline 220 \end{array}$	Add the lengths of each side, making sure to align all numbers according to place value.

The perimeter is 220 feet.

Exercise

Find the perimeter of the trapezoid in feet.



A. 2,200 feet
B. 1,200 feet
C. 200 feet
D. 3,200 feet

Answer

A. Correct. You added the lengths of the sides together successfully.
B. Incorrect. You probably did not add the two diagonal sides of the trapezoid. The correct answer is 2,200 feet.
C. Incorrect. You probably subtracted the total length of the two sides, 1,000 feet, from the total length of the top and bottom, 1,200. The correct answer is 2,200 feet.
D. Incorrect. You may have added some of the sides more than once. The correct answer is 2,200 feet.

Solving Application Problems

Addition is useful for many kinds of problems. When you see a problem written in words, look for key words that let you know you need to *add* numbers.

Example

A woman preparing an outdoor market is setting up a stand with 321 papayas, 45 peaches, and 213 mangos. How many pieces of fruit in total does the woman have on her stand?

Solution

$\begin{array}{r} 321 \\ 45 \\ +213 \\ \hline \end{array}$	The words “how many… in total” suggest that you need to add the numbers of the different kinds of fruits. Use any method you like to add the numbers. Below, the partial sums method is used.
Step 1: Add Hundreds	
$\begin{array}{rr} 321 & 300 \\ 045 & 0 \\ 213 & +200 \\ \hline & 500 \end{array}$	Add the numbers represented by the digits in the hundreds place first, the 3, 0 and 2. This gives a sum of 500.
Step 2: Add Tens	
$\begin{array}{rr} 321 & 20 \\ 045 & 40 \\ 213 & +10 \\ \hline & 70 \end{array}$	Next, add the numbers represented by the digits from the tens place, the 2, 4, and 1. The sum is 70.
Step 3: Add Ones	
$\begin{array}{rr} 321 & 1 \\ 045 & 5 \\ 213 & +3 \\ \hline & 9 \end{array}$	Add the numbers from the ones, the 1, 5, and 3.

Step 4: Add Parts	
$\begin{array}{r} 500 \\ 70 \\ +\quad 9 \\ \hline 579 \end{array}$	Add together the three previous sums. The final sum is 579.

The woman has 579 pieces of fruit on her stand.

Example

Lynn has 23 rock CDs, 14 classical music CDs, 8 country and western CDs, and 6 movie soundtrack CDs. How many CDs does she have in all?

Solution

$\begin{array}{r} 23 \\ 14 \\ 8 \\ +6 \\ \hline \end{array}$	The words “how many… in all” suggest that addition is the way to solve this problem. To find how many CDs Lynn has, you need to add the number of CDs she has for each music style.
$\begin{array}{r} {}^{2} \\ 23 \\ 14 \\ 8 \\ +\quad 6 \\ \hline 51 \end{array}$	Use whatever method you prefer to find the sum of the numbers.

Lynn has 51 CDs.

The following phrases also appear in problem situations that require addition.

Phrase	Example problem
Add to	Jonah was planning a trip from Boston to New York City. The distance is 218 miles. His sister wanted him to visit her in Springfield, Massachusetts, on his way. Jonah knew this would **add** 17 miles **to** his trip. How long is his trip if he visits his sister?
Plus	Carrie rented a DVD and returned it one day late. The store charged $5 for a two-day rental, **plus** a $3 late fee. How much did Carrie pay for the rental?
Increased by	One statistic that is important for football players in offensive positions is *rushing*. After four games, one player had rushed 736 yards. After two more games, the number of yards rushed by this player **increased by** 352 yards. How many yards had he rushed after the six games?
More than	Lavonda posted 38 photos to her social network profile. Chris posted 27 **more** photos to his **than** Lavonda. How many photos did Chris post?

Example

Lena was planning a trip from her home in Amherst to the Museum of Science in Boston. The trip is 91 miles. She had to take a detour on the way, which added 13 miles to her trip. What is the total distance she traveled?

Solution

The word “added” suggests that addition is the way to solve this problem.

To find the total distance, you need to add the two distances.

$$\begin{array}{r} 91 \\ +\ \ 13 \\ \hline 104 \end{array}$$

The total distance is 104 miles.

It can help to seek out words in a problem that imply what operation to use. See if you can find the key word(s) in the following problem that provide you clues on how to solve it.

? Exercise

A city was struck by an outbreak of a new flu strain in December. To prevent another outbreak, 3,462 people were vaccinated against the new strain in January. In February, 1,298 additional people were vaccinated. How many people in total received vaccinations over these two months?

A. 2,164
B. 4,760
C. 4,660
D. 4,750

Answer

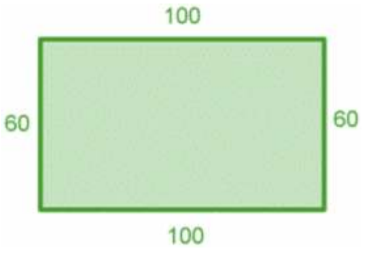
A. Incorrect. You probably subtracted instead of adding. The correct answer is 4,760.
B. Correct. You recognized this as an addition problem and successfully carried out your addition process.
C. Incorrect. You probably did not regroup to the hundreds place, or added the hundreds places incorrectly. The correct answer is 4,760.
D. Incorrect. You probably did not regroup to the tens place, or added the tens place incorrectly. The correct answer is 4,760.

Drawing a diagram to solve problems is very useful in fields such as engineering, sports, and architecture.

✓ Example

A coach tells her athletes to run one lap around a soccer field. The length of the soccer field is 100 yards, while the width of the field is 60 yards. Find the total distance that each athlete will have run after completing one lap around the perimeter of the field.

Solution

	The words “total distance” and “perimeter” both tell you to add. Draw the soccer field and label the various sides so you can see the numbers you are working with to find the perimeter.
$\begin{array}{r} {\scriptstyle 1} \\ 100 \\ 100 \\ 60 \\ +\ \ 60 \\ \hline 20 \end{array}$	There is a zero in the ones place, and the sum of 6 and 6 in the tens place is 12 tens. Place 2 tens in the tens place in the answer, and regroup 10 tens as 1 hundred.

$\begin{array}{r} {\color{blue}1} \\ 100 \\ 100 \\ 60 \\ +\quad 60 \\ \hline {\color{blue}3}20 \end{array}$	By adding the 1 hundred to the other digits in the hundreds place, you end up with a 3 in the hundreds place of the answer.

Each athlete will have run 320 yards.

Summary

You can add numbers with more than one digit using any method, including the partial sums method. Sometimes when adding, you may need to regroup to the next greater place value position. Regrouping involves grouping ones into groups of tens, grouping tens into groups of hundreds, and so on. The perimeter of a polygon is found by adding the lengths of each of its sides.

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1.2.2: Subtracting Whole Numbers and Applications

Learning Objectives

- Subtract whole numbers without regrouping.
- Subtract whole numbers with regrouping.
- Solve application problems using subtraction.

Introduction

Subtracting involves finding the difference between two or more numbers. It is a method that can be used for a variety of applications, such as balancing a checkbook, planning a schedule, cooking, or travel. Suppose a government official is out of the U.S. on business for 142 days a year, including travel time. The number of days per year she is in the U.S. is the difference of 365 days and 142 days. Subtraction is one way of calculating the number of days she would be in the U.S. during the year.

When subtracting numbers, it is important to line up your numbers, just as with addition. The **minuend** is the greater number from which the lesser number is subtracted. The **subtrahend** is the number that is subtracted from the minuend. A good way to keep minuend and subtrahend straight is that since subtrahend has “subtra” in its beginning, it goes next to the subtraction sign and is the number being subtracted. The **difference** is the quantity that results from subtracting the subtrahend from the minuend. In 86 - 52 = 34, 86 is the *minuend*, 52 is the *subtrahend*, and 34 is the *difference*.

Subtracting Whole Numbers

When writing a subtraction problem, the minuend is placed above the subtrahend. This can be seen in the example below, where the minuend is 10 and the subtrahend is 7.

Example

10 - 7 = ?

Solution

$$\begin{array}{r} 10 \\ - \quad 7 \\ \hline 3 \end{array}$$

10 - 7 = 3

When both numbers have more than one digit, be sure to work with one place value at a time, as in the example below.

Example

689 - 353 = ?

Solution

$\begin{array}{r} 689 \\ -353 \\ \hline \end{array}$	First, set up the problem and align the numbers by place value.
$\begin{array}{r} 689 \\ -353 \\ \hline 6 \end{array}$	Then, subtract the digits in the ones place, 9 - 3.
$\begin{array}{r} 689 \\ -353 \\ \hline 36 \end{array}$	Next, subtract the digits in the tens place, 8 - 5.
$\begin{array}{r} 689 \\ -353 \\ \hline 336 \end{array}$	Finally, subtract the digits in the hundreds place, 6 - 3.

689 - 353 = 336

Lining up numbers by place value becomes especially important when you are working with larger numbers that have more digits, as in the example below.

Example

9,864 - 743 = ?

Solution

$\begin{array}{r} 9864 \\ -\quad 743 \\ \hline \end{array}$	First, set up the problem and align the numbers by place value.
$\begin{array}{r} 9864 \\ -\quad 743 \\ \hline 1 \end{array}$	Then, subtract the digits in the ones place, 4 - 3.
$\begin{array}{r} 9864 \\ -\quad 743 \\ \hline 21 \end{array}$	Next, subtract the digits in the tens place, 6 - 4.
$\begin{array}{r} 9864 \\ -\quad 743 \\ \hline 121 \end{array}$	Now, subtract the digits in the hundreds place, 8 - 7.
$\begin{array}{r} 9864 \\ -\quad 743 \\ \hline 9121 \end{array}$	There is no digit to subtract in the thousands place, so keep the 9.

9,864 - 743 = 9,121

Exercise

Subtract: 2,489 - 345

A. 2,144
B. 1,355
C. 2,834
D. 1,134

Answer

A. Correct. You successfully subtracted 345 from 2,489.
B. Incorrect. You probably did not line up your numbers correctly by place value. The correct answer is 2,144.
C. Incorrect. You probably added when you should have subtracted. The correct answer is 2,144.
D. Incorrect. You probably did not subtract correctly in the tens and thousands places. The correct answer is 2,144.

Subtracting Whole Numbers with Regrouping

You may need to regroup when you subtract. When you **regroup**, you rewrite the number so you can subtract a greater digit from a lesser one.

When you're subtracting, just regroup to the next greater place-value position in the minuend and add 10 to the digit you're working with. As you regroup, cross out the regrouped digit in the minuend and place the new digit above it. This method is demonstrated in the example below.

Example

3,225 - 476 = ?

Solution

$\begin{array}{r} 3225 \\ -\quad 476 \\ \hline \end{array}$	First, set up the problem and align the digits by place value.
$\begin{array}{rrrr} & & 1 & 15 \\ 3 & 2 & \cancel{2} & \cancel{5} \\ - & 4 & 7 & 6 \\ \hline & & & 9 \end{array}$	Since you can’t subtract 6 from 5, regroup, so 2 tens and 5 ones become 1 ten and 15 ones. Now you can subtract 6 from 15 to get 9.
$\begin{array}{rrrr} & 1 & 11 & 15 \\ 3 & \cancel{2} & \cancel{2} & \cancel{5} \\ - \; & 4 & 7 & 6 \\ \hline & & 4 & 9 \end{array}$	Next, you need to subtract 7 tens from 1 ten. Regroup 2 hundreds as 1 hundred, 10 tens and add the 10 tens to 1 ten to get 11 tens. Now you can subtract 7 from 11 to get 4.
$\begin{array}{rrrr} 2 & 11 & 11 & 15 \\ \cancel{3} & \cancel{2} & \cancel{2} & \cancel{5} \\ - \; & 4 & 7 & 6 \\ \hline & 7 & 4 & 9 \end{array}$	To subtract the digits in the hundreds place, regroup 3 thousands as 2 thousands, 10 hundreds and add the 10 hundreds to the 1 hundred that is already in the hundreds place. Now, subtract 4 from 11 to get 7.
$\begin{array}{rrrr} 2 & 11 & 11 & 15 \\ \cancel{3} & \cancel{2} & \cancel{2} & \cancel{5} \\ - \; & 4 & 7 & 6 \\ \hline 2 & 7 & 4 & 9 \end{array}$	Since there is no digit in the thousands place of the subtrahend, bring down the 2 in the thousands place into the answer.

3,225 - 476 = 2,749

? Exercise

Subtract: 1,610 - 880

A. 1,522
B. 2,490
C. 730
D. 620

Answer

A. Incorrect. You probably did not line up the numbers by place value when you were subtracting. The correct answer is 730.
B. Incorrect. You probably added the numbers instead of subtracting. The correct answer is 730.
C. Correct. You successfully subtracted 880 from 1,610.
D. Incorrect. You probably did not regroup correctly. The correct answer is 730.

Checking Your Work

You can check subtraction by adding the difference and the subtrahend. The sum should be the same as the minuend.

✓ Example

Check to make sure that 7 subtracted from 12 is equal to 5.

Solution

12 - 7 = 5	Write out the original equation. The minuend is 12, the subtrahend is 7, and the difference is 5.
$\begin{array}{r} 5 \\ +\quad 7 \\ \hline 12 \end{array}$	Next, add the difference, 5, to the subtrahend, 7, which results in the number 12. This confirms that your answer is correct.

The answer of 5 is correct.

Checking your work is very important and should always be performed when time permits.

Subtracting Numbers Using the Expanded Form

An alternative method to subtract involves writing numbers in expanded form, as shown in the examples below. If you have 4 tens and want to subtract 1 ten, you can just think (4 - 1) tens and get 3 tens. Let's see how that works.

Example

45 - 12 = ?

Solution

$45 = 40 + 5$ $12 = 10 + 2$	Let's write the numbers in expanded form so you can understand what they really mean.
$\begin{array}{r} 45 = 40 + 5 \\ \underline{12 = 10 + 2} \\ 30 \end{array}$	For the tens, the minuend is 40, or 4 tens. The subtrahend is 10, or 1 ten. Since 4 - 1 = 3, 4 tens -1 ten =3 tens, or 30.
$\begin{array}{r} 45 = 40 + 5 \\ \underline{12 = 10 + 2} \\ 30 + 3 \end{array}$	Now, the ones. 5 - 2 = 3. So, 30 + 3 = 33.

45 - 12 = 33

Now let's use this method in the example below, which asks for the difference of 467 and 284. In the tens place of this problem, you need to subtract 8 from 6. What can you do?

Example

467 - 284 = ?

Solution

Step 1: Separate by place value	
$\begin{array}{r} 4 \text{ hundreds} + 6 \text{ tens} + 7 \text{ ones} \\ -2 \text{ hundreds} + 8 \text{ tens} + 4 \text{ ones} \end{array}$	Write both the minuend and the subtrahend in expanded form.
Step 2: Identify impossible differences	
6 - 8 = [ ]	Identify differences that are not whole numbers. Since 8 is greater than 6, you won't get a whole number difference.
Step 3: Regroup	
$\begin{array}{r} 3 \text{ hundreds } + 16 \text{ tens } + 7 \text{ ones} \\ \underline{-2 \text{ hundreds } + 8 \text{ tens } + 4 \text{ ones}} \\ 1 \text{ hundred } + 8 \text{ tens } + 3 \text{ ones} \end{array}$	Regroup one of the hundreds from the 4 hundreds into 10 tens and add it to the 6 tens. Now you have 16 tens. Subtracting 8 tens from 16 tens yields a difference of 8 tens.
Step 4: Combine the parts	
$1 \text{ hundred } + 8 \text{ tens } + 3 \text{ ones } = 183$	Combining the resulting differences for each place value yields a final answer of 183.

467 - 284 = 183

? Exercise

A woman who owns a music store starts her week with 965 CDs. She sells 452 by the end of the week. How many CDs does she have remaining?

A. 313

B. 513
C. 510
D. 1,417

Answer

A. Incorrect. You probably made an error when you were subtracting digits in the hundreds place. The correct answer is 513.
B. Correct. You successfully subtracted 452 from 965. In expanded form, 513 is 5 hundreds, 1 ten, and 3 ones.
C. Incorrect. You probably made an error when you were subtracting digits in the ones place. The correct answer is 513.
D. Incorrect. You probably added instead of subtracted. The correct answer is 513.

Example

45 - 17 = ?

Solution

$45 = 40 + 5$ $17 = 10 + 7$	When you try to subtract 17 from 45, you would first try to subtract 7 from 5. But 5 is less than 7. Let’s write the numbers in expanded form so you can see what they really mean.
$45 = 30 + 15$ $\underline{17 = 10 + 7}$	Now, regroup 4 tens as 3 tens and 10 ones. Add the 10 ones to 5 ones to get 15 ones, which is greater than 7 ones, so you can subtract.
$\begin{array}{rrcr} 45 = & 30 & + & 15 \\ -17 = & 10 & + & 7 \\ \hline & 20 & + & 8 \end{array}$	Finally, subtract 7 from 15, and 10 from 30 and add the results: 20 + 8 = 28.

45 - 17 = 28

Solve Application Problems Using Subtraction

You are likely to run into subtraction problems in everyday life, and it helps to identify key phrases in a problem that indicate that subtraction is either used or required. The following phrases appear in problem situations that require subtraction.

Phrase or word	Example problem
Less than	The cost of gas is 42 cents per gallon **less than** it was last month. The cost last month was 280 cents per gallon. How much is the cost of gas this month?
Take away	Howard made 84 cupcakes for a neighborhood picnic. People **took away** 67 cupcakes. How many did Howard have left?
Decreased by	The temperature was 84° Fahrenheit in the early evening. It **decreased by** \(\ 15^{\circ}\) overnight. What was the temperature in the morning?
Subtracted from	Jeannie works in a specialty store on commission. When she sells something for $\$75$, she **subtracts** $\$15$ **from** the $\$75$ and gives the rest to the store. How much of the sale goes to the store?
The difference	What is **the difference** between this year’s rent of $\$1,530$ and last year’s rent of $\$1,450$?

Fewer than	The number of pies sold at this year’s bake sale was 15 **fewer** than the number sold at the same event last year. Last year, 32 pies were sold. How many pies were sold this year?

When translating a phrase such as “5 fewer than 39" into a mathematical expression, the order in which the numbers appears is critical. Writing 5 - 39 would not be the correct translation. The correct way to write the expression is 39 - 5. This results in the number 34, which is 5 fewer than 39. The chart below shows how phrases with the key words above can be written as mathematical expressions.

Phrase	Expression
three subtracted from six	6 - 3
the difference of ten and eight	10 - 8
Nine fewer than 40	40 - 9
Thirty-nine decreased by fourteen	39 - 14
Eighty-five take away twelve	85 - 12
Four less than one hundred eight	108 - 4

✓ Example

Each year, John is out of the U.S. on business for 142 days, including travel time. The number of days per year he is in the U.S. is the difference of 365 days and 142 days. How many days during the year is John in the U.S.?

Solution

$\begin{array}{r} 365 \\ -142 \\ \hline \end{array}$	The words “the difference of” suggest that you need to subtract to answer the problem. First, write out the problem based on the information given and align numbers by place value, 365 - 142.
$\begin{array}{r} 365 \\ -142 \\ \hline 3 \end{array}$	Then, subtract numbers in the ones place, 5 - 2.
$\begin{array}{r} 365 \\ -142 \\ \hline 23 \end{array}$	Subtract numbers in the tens place, 6-4.
$\begin{array}{r} 365 \\ -142 \\ \hline 223 \end{array}$	Finally, subtract numbers in the hundreds place, 3-1.

John is in the U.S. for 223 days during the year.

? Exercise

To make sure he was paid up for the month on his car insurance, Dave had to pay the difference of the amount on his monthly bill, which was \$289, and what he had paid earlier this month, which was \$132. Write the difference of \$289 and \$132 as a mathematical expression.

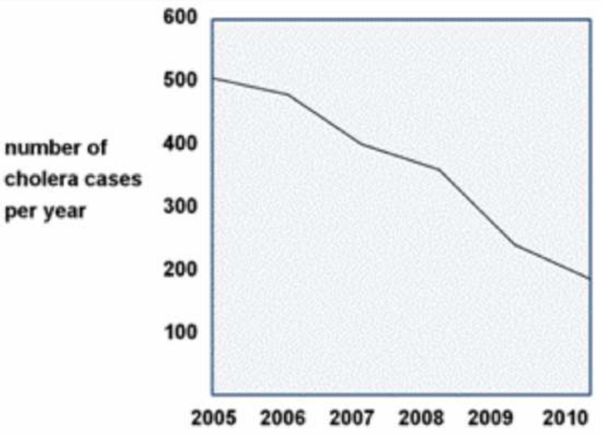
A. 132-289
B. 289+132
C. 132+289
D. 289-132

Answer

A. Incorrect. Dave had paid \$132, so that \$132 can be taken away from the full \$289 he owed for the month. The correct answer is 289-132.
B. Incorrect. Dave owes the difference of 132 and 289, not the sum of 289 and 132. The correct answer is 289-132.
C. Incorrect. Dave owes the difference of 132 and 289, not the sum of 132 and 289. The correct answer is 289-132.
D. Correct. The difference of 289 and 132 can be written as 289-132.

✓ Example

An African village is now getting cleaner water than it used to get. The number of cholera cases in the village has declined over the past five years. Using the graph below, determine the difference between the number of cholera cases in 2005 and the number of cases in 2010.



Solution

$\begin{array}{r}500\\-200\\\hline 300\end{array}$	The words “the difference” suggest that you need to subtract to answer the problem. First, use the graph to find the number of cholera cases per year for the two years: 500 in 2005 and 200 in 2010. Then write the subtraction problem and align numbers by place value. Subtract the numbers as you usually would.

500-200=300 cases

Summary

Subtraction is used in countless areas of life, such as finances, sports, statistics, and travel. You can identify situations that require subtraction by looking for key phrases, such as *difference* and *fewer than*. Some subtraction problems require regrouping to the next greater place value, so that the digit in the minuend becomes greater than the corresponding digit in the subtrahend. Subtraction problems can be solved without regrouping, if each digit in the minuend is greater than the corresponding digit in the subtrahend.

In addition to subtracting using the standard algorithm, subtraction can also can be accomplished by writing the numbers in expanded form so that both the minuend and the subtrahend are written as the sums of their place values.

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1.2.3: Estimation

Learning Objectives

- Use rounding to estimate sums and differences.
- Use rounding to estimate the solutions for application problems.

Introduction

An **estimate** is an answer to a problem that is close to the solution, but not necessarily exact. Estimating can come in handy in a variety of situations, such as buying a computer. You may have to purchase numerous devices: a computer tower and keyboard for $1,295, a monitor for $679, the printer for $486, the warranty for $196, and software for $374. Estimating can help you know *about* how much you'll spend without actually adding those numbers exactly.

Estimation usually requires **rounding**. When you round a number, you find a new number that's close to the original one. A rounded number uses zeros for some of the place values. If you round to the nearest ten, you will have a zero in the ones place. If you round to the nearest hundred, you will have zeros in the ones and tens places. Because these place values are zero, adding or subtracting is easier, so you can find an estimate to an exact answer quickly.

It is often helpful to estimate answers before calculating them. Then if your answer is not close to your estimate, you know something in your problem-solving process is wrong.

Using Rounding to Estimate Sums and Differences

Suppose you must add a series of numbers. You can round each addend to the nearest hundred to estimate the sum.

Example

Estimate the sum 1,472+398+772+164 **by rounding each number to the nearest hundred.**

Solution

$\begin{array}{rr} 1{,}472 \text{ rounds to} & 1{,}500 \\ 398 \text{ rounds to} & 400 \\ 772 \text{ rounds to} & 800 \\ 164 \text{ rounds to} & 200 \end{array}$	First, round each number to the nearest hundred.
$\begin{array}{r} 1{,}500 \\ 400 \\ 800 \\ +\quad 200 \\ \hline 2{,}900 \end{array}$	Then, add the rounded numbers together.

The estimate is 2,900

In the example above, the exact sum is 2,806. Note how close this is to the estimate, which is 94 greater.

In the example below, notice that rounding to the nearest ten produces a far more accurate estimate than rounding to the nearest hundred. In general, rounding to the lesser place value is more accurate, but it takes more steps.

Example

Estimate the sum 1,472+398+772+164 by first rounding each number to the nearest ten.

Solution

$\begin{array}{rlr} 1{,}472 & \text{rounds to} & 1{,}470 \\ 398 & \text{rounds to} & 400 \\ 772 & \text{rounds to} & 770 \\ 164 & \text{rounds to} & 160 \end{array}$	First, round each number to the nearest ten.

$\begin{array}{r} 12 \\ 1470 \\ 400 \\ +\quad 160 \\ \hline 00 \end{array}$	Next, add the ones and then the tens. Here, the sum of 7, 7, and 6 is 20. Regroup.
$\begin{array}{r} 12 \\ 1470 \\ 400 \\ 770 \\ +\quad 160 \\ \hline 800 \end{array}$	Now, add the hundreds. The sum of the digits in the hundreds place is 18. Regroup.
$\begin{array}{r} 12 \\ 1470 \\ 400 \\ 770 \\ +\quad 160 \\ \hline 2800 \end{array}$	Finally, add the thousands. The sum in the thousands place is 2.

The estimate is 2,800.

Note that the estimate is 2,800, which is only 6 less than the actual sum of 2,806.

Exercise

In three months, a freelance graphic artist earns $1,290 for illustrating comic books, $2,612 for designing logos, and $4,175 for designing web sites. Estimate how much she earned in total by first rounding each number to the nearest hundred.

A. $8,200
B. $7,900
C. $8,000
D. $8,100

Answer

A. Incorrect. You probably rounded one or more numbers up when you should have rounded them down. The correct answer is $8,100
B. Incorrect. You probably rounded one or more numbers down when you should have rounded them up. The correct answer is $8,100
C. Incorrect. You probably rounded each number to the nearest thousand instead of to the nearest hundred before adding. The correct answer is $8,100
D. Correct. You probably rounded the numbers to $1,300, $2,600 and $4,200 and added them together successfully.

You can also estimate when you subtract, as in the example below. Because you round, you do not need to subtract in the tens or hundreds places.

Example

Estimate the difference of 5,876 and 4,792 by first rounding each number to the nearest hundred.

Solution

$\begin{array}{lll} 5,876 & \text{rounds to} & 5,900 \\ 4,792 & \text{rounds to} & 4,800 \end{array}$	First, round each number to the nearest hundred.
$\begin{array}{r} 5,900 \\ -4,800 \\ \hline 1,100 \end{array}$	Subtract. No regrouping is needed since each number in the minuend is greater than or equal to the corresponding number in the subtrahend.

The estimate is 1,100.

The estimate is 1,100, which is 16 greater than the actual difference of 1,084.

? Exercise

Estimate the difference of 474,128 and 262,767 by rounding to the nearest thousand.

A. 212,000
B. 211,000
C. 737,000
D. 447,700

Answer

A. Incorrect. You probably rounded 474,128 up to 475,000 when you should have rounded it down to 474,000. The correct answer is 211,000.
B. Correct. You most likely rounded the numbers to 474,000 and 263,000 and subtracted successfully.
C. Incorrect. You probably added instead of subtracted. The correct answer is 211,000.
D. Incorrect. You probably used 26,300 instead of 263,000 as the number that you subtracted from 474,000. The correct answer is 211,000.

Solving Application Problems by Estimating

Estimating is handy when you want to be sure you have enough money to buy several things.

✓ Example

When buying a new computer, you find that the computer tower and keyboard cost $1,295, the monitor costs $679, the printer costs $486, the 2-year warranty costs $196, and a software package costs $374. Estimate the total cost by first rounding each number to the nearest hundred.

Solution

$\begin{array}{rlr} 1,295 & \text{rounds to} & 1,300 \\ 679 & \text{rounds to} & 700 \\ 486 & \text{rounds to} & 500 \\ 196 & \text{rounds to} & 200 \\ 374 & \text{rounds to} & 400 \end{array}$	First, round each number to the nearest hundred.
$\begin{array}{r} \overset{2}{} \\ 1300 \\ 700 \\ 500 \\ 200 \\ +\ \ 400 \\ \hline 3,100 \end{array}$	Add. After adding all of the rounded values, the estimated answer is $3,100.

The total cost is approximately $3,100.

Estimating can also be useful when calculating the total distance one travels over several trips.

✓ Example

James travels 3,247 meters to the park, then travels 582 meters to the store. He then travels 1,634 meters back to his house. Find the total distance traveled by first rounding each number to the nearest ten.

Solution

$\begin{array}{rcr} 3247 & \text{rounds to} & 3250 \\ 582 & \text{rounds to} & 580 \\ 1634 & \text{rounds to} & 1630 \end{array}$	First, round each number to the nearest ten.
$\begin{array}{rr} & 1 \\ & 3250 \\ & 580 \\ + & 1630 \\ \hline & 60 \end{array}$	Adding the numbers in the tens place gives 16, so you need to regroup.
$\begin{array}{rr} & 11 \\ & 3250 \\ & 580 \\ + & 1630 \\ \hline & 460 \end{array}$	Adding the numbers in the hundreds place gives 14, so regroup.
$\begin{array}{rr} & 11 \\ & 3250 \\ & 580 \\ + & 1630 \\ \hline & 5,460 \end{array}$	Adding the numbers in the thousands place gives 5.

The total distance traveled was approximately 5,460

In the example above, the final estimate is 5,460 meters, which is 3 less than the actual sum of 5,463 meters.

Estimating is also effective when you are trying to find the difference between two numbers. Problems dealing with mountains like the example below may be important to a meteorologist, a pilot, or someone who is creating a map of a given region. As in other problems, estimating beforehand can help you find an answer that is close to the exact value, preventing potential errors in your calculations.

✓ Example

One mountain is 10,496 feet high and another mountain is 7,421 feet high. Find the difference in height by first rounding each number to the nearest 100.

Solution

$\begin{array}{rcr} 10,496 & \text{rounds to} & 10,500 \\ 7,421 & \text{rounds to} & 7,400 \end{array}$	First, round each number to the nearest hundred.
$\begin{array}{rr} & 10,500 \\ - & 7,400 \\ \hline & 3,100 \end{array}$	Then, align the numbers and subtract. The final estimate is 3,100, which is 25 greater than the actual value of 3,075.

The estimated difference in height between the two mountains is 3,100 feet.

? Exercise

A space shuttle traveling at 17,581 miles per hour decreases its speed by 7,412 miles per hour. Estimate the speed of the space shuttle after it has slowed down by rounding each number to the nearest hundred.

A. 10,100 miles per hour
B. 10,200 miles per hour
C. 25,000 miles per hour
D. 25,100 miles per hour

Answer

A. Incorrect. You probably rounded the subtrahend up or the minuend down. You should have rounded the subtrahend of 7,412 down to 7,400 and the minuend of 17,581 up to 17,600. The correct answer is 10,200.

B. Correct. You correctly rounded both numbers and subtracted them successfully.
C. Incorrect. You probably added when you should have subtracted. The correct answer is 10,200
D. Incorrect. You probably rounded incorrectly and added when you should have subtracted. The correct answer is 10,200

Summary

Estimation is very useful when an exact answer is not required. You can use estimation for problems related to travel, finances, and data analysis. Estimating is often done before adding or subtracting by rounding to numbers that are easier to think about. Following the rules of rounding is essential to the practice of accurate estimation.

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SECTION OVERVIEW

1.3: Multiplying and Dividing Whole Numbers

Topic hierarchy

1.3.1: Multiplying Whole Numbers and Applications

1.3.2: Dividing Whole Numbers and Applications

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1.3.1: Multiplying Whole Numbers and Applications

Learning Objectives

- Use three different ways to represent multiplication.
- Multiply whole numbers.
- Multiply whole numbers by a power of 10.
- Use rounding to estimate products.
- Find the area of a rectangle.
- Solve application problems using multiplication.

Introduction

Instead of adding the same number over and over again, an easier way to reach an answer is to use multiplication. Suppose you want to find the value in pennies of 9 nickels. You can use addition to figure this out. Since a nickel is worth 5 pennies, or 5 cents, you can find the value of 9 nickels by adding 5+5+5+5+5+5+5+5+5. This repeated addition shows that 9 nickels have a value of 45 cents.

All of this addition can become very tiring. So, the math operation called *multiplication* can help perform repeated addition of whole numbers much more quickly. To find the value of these nickels, you could write a multiplication equation: $5 \cdot 9 = 45$.

$5 \cdot 9 = 45$ is read “5 times 9 equals 45" or "5 multiplied by 9 is equal to 45." The numbers that are being multiplied are called **factors**. The factors in this example are 5 and 9. The result of the multiplication (or the answer) is called the **product**. The product for $5 \cdot 9$ is 45.

In addition to showing multiplication as $5 \cdot 9 = 45$, you can show multiplication by using the x sign, $5 \times 9 = 45$, and also with parentheses, (5) (9) = 45 or 5(9)=45.

Three Ways to Write Multiplication

Using a multiplication or times sign: $2 \times 3 = 6$

Using a dot: $2 \cdot 3 = 6$ (this dot is NOT a decimal point)

Using parentheses: (2)(3)=6 or 2(3)=6

When you are adding the same number over and over again, you can use multiplication. You take the number that you are adding and rewrite it as a multiplication problem, multiplying it by the number of times you are adding it. For example, if you were serving 2 cookies each to 13 children, you could add 2 thirteen times or you could use multiplication to find the answer.

$2+2+2+2+2+2+2+2+2+2+2+2+2 = 2 \cdot 13 = 26$

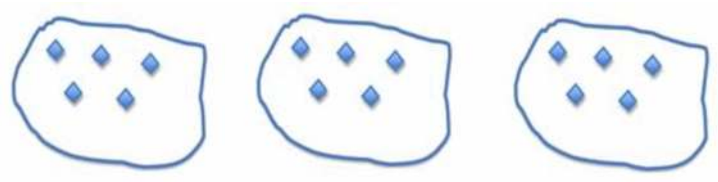
You could also write this using parentheses: $2(13) = 26$

What Is Multiplication?

In order to understand what multiplication is, consider three different ways to think about multiplication of whole numbers.

Approach 1: Set Model

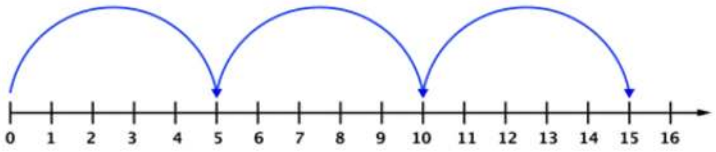
Multiplication is a way of writing repeated addition. When you read the problem $3 \cdot 5$ you could think of this as 3 groups of 5 things: 3 plates with 5 cookies on each plate; 3 baskets, each with 5 oranges in it; or 3 piles with 5 coins in each pile. We could show this as a picture:



$3 \cdot 5 = 3$ groups of $5 = 15$

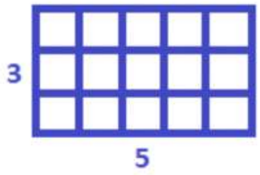
Approach 2: Number Line Model

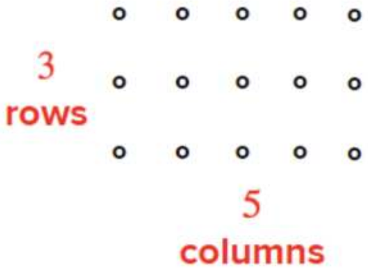
Multiplication can also be shown on a number line. The problem, $3 \cdot 5$ is modeled on the number line below. You can see that the arrows cover a distance of 5 units at a time. After 3 “jumps” on the number line, the arrow ends at 15.



Approach 3: Area Model

Another way of thinking about multiplication is to think about an array or area model to represent multiplication. You could think of $3 \cdot 5$ as 3 rows of 5 things. This might be a box of chocolates that has 3 rows of 5 chocolates, or a meeting room that is set up with 3 rows of 5 chairs. The pictures below show two rectangular arrangements of $3 \cdot 5$.





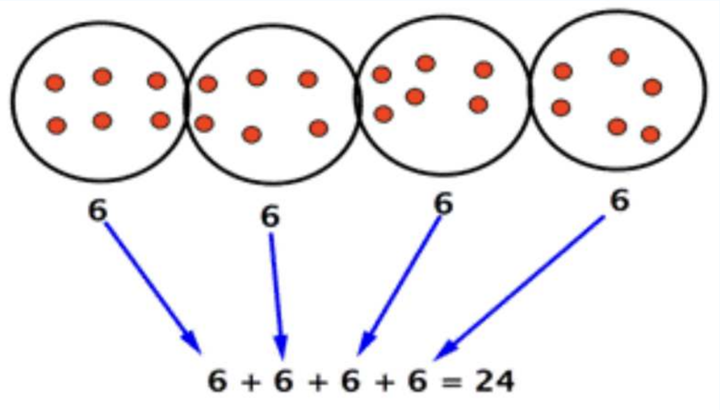
Do you see how both pictures represent the product 15? The picture on the left shows an area of 3 by 5. If you count the small squares that make up the rectangle, they total 15. Similarly, in the picture on the right, you see that 3 rows of 5 circles is equal to 15 circles.

✓ Example

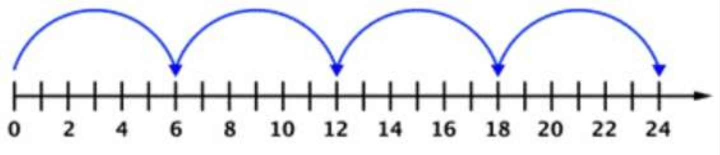
What is the product of $4 \cdot 6$? Use the set model, number line model and area model to represent the multiplication problem.

Solution

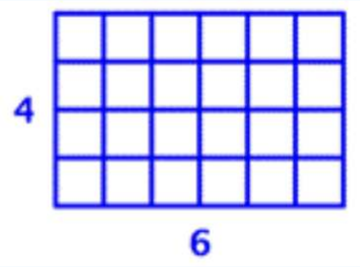
Set Model:



Number Line model:


0
2
4
6
8
10
12
14
16
18
20
22
24

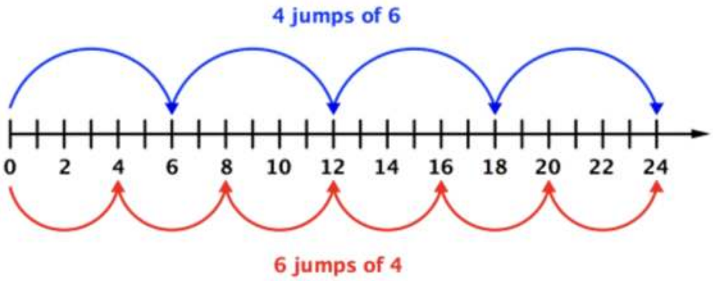
Area Model:



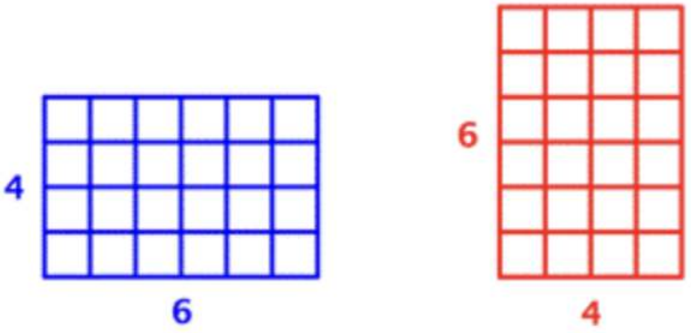
$4 \cdot 6 = 24$

If you switch the order in which you multiply two numbers, the product will not change. This is true for any two numbers you multiply. Think about the problem shown above.

You could make 6 jumps of 4 or 4 jumps of 6 on the number line and end up at 24.


4 jumps of 6
0
2
4
6
8
10
12
14
16
18
20
22
24
6 jumps of 4

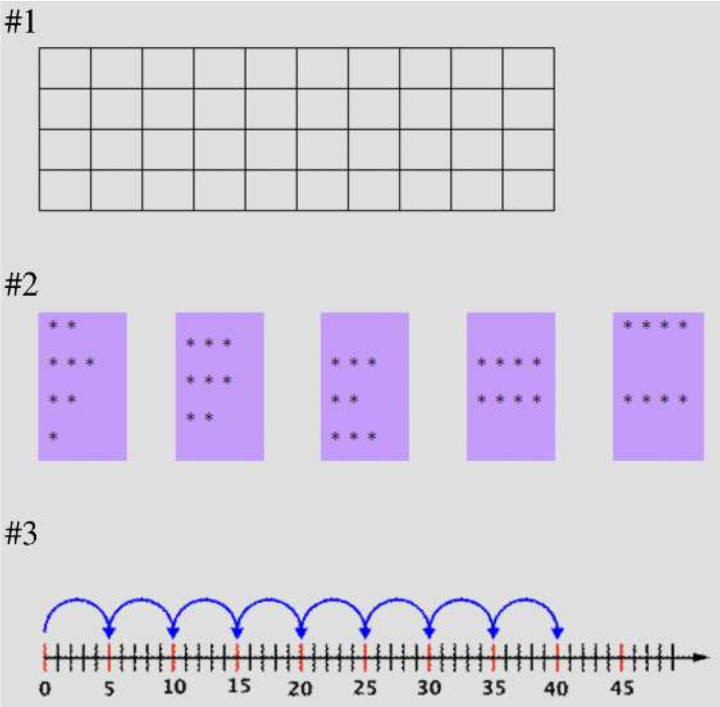
Or, you could make 6 rows of 4 or 4 rows of 6 and still have 24 squares.



$6 \cdot 4 = 24$ and $4 \cdot 6 = 24$.

? Exercise

Tanisha modeled $5 \cdot 8$ using the following models. Which models are accurate representations of the multiplication of these two factors?



A. All three models accurately represent $5 \cdot 8$.
B. Only models #1 and #3 accurately represent $5 \cdot 8$.
C. Only models #2 and #3 accurately represent $5 \cdot 8$.
D. None of the models accurately represents $5 \cdot 8$.

Answer

A. All three models accurately represent $5 \cdot 8$. Incorrect. While the first model has the same product, it models the factors $4 \cdot 10$. The correct answer is Only models #2 and #3 accurately represent $5 \cdot 8$.
B. Only models #1 and #3 accurately represent $5 \cdot 8$. Incorrect. Model #1 models $4 \cdot 10$, not $5 \cdot 8$. The correct answer is Only models #2 and #3 accurately represent $5 \cdot 8$.
C. Only models #2 and #3 accurately represent $5 \cdot 8$. Correct. Model #2 shows 5 groups of 8, which equals 40. Model #3 shows skip counting 8 times by 5 to get to 40.
D. None of the models accurately represents $5 \cdot 8$. Incorrect. Models #2 and #3 represent the multiplication $5 \cdot 8$. The correct answer is Only models #2 and #3 accurately represent $5 \cdot 8$.

Multiplying Greater Numbers

Let's go back to the question posed at the opening of this topic of study. How can you use multiplication to figure out the total cost of 6 baseball caps that cost $14 each? (You do not have to pay sales tax). You can figure out the cost by multiplying $14 \cdot 6$.

One way to do this computation is to break 14 down into parts and multiply each part by 6.

$$\begin{aligned} 14 &= 10 + 4 \quad \text{So,} \\ 14 \cdot 6 &= 10 \cdot 6 + 4 \cdot 6 \\ &= 60 + 24 \\ &= 84 \end{aligned}$$

You may recall the multiplication computed as $14 \cdot 6$:

$$\begin{array}{r} {}^{2} \\ 14 \\ \times \quad 6 \\ \hline 84 \end{array}$$

In this notation, some of the steps are written down with special notation. The product of 6 and 4(24) is written by putting the 4 in the ones place and writing a 2 up above the 1. This 2 actually stands for 20. Then, 6 is multiplied by 1. We are actually multiplying 6 by 1 ten and adding 20 to get the 80 in 84.

An example of multiplying two two-digit numbers is shown below. When performing this multiplication, each part of each number is multiplied by the other number. The numeric notation and an accompanying description are provided.

Example

$47 \cdot 52$

Solution

$\begin{array}{r} 47 \\ \times\quad 52 \\ \hline \end{array}$	Stack the numbers with the place values aligned.
$\begin{array}{r} 1 \\ 47 \\ \times\quad 52 \\ \hline 4 \end{array}$	Multiply the ones. $2 \times 7 = 14$ ones. Write 4 ones in the ones place and regroup 10 ones into the tens place.
$\begin{array}{r} 1 \\ 47 \\ \times\quad 52 \\ \hline 94 \end{array}$	Multiply 2 ones by 4 tens, and add the regrouped 1 ten. 2 ones times 4 tens + 1 ten = 9 tens
$\begin{array}{r} 3 \\ 47 \\ \times\quad 52 \\ \hline 94 \\ 50 \end{array}$	Multiply the tens. $5 \times 7 = 35$ tens. Write 5 tens in the tens place and regroup.
$\begin{array}{r} 3 \\ 47 \\ \times 52 \\ \hline 94 \\ 2350 \end{array}$	Multiply: 5 tens times 4 tens = 20 hundreds. Add the regrouped 3, which is 3 hundred.
$\begin{array}{r} 3 \\ 47 \\ \times\quad 52 \\ \hline 194 \\ 2350 \\ \hline 2,444 \end{array}$	Add the two lines. 94+2,350

$47 \cdot 52 = 2,444$

Notice that you are multiplying each of the parts of each number by the parts of the other number. You are doing so in a systematic way, from ones places to tens places. You are also using notation to keep track of what you have regrouped. These are shown with raised numbers.

To keep your columns straight and your work organized, consider using grid paper or lined paper turned sideways so the lines form columns. Here is an example of a problem written on grid paper:

		2	3
	x	1	2
		4	6
+	2	3	0
	2	7	6

When you are multiplying whole numbers, be sure to line up the digits by their place values. In the example above, the digits in the ones place are lined up: the 2 in 12 is directly below the 3 in 23.

Multiplying Whole Numbers by 10

When you multiply numbers by 10 or powers of 10(100 ; 1,000 ; 10,000 ; 100,000), you'll discover some interesting patterns. These patterns occur because our number system is based on ten: ten *ones* equal ten; ten *tens* equal one hundred; ten *hundreds* equal one thousand. Learning about these patterns can help you compute easily and quickly.

Consider the example of $25 \cdot 100$. First, let's use the standard algorithm method to multiply these numbers.

Example

$25 \cdot 100$

Solution

$$\begin{array}{r} 100 \\ \times \quad 25 \\ \hline 500 \\ 2000 \\ \hline 2,500 \end{array}$$

$25 \cdot 100 = 2,500$

Using the standard algorithm, we calculated $25 \cdot 100 = 2,500$.

Look at the table below to find a pattern in the factors and products. See how the number of zeros in the power of 10(10,100, 1,000, etc.) relates to the number of zeros in the product.

Factors	Product
$5 \cdot 10$	= 50
$5 \cdot 100$	= 500
$5 \cdot 1,000$	= 5,000
$5 \cdot 10,000$	= 50,000

You can see that the number of zeros in the product matches the number of zeros in the power of 10(10,100,1,000, etc.). Will this always be true or is it true only in certain situations? Look at two more patterns:

Factors	Product
$10 \cdot 10$	= 100
$10 \cdot 100$	= 1,000
$10 \cdot 1,000$	= 10,000
$10 \cdot 10,000$	= 100,000

Factors	Product
$120 \cdot 10$	= 1,200
$120 \cdot 100$	= 12,000
$120 \cdot 1,000$	= 120,000
$120 \cdot 10,000$	= 1,200,000

Notice that in these last two examples, both factors had zeros in them. The number of zeros in the product is equal to the sum of the number of zeros at the end of each of the factors.

The example below illustrates how to multiply $140 \cdot 3000$.

Example

$140 \cdot 3000$

Solution

$\begin{array}{r} 1 \\ 14 \\ \times \quad 3 \\ \hline 42 \end{array}$	Identify the non-zero parts of the factors and multiply these parts. Multiply 3 ones by 4 ones. $4 \cdot 3 = 12$.
420,000	Count the number of zeros in each factor. 140 has **one** zero; 3,000 has **three** zeros. 1+3=4 Write another 4 zeros after the 42.

$140 \cdot 3000 = 420,000$

Multiplying by Ten

When you multiply a whole number by 10 or a power of 10, first multiply the nonzero parts of the numbers. Then include the number of zeros at the end of the product equal to the total number of zeros at the end of the factors.

$13 \cdot 100 = 1,300$

$180 \cdot 2,000 = 360,000$

Exercise

An apple orchard sold 100 bags of apples. If there are 30 apples in each bag, how many apples did the orchard sell?

A. 130
B. 300
C. 30,000
D. 3,000

Answer

A. Incorrect. You need to multiply 100 and 30, not add. The correct answer is 3,000.
B. Incorrect. $30 \cdot 100 = 3,000$. You did not include the correct number of zeros. The correct answer is 3,000.
C. Incorrect. $30 \cdot 100 = 3,000$. You included too many zeros. The correct answer is 3,000.
D. Correct. $30 \cdot 100 = 3,000.3 \cdot 1 = 3$ and add three zeros since there is one zero in 30 and there are two zeros in 100.

Using Rounding to Estimate Products

Sometimes you don’t need an exact product because an estimate is enough. If you’re shopping, stopping to make a calculation with pencil and paper, or even a calculator, is inconvenient. Usually, shoppers will round numbers up so they will be sure that they have enough money for their purchases.

Estimating products is also helpful for checking an answer to a multiplication problem. If your actual calculation is quite different from your estimate, there is a good chance you have made a place value and/or regrouping mistake.

To estimate a product, you often round the numbers first. When you round numbers, you are always rounding to a particular place value, such as the nearest thousand or the nearest ten. If you are rounding a number to the nearest ten, you round it to the ten that is closest to the original number. An example of this is rounding 317 to the nearest ten. In this case, you round 317 to 320. If the number is half way in between (315), generally round up to 320.

Rounding factors can make it easy to multiply in your head. Let’s consider the multiplication problem $145 \cdot 29$. To estimate this product by rounding, you can round to the nearest ten.

Example

Use rounding to estimate the product of $145 \cdot 29$.

Solution

$150 \cdot 30$	Round numbers to the nearest ten.
$15 \cdot 3 = 45$	Multiply the non-zero numbers.
4,500	Count the zeros in the factors and include that many zeros after the 45.

The estimate of $145 \cdot 29$ is $4,500$.

You can use a calculator to see if your estimate seems reasonable. Or you can use estimation to make sure that the answer that you got on a calculator is reasonable. (Have you ever input the wrong numbers?)

	Key entries: 145 x 29 = Result: 4,205 The exact product and the estimate are close enough to give you confidence in your calculations.

Exercise

A factory produces 58 packages of cookies in one hour. There are 32 cookies in each package. Which is the best estimate of the number of cookies the factory produces in one hour?

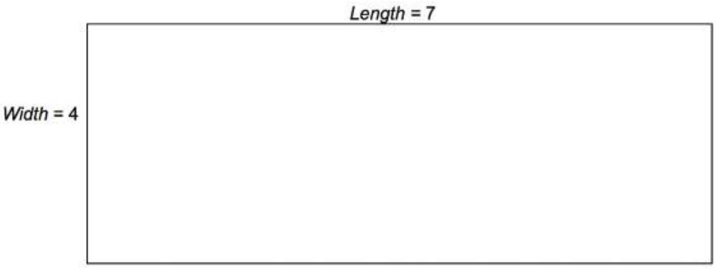
A. 1,800
B. 1,500
C. 18,000
D. 180

Answer

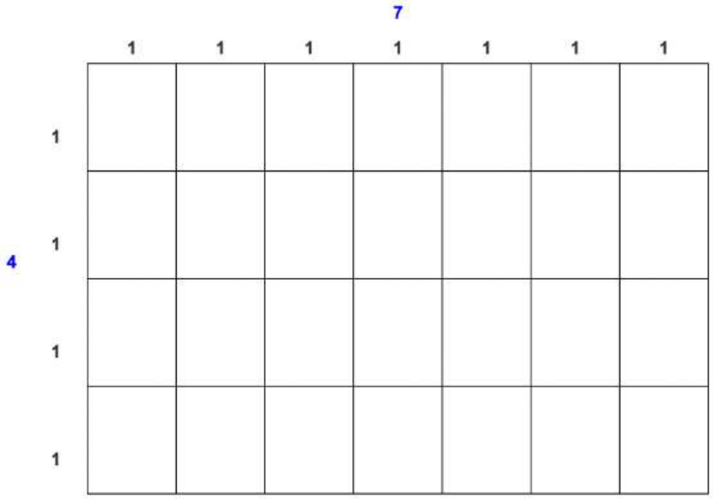
A. Correct. Multiplying $60 \cdot 30$ would give a good estimate. $60 \cdot 30 = 1,800$.
B. Incorrect. $50 \cdot 30 = 1,500$, but 58 rounds to 60, not 50.The correct answer is 1,800.
C. Incorrect. $60 \cdot 30 = 1,800$. The correct answer is 1,800.
D. Incorrect. $60 \cdot 30 = 1,800$. The correct answer is 1,800.

Finding the Area of a Rectangle

The formula for the area of a rectangle uses multiplication: $\text{length} \cdot \text{width} = \text{area}$. Applying what you know about multiplication, you can find the area of any rectangle if you know its dimensions (length and width). Consider the rectangle that is 4 by 7 shown below. Its length is 7 and its width is 4.



You can divide the rectangle into units by making 7 columns and 4 rows.



You can see that dividing the rectangle in this way results in 28 squares. You could say that the area of the rectangle is 28 square units. You could also find the area by multiplying $7 \cdot 4$. (Note: Area is always measured in square units: square inches, square centimeters, square feet, etc.)

Consider an example of a larger rectangle, like one that is found on a soccer field. At each end of a soccer field, centered at the goal, is a large rectangle. This rectangle is called the *penalty box* because fouls committed within the lines of this rectangle may result in a penalty kick. On a regulation soccer field, the penalty box is 44 yards by 18 yards. What is the area of a penalty box?

✓ Example

$44 \text{ yards } \cdot 18 \text{ yards}$

Solution

$$\begin{array}{r} {}^{3} \\ 44 \\ \times \quad 18 \\ \hline 352 \\ + \quad 440 \\ \hline 792 \end{array}$$

The area of the penalty box is 792 square yards.

? Exercise

What is the area of a rectangle whose length is 23 feet and whose width is 7 feet?

A. 30 feet
B. 161 feet
C. 161 square feet
D. 1,421 square feet

Answer

A. Incorrect. To find the area, you multiply, not add, the length and width. The correct answer is 161 square feet.
B. Incorrect. Area is measured in square units. The correct answer is 161 square feet.
C. Correct. $23 \cdot 7 = 161$
D. Incorrect. There is a place-value mistake. When you multiply $7 \cdot 3$, you need to regroup 20 ones to 2 tens and add it to the product of 2 and 7. The correct answer is 161 square feet.

Using Multiplication in Problem Solving

Multiplication is used in solving many types of problems. Below are two examples that use multiplication in their solutions to the problem.

Example

A case of cat food has two layers. Each layer has 4 rows of 6 cans. How many cans are in a case of cat food?

Solution

$\begin{array}{r} 6 \\ \times \quad 4 \\ \hline 24 \end{array}$	First find the number of cans in one layer. You can multiply to figure this out.
$\begin{array}{r} 24 \\ \times \quad 2 \\ \hline 48 \end{array}$	Since there are two layers, you multiply the number of cans in one layer by 2.

There are 48 cans in a case of cat food.

Example

A theater has 45 rows with 40 seats in each row. How many seats are there in the theater?

Solution

$\begin{array}{lr} & 2 \\ & 45 \\ \times & 40 \\ \hline & 00 \\ + & 1800 \\ \hline & 1,800 \end{array}$	You can solve this problem by adding 40, 45 times, but that would take a lot of work. Multiplication is the way to go.

There are 1,800 seats in the theater.

Exercise

A lawn care company charges $35 to mow a lawn. If the company mows 32 lawns, how much money will it make?

A. $9,760
B. $1,120
C. $130.00
D. $67.00

Answer

A. Incorrect. This answer is too large. By estimation, $40 \cdot 30 = 1200$. There must be an error in regrouping or place value. The correct answer is $1,120.
B. Correct. $35 \cdot 32 = 1,200$
C. Incorrect. The answer should be a larger number. By estimation, $40 \cdot 30 = 1200$. There must be an error in regrouping or place value. The correct answer is $1,120.

D. Incorrect. You must multiply, not add, 32 and 35. The correct answer is $1,120.

Summary

Multiplication can make repeated addition easier to compute in calculations and problem solving. Multiplication can be written using three symbols: parentheses, a times sign, or a multiplication dot. To perform multiplication with two-digit factors or greater, you can use the standard algorithm where you multiply each of the numbers in each factor by the numbers in the other factor. Using strategies such as short cuts for multiplying by powers of 10 and estimation to check your answers can make multiplication easier as well as reduce errors.

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1.3.2: Dividing Whole Numbers and Applications

Learning Objectives

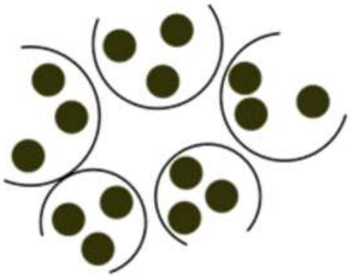
- Use three different ways to represent division.
- Divide whole numbers.
- Perform long division.
- Divide whole numbers by a power of 10.
- Recognize that division by 0 is not defined.
- Solve application problems using division.

Introduction

Some people think about division as “fair sharing” because when you divide a number you are trying to create equal parts. Division is also the **inverse operation** of multiplication because it “undoes” multiplication. In multiplication, you combine equal sets to create a total. In division, you separate a whole group into sets that have the same amount. For example, you could use division to determine how to share 40 empanadas among 12 guests at a party.

What is Division?

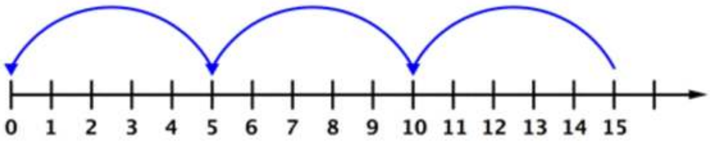
Division is splitting into equal parts or groups. For example, one might use division to determine how to share a plate of cookies evenly among a group. If there are 15 cookies to be shared among five people, you could divide 15 by 5 to find the “fair share” that each person would get. Consider the picture below.



15 cookies split evenly across 5 plates results in 3 cookies on each plate. You could represent this situation with the equation:

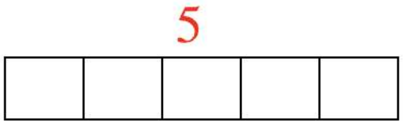
$$15 \div 5 = 3$$

You could also use a number line to model this division. Just as you can think of multiplication as repeated addition, you can think of division as repeated subtraction. Consider how many jumps you take by 5s as you move from 15 back to 0 on the number line.

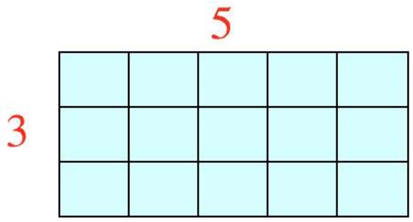


Notice that there are 3 jumps that you make when you skip count by 5 from 15 back to 0 on the number line. This is like subtracting 5 from 15 three times. This repeated subtraction can be represented by the equation: $15 \div 5 = 3$.

Finally, consider how an area model can show this division. Ask yourself, if you were to make a rectangle that contained 15 squares with 5 squares in a row, how many rows would there be in the rectangle? Start by making one row of 5 squares:



Then add two more rows of 5 squares so you have 15 squares.



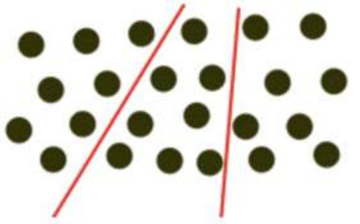
The number of rows is 3. So, 15 divided by 5 is equal to 3.

Example

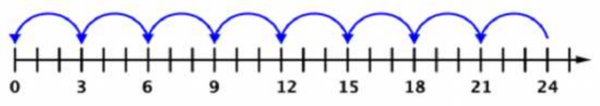
Find $24 \div 3$ using a set model and a number line model.

Solution

Set Model:



Number line model:



$24 \div 3 = 8$

Ways to Represent Division

As with multiplication, division can be written using a few different symbols. We showed this division written as $15 \div 5 = 3$, but it can also be written two other ways:

$$\begin{array}{r}3\\ 5\backslash\text{longdiv}15\end{array}$$

$\frac{15}{5} = 3$

Each part of a division problem has a name. The number that is being divided up, that is the total, is called the **dividend**. In the work in this topic, this number will be the larger number, but that is not always true in mathematics. The number that is dividing the dividend is called the **divisor**. The answer to a division problem is called the **quotient**.

The blue box below summarizes the terminology and common ways to represent division.

Three Ways to Represent Division

$12 \div 3 = 4$ (**with a division symbol**; this equation is read "12 divided by 3 equals 4."

$\begin{array}{r}4\\ 3\backslash\text{longdiv}12\end{array}$ (**with a division or long division symbol**; this expression is read "12 divided by 3 equals 4." Notice here, though, that you have to start with what is underneath the symbol. This may take some getting used to since you are reading from right to left and bottom to top!)

$\frac{12}{3} = 4$ (**with a fraction bar**; this expression can also be read "12 divided by 3 equals 4." In this format, you read from top to bottom.)

In the examples above, 12 is the **dividend**, 3 is the **divisor** and 4 is the **quotient.**

$$\text{Dividend} \div \text{Divisor} = \text{Quotient}$$

$$\begin{array}{l} \qquad\qquad\qquad\text{Quotient} \\ \text{Divisor}\backslash\text{longdiv}\text{Dividend} \end{array}$$

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient}$$

? Exercise

Which of the following expressions represent dividing $56 equally among 7 people?

#1: $\frac{7}{56}$

#2: $56 \div 7$

#3: $56\backslash\text{longdiv}7$

A. #2 represents the situation.
B. All three expressions represent the situation.
C. #1 represents the situation.
D. #3 represents the situation.

Answer

A. Correct. #2 is the only expression that represents 56 divided by 7.
B. Incorrect. #1 and #3 represent 7 divided by 56, not 56 divided by 7. The correct answer is #2 is the only expression that represents the situation.
C. Incorrect. This expression represents 7 divided by 56, not 56 divided by 7. The correct answer is #2 is the only expression that represents the situation.
D. Incorrect. This expression represents 7 divided by 56, not 56 divided by 7. The correct answer is #2 is the only expression that represents the situation.

Dividing Whole Numbers

Once you understand how division is written, you are on your way to solving simple division problems. You will need your multiplication facts to perform division. If you do not have them memorized, you can guess and check or use a calculator.

Consider the following problems:

$10 \div 5 = ?$

$48 \div 2 = ?$

$30 \div 5 = ?$

In the first problem, $10 \div 5$, you could ask yourself, “how many fives are there in ten?” You can probably answer this easily. Another way to think of this is to consider breaking up 10 into 5 groups and picturing how many would be in each group.

$10 \div 5 = 2$

To solve $48 \div 2$, you might realize that dividing by 2 is like splitting into two groups or splitting the total in half. What number could you double to get 48?

$48 \div 2 = 24$

To figure out $30 \div 5$, you could ask yourself, how many times do you have to skip count by 5 to get from 0 to 30? 5, 10, 15, 20, 25, 30. You have to skip count 6 times to get to 30.

$30 \div 5 = 6$

? Exercise

Compute $35 \div 5$.

Answer

7

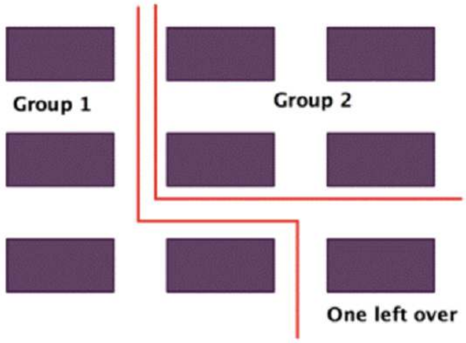
Exercise

Compute $32 \div 4$.

Answer

8

Sometimes when you are dividing, you cannot easily share the number equally. Think about the division problem $9 \div 2$. You could think of this problem as 9 pieces of chocolate being split between 2 people. You could make two groups of 4 chocolates, and you would have one chocolate left over.



In mathematics, this left over part is called the **remainder**. It is the part that *remains* after performing the division. In the example above, the remainder is 1. We can write this as:

$9 \div 2 = 4 \quad \text{R1}$

We read this equation: “Nine divided by two equals four with a remainder of 1."

You might be thinking you could split that extra piece of chocolate in parts to share it. This is great thinking! If you split the chocolate in half, you could give each person another half of a piece of chocolate. They would each get $4\frac{1}{2}$ pieces of chocolate. We are not going to worry about expressing remainders as fractions or decimals right now. We are going to use the remainder notation with the letter R. Here’s an example:

Example

$45 \div 6$

Solution

$6 \cdot 7 = 42$	How many sixes are there in 45? Try 7.
$45 - 42 = 3$	3 is not enough for another 6. So, 3 is the remainder.

$45 \div 6 = 7 \quad \text{R3}$

Since multiplication is the inverse of division, you can check your answer to a division problem with multiplication. To check the answer 7 R3, first multiply 6 by 7 and then add 3.

$6 \cdot 7 = 42$

$42 + 3 = 45$, so the quotient 7 R3 is correct.

Exercise

Compute $67 \div 7$.

A. 9
B. 9 R4
C. 60
D. 10

Answer

A. Incorrect. $9 \cdot 7 = 63$. There is a remainder of 4. The correct answer is 9 R4.
B. Correct. $9 \cdot 7 = 63$ and there are 4 left over.
C. Incorrect. This is a division, not subtraction, problem. The correct answer is 9 R4.
D. Incorrect. $70 \div 7 = 10$, so the answer to $67 \div 7$ cannot be; $9 \cdot 7 = 63$ and there are 4 left over. The correct answer is 9 R4.

Performing Long Division

Long division is a method that is helpful when you are performing division that you cannot do easily in your head, such as division involving larger numbers. Below is an example of a way to write out the division steps.

Example

$68 \div 4$

Solution

4\longdiv68	Rewrite the division.
1 4\longdiv68 -4 28	Divide the tens. What is 6 divided by 4? $4 \cdot 1 = 4$, so write a 1 above the 6. Subtract 4 from 6 and bring down the next digit of the dividend, 8. What is 28 divided by 4?
17 4\longdiv68 -4 28 -28 0	$7 \cdot 4 = 28$, so write a 7 above the 8. There is no remainder.
\(\ \begin{array}{r} 17 \cdot 4 \ ^217 \ \times 4 \ \hline 68 \end{array}\)	Check your answer using multiplication.

$68 \div 4 = 17$

Example

$6,707 \div 233$

Solution

2 233\longdiv6707 -466	Examine the first 3 digits of the dividend and determine how many 233s are in it. Use guess and check. Try: $2 \cdot 233 = 466$ Try: $3 \cdot 233 = 699$ (too large)
2 233\longdiv6707 -466 2047	Subtract 466 from 670 and bring down the next digit of the dividend, 7.
28 233\longdiv6707 -466 2047 -1864 183	How many 233s are in 2,047? It looks like close to 10 because $233 \cdot 10 = 2,330$. Try $9.233 \cdot 9$ equals 2,097. 22 233 $\times \quad 9$ $2,097$ 2,097 (Too large) Must be 8! 22 233 $\times \quad 8$ $1,864$
$233 \cdot 28 = 6,524$ $6,524 + 183 = 6,707$	Check your answer using multiplication. First, multiply $233 \cdot 28$. Then, add the remainder.

$6,707 \div 233 = 28 \quad \text{R183}$

Exercise

Compute $417 \div 34$.

A. 451
B. 12
C. 12 R9
D. 13

Answer

A. Incorrect. This is a division problem, not an addition problem. The correct answer is 12 R9.
B. Incorrect. $12 \cdot 34 = 408$. The correct answer is 12 R9.
C. Correct. $12 \cdot 34 = 408$ and $408 + 9 = 417$
D. Incorrect. $13 \cdot 34 = 442$. The correct answer is 12 R9.

Dividing Whole Numbers by a Power of 10

Just as multiplication by powers of 10 results in a pattern, there is a pattern with division by powers of 10. Consider three quotients: $20 \div 10; 200 \div 10; 2,000 \div 10$.

Think about $20 \div 10$. There are 2 tens in twenty, so $20 \div 10 = 2$. The computations for $200 \div 10$ and $2,000 \div 10$ are shown below.

Example

$200 \div 10$

Solution

10\longdiv200	Rewrite the problem.
10\longdiv200	Divide the first digit of the dividend, 2, by the divisor. Since $2 \div 10$ does not give a whole number, go to the next digit, 0.
2 10\longdiv200	$20 \div 10 = 2$
2 10\longdiv200 -20 0	$2 \cdot 10 = 20$ $20 - 20 = 0$
2 10\longdiv200 -20 00	Bring down the next digit of the dividend, which is 0.
20 10\longdiv200 -20 00 0 0	Since 10 still does not go into 00 and we have nothing left to bring down, multiply the 0 by 10. $0 \cdot 10 = 0$ $0 - 0 = 0$ We have no remainder.

$200 \div 10 = 20$

Example

$2000 \div 10$

Solution

10\longdiv2000	Rewrite the problem.
10\longdiv2000	Divide the first digit of the dividend, 2, by the divisor. Since $2 \div 10$ does not give a whole number, go to the next digit, 0.
2 10\longdiv2000	$20 \div 10 = 2$
2 10\longdiv2000 -20 0	$2 \cdot 10 = 20$ $20 - 20 = 0$
2 10\longdiv2000 -20 00	Bring down the next digit, 0, of the dividend.
20 10\longdiv2000 -20 000	Since 10 does not go into 00, add a 0 to the quotient and bring down the next digit, 0.
200 10\longdiv2000 -20 000 0 0	Since 10 still does not go into 000 and we have nothing left to bring down, add a 0 to the quotient, multiply the 0 by 10. $0 \cdot 10 = 0$ $0 - 0 = 0$ We have no remainder.

$2,000 \div 10 = 200$

Examine the results of these three problems to try to determine a pattern in division by 10.

$$\begin{aligned} 20 \div 10 &= 2 \\ 200 \div 10 &= 20 \\ 2,000 \div 10 &= 200 \end{aligned}$$

Notice that the number of zeros in the quotient decreases when a dividend is divided by 10: 20 becomes 2; 200 becomes 20 and 2,000 become 200. In each of the examples above, you can see that there is one fewer 0 in the quotient than there was in the dividend.

Continue another example of division by a power of 10.

✓ Example

$2,000 \div 100$

Solution

100\longdiv2000	Rewrite the problem.
100\longdiv2000	Divide the first digit of the dividend, 2, by the divisor. Since $2 \div 100$ does not give a whole number, go to the next digit, 0.
100\longdiv2000	Divide the first two digits of the dividend, 20, by the divisor. Since $20 \div 100$ does not give a whole number, go to the next digit, 0.
100\longdiv2000	$200 \div 100 = 2$
2 100\longdiv2000 -200 0	$2 \cdot 100 = 200$ $200 - 200 = 0$
2 100\longdiv2000 -200 00	Bring down the next digit, 0, of the dividend.
20 100\longdiv2000 -200 00 0 0	Since 100 still does not go into 00 and we have nothing left to bring down, add a 0 to the quotient, multiply the 0 by 10. $0 \cdot 10 = 0$ $0 - 0 = 0$ We have no remainder.

$2,000 \div 100 = 20$

Consider this set of examples of division by powers of 10. What pattern do you see?

$$\begin{aligned} 20 &\div 10 = 2 \\ 200 &\div 10 = 20 \\ 2,000 &\div 10 = 200 \\ 2,000 &\div 100 = 20 \\ 2,000 &\div 1,000 = 2 \end{aligned}$$

Notice that when you divide a number by a power of 10, the quotient has fewer zeros. This is because division by a power of 10 has an effect on the place value. For example, when you perform the division $18,000 \div 100 = 180$, the quotient, 180, has two fewer zeros than the dividend, 18,000. This is because the power of 10 divisor, 100, has two zeros.

Exercise

Compute $135,000 \div 100$.

A. 13,500
B. 134,900
C. 13,500,000
D. 1,350

Answer

A. Incorrect. This answer is too large. $13,500 \cdot 100 = 1,350,000$. The correct answer is 1,350.
B. Incorrect. This is a division, not a subtraction, problem. The correct answer is 1,350.
C. Incorrect. This is a division, not a multiplication, problem. The correct answer is 1,350.
D. Correct. 1,350 \cdot 100=135,000.

Division by Zero

You know what it means to divide by 2 or divide by 10, but what does it mean to divide a quantity by 0? Is this even possible? Can you divide 0 by a number? Consider the two problems written below.

$\frac{0}{8}$ and $\frac{8}{0}$

We can read the first expression, "zero divided by eight" and the second expression, "eight divided by zero." Since multiplication is the inverse of division, we could rewrite these as multiplication problems.

$0 \div 8 = ?$

$? \cdot 8 = 0$

The quotient must be 0 because $0 \cdot 8 = 0$.

$\frac{0}{8} = 0$

Now let's consider $\frac{8}{0}$.

$8 \div 0 = ?$

$? \cdot 0 = 8$

This is not possible. There is no number that you could multiply by zero and get eight. Any number multiplied by zero is always zero. There is no quotient for $\frac{8}{0}$. There is no quotient for any number when it is divided by zero.

Division by zero is an operation for which you cannot find an answer, so it is not allowed. We say that division by 0 is undefined.

Using Division in Problem Solving

Division is used in solving many types of problems. Below are three examples from real life that use division in their solutions.

Example

Luana made 40 empanadas for a party. If the empanadas are divided equally among 12 guests, how many will each guest have? Will there be any leftover empanadas?

Solution

$40 \div 12$	Since each guest will have an equal share, we can use division.
3 12\longdiv40 -36 4	Use trial and error. Try 3. $12 \cdot 3 = 36$ When 40 empanadas are divided equally among 12 people, there are 4 left over.

Each guest will have 3 empanadas. There will be 4 empanadas left over.

Example

A case of floor tiles has 12 boxes in it. The case costs $384. How much does one box cost?

Solution

$384 \div 12$	Since the boxes each cost the same amount, you want to divide $384 into 12 equal parts.
12\longdiv384	Perform the division. Try to divide the first digit in the dividend by the divisor. 12 will not divide into 3, so go to the next digit.
3 12\longdiv384 -36	Perform $38 \div 12$. Pick a quotient and test it. Try 3. $3 \cdot 12 = 36$.
3 12\longdiv384 -36 2	Subtract 36 from 38.
32 12\longdiv384 -36 24 -24 0	Bring the next digit of the dividend down and perform division. $12 \cdot 2 = 24$ $24 - 24 = 0$
32 $\times$ 12 64 $+$ 320 384	Does $32 \cdot 12$ equal 384? Check your answer by multiplying. Yes! The answer is correct!

Each box of tiles costs $32

Example

A banana grower is shipping 4,644 bananas. There are 86 crates, each containing the same number of bananas. How many bananas are in each crate?

Solution

$4,644 \div 86$	Since each crate, or box, has the same number of bananas, you can take the total number of bananas and divide by the number of crates. Rewrite the division.
86\longdiv4644	Use trial and error to determine what $464 \div 86$ equals. Try 5: $^{3}86$ $\times$ 5 430
5 86\longdiv4644 -430 344	$464 - 430 = 34$ Then, bring down the next digit of the dividend, 4.

$\begin{array}{r} 54 \\ 86\backslash\text{longdiv}4644 \\ -430 \\ \hline 344 \\ -344 \\ \hline 0 \end{array}$	Use trial and error to determine the quotient of 344 and 86. Try 4: $\begin{array}{r} {}^{2}86 \\ \times 4 \\ \hline 344 \end{array}$
$\begin{array}{r} 86 \\ \times 54 \\ \hline 344 \\ +4300 \\ \hline 4644 \end{array}$	Check your answer by multiplying. Yes! The answer is correct!

Each crate contains 54 bananas.

Exercise

A theater has 1,440 seats. The theater has 30 rows of seats. How many seats are in each row?

A. 1,410
B. 48
C. 43,200
D. 480

Answer

A. Incorrect. This answer is too large. Use division, $1440 \div 30$, not subtraction for this problem. The correct answer is 48.
B. Correct. $1440 \div 30 = 48$.
C. Incorrect. The answer is too large. Use division, $1440 \div 30$, not multiplication, for this problem. The correct answer is 48.
D. Incorrect. There is a place-value error. The correct answer is 48.

Summary

Division is the inverse operation of multiplication, and can be used to determine how to evenly share a quantity among a group. Division can be written in three different ways: using a fraction bar, using a division symbol, and using long division. Division can be represented as splitting a total quantity into sets of equal quantities, as skip subtracting on the number line, and as a dimension with an area model. Remainders may result when performing division and they can be represented with the letter R, followed by the number remaining. Since division is the inverse operation of multiplication, you need to know your multiplication facts in order to do division. For larger numbers, you can use long division to find the quotient.

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SECTION OVERVIEW

1.4: Properties of Whole Numbers

Topic hierarchy

1.4.1: Properties and Laws of Whole Numbers

1.4.2: The Distributive Property

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1.4.1: Properties and Laws of Whole Numbers

Learning Objectives

- Simplify by using the addition property of 0.
- Simplify by using the multiplication property of 1.
- Identify and use the commutative law of addition.
- Identify and use the commutative law of multiplication.
- Identify and use the associative law of addition.
- Identify and use the associative law of multiplication.

Introduction

Mathematics often involves simplifying numerical expressions. When doing so, you can use laws and properties that apply to particular operations. The multiplication property of 1 states that any number multiplied by 1 equals the same number, and the addition property of zero states that any number added to zero is the same number.

Two important laws are the commutative laws, which state that the order in which you add two numbers or multiply two numbers does not affect the answer. You can remember this because if you *commute* to work you go the same distance driving to work and driving home as you do driving home and driving to work. You can move numbers around in addition and multiplication expressions because the order in these expressions does not matter.

You will also learn how to simplify addition and multiplication expressions using the associative laws. As with the commutative laws, there are associative laws for addition and multiplication. Just like people may associate with people in different groups, a number may *associate* with other numbers in one group or another. The associative laws allow you to place numbers in different groups using parentheses.

Addition and Multiplication Properties of 0 and 1

The **addition property of 0** states that for any number being added to 0, the sum equals that number. Remember that you do not end up with zero as an answer; that only happens when you multiply. Your answer is simply the same as your original number.

Example

$62+0=?$

Solution

$62+0=62$	Adding zero to 62 does not add any quantity to the sum, so the number remains 62.

$62+0=62$

Exercise

$112+0=?$

A. 112
B. 0
C. 1
D. 1,120

Answer

A. Correct. Adding zero to a number does not change a number.
B. Incorrect. Your answer would be zero if you multiply 112 by 0, but not if you add 112 to 0. The correct answer is 112.
C. Incorrect. Adding 112 to 0 does not equal 1. The correct answer is 112.

D. Incorrect. You do not put zero in the ones place and move other digits up by one place value. This only occurs if you are multiplying a number by ten. The correct answer is 112.

According to the **multiplication property of 1**, the product of 1 and any number results in that number. The answer is simply identical to the original number.

✓ Example

$2,500 \cdot 1 = ?$

Solution

$2,500 \cdot 1 = 2,500$	Multiplying 2,500 by 1 yields the same number.

$2,500 \cdot 1 = 2,500$

? Exercise

$72,540 \cdot 1 = ?$

A. 725,401
B. 72,541
C. 72,540
D. 72,539

Answer

A. Incorrect. You do not put 1 in the ones place and move other digits up by one place value. The correct answer is 72,540.
B. Incorrect. You do not add one to the number. The correct answer is 72,540.
C. Correct. Multiplying any number by 1 yields the same number, which is in this case 72,540.
D. Incorrect. You do not subtract one from the number. The correct answer is 72,540.

The Commutative Law of Addition

The **commutative law of addition** states that you can change the position of numbers in an addition expression without changing the sum. For example, 3+2 is the same as 2+3.

3+2=5

2+3=5

You likely encounter daily routines in which the order can be switched. For example, when you get ready for work in the morning, putting on your left glove and right glove is commutative. You could put the right glove on before the left glove, or the left glove on before the right glove. Likewise, brushing your teeth and combing your hair is commutative, because it does not matter which one you do first.

Remember that this law only applies to addition, and not subtraction. For example:

8-2 is not the same as 2-8.

Below, you will find examples of expressions that have been changed with the commutative law. Note that expressions involving subtraction cannot be changed.

Original Expression	Rewritten Expression
4+5	5+4
6+728	728+6
9+4+1	9+1+4

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9-1	cannot be changed
72-10	cannot be changed
128-100	cannot be changed

You also will likely encounter real life routines that are not commutative. When preparing to go to work, putting on our clothes has to occur before putting on a coat. Likewise, getting in the car has to occur before putting the key in the ignition. In a store, you would need to pick up the items you are buying before proceeding to the cash register for checkout.

✓ Example

Write the expression 10+25 in a different way, using the commutative law of addition, and show that both expressions result in the same answer.

Solution

10+25=35	Solving the problem yields an answer of 35.
25+10	Using the commutative property, you can switch the 10 and the 25 so that they are in different positions.
25+10=35	Adding 25 to 10 in this new order also yields 35.

10+25=35 and 25+10=35

? Exercise

Rewrite 15+12=27 in a different way, using the commutative law of addition.

A. 15=12+27
B. 12=15+27
C. 15+(12=27)
D. 12+15=27

Answer

A. Incorrect. Only the order of the numbers to be added can change. The correct answer is 12+15=27.
B. Incorrect. Only the order of the numbers to be added can change. The correct answer is 12+15=27.
C. Incorrect. You would never put parentheses around an = sign (and parentheses are not used with the commutative laws). The correct answer is 12+15=27.
D. Correct. The commutative law lets you change the order of the numbers being added.

The Commutative Law of Multiplication

Multiplication also has a commutative law. The **commutative law of multiplication** states that when two or more numbers are being multiplied, their order can be changed without affecting the answer. In the example below, note that 5 multiplied by 4 yields the same result as 4 multiplied by 5. In both cases, the answer is 20.

$5 \cdot 4 = 20$

$4 \cdot 5 = 20$

This example shows how numbers can be switched in a multiplication expression.

✓ Example

Write the expression $30 \cdot 50$ in a different way, using the commutative law of multiplication, and show that both expressions result in the same answer.

Solution

$30 \cdot 50 = 1,500$	Solving the problem yields an answer of 1,500.
$50 \cdot 30$	Using the commutative law, you can switch the 30 and the 50 so that they are in different positions.
$50 \cdot 30 = 1,500$	Multiplying 50 and 30 also yields 1,500.

$50 \cdot 30$ and $30 \cdot 50 = 1,500$

Keep in mind that when you are using the commutative law, only the order is affected. The grouping remains unchanged.

? Exercise

Rewrite $52 \cdot 46$ in a different way, using the commutative law of multiplication.

A. $42 \cdot 56$
B. $5 \cdot 246$
C. $5 \cdot 24 \cdot 6$
D. $46 \cdot 52$

Answer

A. Incorrect. You cannot switch one digit from each number. The correct answer is $46 \cdot 52$.
B. Incorrect. The same two numbers should be multiplied. The numbers themselves should not be changed, only the order in which the numbers appear. The correct answer is $46 \cdot 52$.
C. Incorrect. The same two numbers should be multiplied. The numbers themselves should not be changed, only the order in which the numbers appear. The correct answer is $46 \cdot 52$.
D. Correct. The order of numbers is reversed, and the same two numbers are multiplied.

The Associative Law of Addition

Below are two ways of simplifying and solving an addition problem. Note that you can add numbers in any order. In the first example, 4 is added to 5 to make 9.

$4+5+6=9+6=15$

Here, the same problem is solved, but this time, 5 is added to 6 to make 11. Note that solving it this way yields the same answer.

$4+5+6=4+11=15$

The **associative law of addition** states that numbers in an addition expression can be regrouped using parentheses. You can remember the meaning of the associative law by remembering that when you *associate* with family members, friends, and co-workers, you end up forming groups with them. In the following expression, parentheses are used to group numbers together so that you know what to add first. Note that when parentheses are present, any numbers within parentheses are numbers you will add first. The expression can be re-written with different groups using the associative law.

$(4+5)+6=9+6=15$

$4+(5+6)=4+11=15$

Here, it is clear that the parentheses do not affect the final answer. The answer is the same regardless of where the parentheses are.

✓ Example

Rewrite (5+8)+3 using the associative law of addition. Show that the rewritten expression yields the same answer.

Solution

$(5+8)+3=13+3=16$	The original expression yields an answer of 16.
$5+(8+3)=5+11=16$	Grouping 8 and 3 instead of 5 and 8 results in the same answer of 16.

(5+8)+3=16 and 5+(8+3)=16

When rewriting an expression using the associative law, remember that you are regrouping the numbers and not reversing the order, as in the commutative law.

? Exercise

Rewrite 10+(5+6) using the associative property.

A. (5+6)+10
B. 10+(6+5)
C. (10+5)+6
D. (10+6)+5

Answer

A. Incorrect. The *order* of numbers is not changed when you are rewriting the expression using the associative law of addition. How the numbers are *grouped* should change. The correct answer is (10+5)+6.
B. Incorrect. The order of numbers is not changed when you are rewriting the expression using the associative law of addition. How the numbers are *grouped* should change. The correct answer is (10+5)+6.
C. Correct. Here, the numbers are regrouped. Now 10 and 5 are grouped in parentheses instead of 5 and 6.
D. Incorrect. The order of numbers is not changed when you are rewriting the expression using the associative law of addition. Only how the numbers are *grouped* should change. The correct answer is (10+5)+6.

The Associative Law of Multiplication

Multiplication has an associative law that works exactly the same as the one for addition. The **associative law of multiplication** states that numbers in a multiplication expression can be regrouped using parentheses. The following expression can be rewritten in a different way using the associative law.

$(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$

Here, it is clear that the parentheses do not affect the final answer. The answer is the same regardless of where the parentheses are.

✓ Example

Rewrite $(10 \cdot 200) \cdot 24$ **using the associative law of multiplication, and show that the rewritten expression yields the same answer.**

Solution

$(10 \cdot 200) \cdot 24 = 2000 \cdot 24 = 48,000$	The original expression yields an answer of 48,000.
$10 \cdot (200 \cdot 24) = 10 \cdot 4800 = 48,000$	Grouping 200 and 24 instead of 10 and 200 results in the same answer of 48,000.

$(10 \cdot 200) \cdot 24 = 48,000$ and $10 \cdot (200 \cdot 24) = 48,000$

When rewriting an expression using the associative law, remember that you are regrouping the numbers and not changing the order. Changing the order uses the commutative law.

? Exercise

Rewrite $8 \cdot (7 \cdot 6)$ using the associative property.

A. $(8 \cdot 7) \cdot 6$
B. $(7 \cdot 6) \cdot 8$
C. $(7 \cdot 8) \cdot 6$
D. $(8 \cdot 76)$

Answer

A. Correct. Here, the numbers are regrouped. Now 8 and 7 are grouped in parentheses instead of 7 and 6.
B. Incorrect. The order of numbers is not changed when you are rewriting the expression using the associative law of multiplication. How they are *grouped* should change. The correct answer is $(8 \cdot 7) \cdot 6$.
C. Incorrect. The order of numbers is not changed when you are rewriting the expression using the associative law of multiplication. Only how they are *grouped* should change. The correct answer is $(8 \cdot 7) \cdot 6$.
D. Incorrect. The digits of different numbers are not combined to create new numbers. How the numbers are *grouped* should change. The correct answer is $(8 \cdot 7) \cdot 6$.

Commutative or Associative?

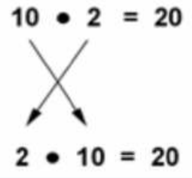
When an expression is being rewritten, you can tell whether it is being rewritten using the commutative or associative laws based on whether the order of the numbers change or the numbers are being regrouped using parentheses.

If an expression is rewritten so that the order of the numbers is changed, the commutative law is being used.

✓ Example

$10 \cdot 2 = 20$ **is rewritten as 2** $\cdot 10 = 20$**. Was this expression rewritten using the commutative law or the associative law?**

Solution



Rewriting the expression involves switching the order of the numbers. Therefore, the commutative law is being used.

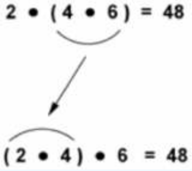
The commutative law is being used to rewrite the expression.

Remember that when you associate with friends and family, typically you are *grouping* yourself with other people. So, if numbers in an expression are regrouped using parentheses and the order of numbers remains the same, then the associative law is being used.

✓ Example

$2 \cdot (4 \cdot 6) = 48$ **is rewritten as** $(2 \cdot 4) \cdot 6 = 48$**. Was this expression rewritten using the commutative law or the associative law?**

Solution



Regrouping using parentheses does not change the order of the numbers. Therefore, the associative law is being used.

The associative law is being used to rewrite the expression.

? Exercise

$12 \cdot (6 \cdot 2) = 144$ is rewritten as $3 \cdot 17 = 51$. Was this expression rewritten using the commutative law or associative law?

A. commutative law
B. associative law

Answer

A. Correct. The order of numbers is being switched, which shows that the commutative law is being used.
B. Incorrect. The associative law involves regrouping numbers using parentheses, which does not occur in this problem. Rather, the order of numbers is switched. The correct answer is commutative law.

Using the Associative and Commutative Laws

The associative and commutative laws are useful when you have an expression with only addition. Using the commutative law, the numbers can be reordered so that the numbers that are easiest to add are next to each other, and using the associative law, you can group them in any way.

For example, here are some of the ways we can add $6+5+4$ using the associative and commutative laws. Note that the answer is always the same.

$(6+5)+4=11+4=15$ (grouping 6 and 5 to add first)

$(5+6)+4=11+4=15$ (reordering 6 and 5)

$5+(6+4)=5+10=15$ (grouping 6 and 4 to add first)

$6+(5+4)=6+9=15$ (grouping 5 and 4 to add first)

$6+(4+5)=6+9=15$ (reordering 4 and 5)

$(6+4)+5=10+5=15$ (grouping 6 and 4 to add first)

✓ Example

Write the expression $13+28+7$ a different way to make it easier to simplify. Then simplify.

Solution

$13+28+7$ $13+7+28$	Using the commutative property, reorder the numbers 7 and 28 since $13+7$ is easier to add than $13+28$.
$20+28$	Using the associative property, group the 13 and 7 together and add them first.
48	Add 20 and 28.

$13+28+7=13+7+28=48$

Sometimes the commutative and associative laws can make the problem easy enough to do in your head.

✓ Example

Jim is buying 8 pears, 7 apples, and 2 oranges. He decided the total number of fruits is $8+7+2$. Use the commutative property to write this expression in a different way. Then find the total.

Solution

$8+7+2$ $8+2+7$	Using the commutative property, reorder 2 and 7.
$10+7$	Using the associative property, group the 8 and 2 together and add them first.
17	Add 10 and 7.

$8+7+2=8+2+7=17$

This also works when you are multiplying more than two numbers. You can use the commutative and associative laws freely if the expression involves only multiplication.

✓ Example

There are 2 trucks in a garage, and each truck holds 60 boxes. There are 5 laptop computers in each box. Find the number of computers in the garage.

Solution

$2 \cdot 60 \cdot 5$	In order to find the answer, you need to multiply the number of trucks times the number of boxes in each truck, and, then by the number of computers in each box.
$2 \cdot 5 \cdot 60$	Using the commutative property, reorder the 5 and the 60. Now you can multiply $2 \cdot 5$ first.
$10 \cdot 60$	Using the associative property, multiply the 2 and the 5, 2 \cdot 5=10.
600	Now it's easier to multiply 10 and 60 to get 600.

There are 600 computers in the garage.

Summary

The addition property of 0 states that for any number being added to zero, the sum is the same number. The multiplication property of 1 states that for any number multiplied by one, that answer is that same number. Zero is called the additive identity, and one is called the multiplicative identity.

When you rewrite an expression by a commutative law, you change the order of the numbers being added or multiplied. When you rewrite an expression using an associative law, you group a different pair of numbers together using parentheses.

You can use the commutative and associative laws to regroup and reorder any number in an expression that involves only addition. You can also use the commutative and associative laws to regroup and reorder any number in an expression that involves only multiplication.

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1.4.2: The Distributive Property

Learning Objectives

- Simplify using the distributive property of multiplication over addition.
- Simplify using the distributive property of multiplication over subtraction.

Introduction

The distributive property of multiplication is a very useful property that lets you simplify expressions in which you are multiplying a number by a sum or difference. The property states that the product of a sum or difference, such as 6(5-2), is equal to the sum or difference of the products; in this case, 6(5)-6(2).

Remember that there are several ways to write multiplication. $3 \times 6 = 3(6) = 3 \cdot 6$

$3 \cdot (2+4) = 3 \cdot 6 = 18$

Distributive Property of Multiplication over Addition

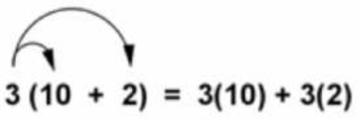
The **distributive property of multiplication over addition** can be used when you multiply a number by a sum. For example, suppose you want to multiply 3 by the sum of 10+2.

3(10+2)=?

According to this property, you can add the numbers and then multiply by 3.

3(10+2)=3(12)=36

Or, you can first multiply each addend by the 3. (This is called **distributing** the 3.) Then, you can add the products.



The multiplication of 3(10) and 3(2) will each be done before you add.

3(10)+3(2)=30+6=36

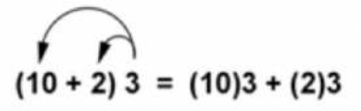
Note that the answer is the same as before.

You probably use this property without knowing that you are using it. When a group (let’s say 5 of you) orders food, and orders the same thing (let’s say you each order a hamburger for $3 each and a soda for $1 each), you can compute the bill (without tax) in two ways. You can figure out how much each of you needs to pay and multiply the sum times the number of you. So, you each pay (3+1) and then multiply times 5. That’s 5(3+1)=5(4)=20. Or, you can figure out how much the 5 hamburgers will cost and the 5 sodas and then find the total. That’s 5(3)+5(1)=15+5=20. Either way, the answer is the same, $20.

The two methods are represented by the equations below. On the left side, we add 10 and 2, and then multiply by 3. The expression is rewritten using the distributive property on the right side, where we distribute the 3, then multiply each by 3 and add the results. Notice that the result is the same in each case.

$$\begin{aligned} 3(10+2) &= 3(10)+3(2) \\ 3(12) &= 30+6 \\ 36 &= 36 \end{aligned}$$

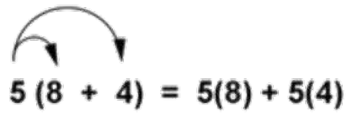
The same process works if the 3 is on the other side of the parentheses, as in the example below.



Example

Rewrite the expression 5(8+4) using the distributive property of multiplication over addition. Then simplify the result.

Solution

	In the original expression, the 8 and the 4 are grouped in parentheses. Using arrows, you can see how the 5 is distributed to each addend. The 8 and 4 are each multiplied by 5.
40+20=60	The resulting products are added together, resulting in a sum of 60.

5(8+4)=5(8)+5(4)=60

Exercise

Rewrite the expression 30(2+4) using the distributive property of addition.

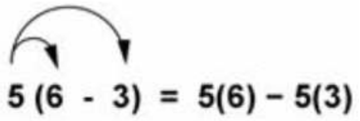
A. 30(2+4)+30(2+4)
B. 30(2)+30(4)
C. 30(6)
D. 30(24)

Answer

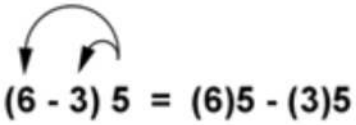
A. Incorrect. This would be doubling your original value. To distribute the 30, multiply the 2 by 30 and the 4 by 30. The correct answer is 30(2)+30(4).
B. Correct. The number 30 is distributed to both the 2 and the 4, so that both 2 and 4 are multiplied by 30.
C. Incorrect. The number 30 is not distributed in this answer. To distribute the 30, multiply the 2 by 30 and the 4 by 30. The correct answer is 30(2)+30(4).
D. Incorrect. The digits 2 and 4 should not be combined to form 24 because the addition process is incorrect. The number 30 is not distributed in this answer. To distribute the 30, multiply the 2 by 30 and the 4 by 30. The correct answer is 30(2)+30(4).

Distributive Property of Multiplication over Subtraction

The **distributive property of multiplication over subtraction** is like the distributive property of multiplication over addition. You can subtract the numbers and then multiply, or you can multiply and then subtract as shown below. This is called “distributing the multiplier.”



The same number works if the 5 is on the other side of the parentheses, as in the example below.



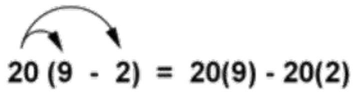
In both cases, you can then simplify the distributed expression to arrive at your answer. The example below, in which 5 is the outside multiplier, demonstrates that this is true. The expression on the right, which is simplified using the distributive property, is shown to be equal to 15, which is the resulting value on the left as well.

$$\begin{aligned} 5(6-3) &= 5(6)-5(3) \\ 5(3) &= 30-15 \\ 15 &= 15 \end{aligned}$$

✓ Example

Rewrite the expression 20(9-2) using the distributive property of multiplication over subtraction. Then simplify.

Solution

	In the original expression, the 9 and the 2 are grouped in parentheses. Using arrows, you can see how the 20 is distributed to each number so that the 9 and 2 are both multiplied by 20 individually.
180-40=140	Here, the resulting product of 40 is subtracted from the product of 180, resulting in an answer of 140.

20(9-2)=20(9)-20(2)=140

? Exercise

Rewrite the expression 10(15-6) using the distributive property of subtraction.

A. 10(6)-10(15)
B. 10(9)
C. 10(6-15)
D. 10(15)-10(6)

Answer

A. Incorrect. Here, a greater number would be subtracted from a lesser number, and the answer would not be a whole number. The correct answer is 10(15)-10(6).
B. Incorrect. The numbers in parentheses were subtracted before the number 10 could be distributed. The correct answer is 10(15)-10(6).
C. Incorrect. You probably used the commutative law instead of the distributive property. The correct answer is 10(15)-10(6).
D. Correct. The 10 is correctly distributed so that it is used to multiply the 15 and the 6 separately.

Summary

The distributive properties of addition and subtraction can be used to rewrite expressions for a variety of purposes. When you are multiplying a number by a sum, you can add and then multiply. You can also multiply each addend first and then add the products. This can be done with subtraction as well, multiplying each number in the difference before subtracting. In each case, you are distributing the outside multiplier to each number in the parentheses, so that multiplication occurs with each number before addition or subtraction occurs. The distributive property will be useful in future math courses, so understanding it now will help you build a solid math foundation.

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SECTION OVERVIEW

1.5: Exponents, Square Roots, and the Order of Operations

Topic hierarchy

1.5.1: Understanding Exponents and Square Roots

1.5.2: Order of Operations

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1.5.1: Understanding Exponents and Square Roots

Learning Objectives

- Evaluate expressions containing exponents.
- Write repeated factors using exponential notation.
- Find a square root of a perfect square.

Introduction

Exponents provide a special way of writing repeated multiplication. Numbers written in this way have a specific form, with each part providing important information about the number. Writing numbers using exponents can save a lot of space, too. The inverse operation of multiplication of a number by itself is called finding the square root of a number. This **operation** is helpful for problems about the area of a square.

Understanding Exponential Notation

Exponential notation is a special way of writing repeated factors, for example $7 \cdot 7$. Exponential notation has two parts. One part of the notation is called the **base**. The base is the number that is being multiplied by itself. The other part of the notation is the **exponent**, or power. This is the small number written up high to the right of the base. The exponent, or power, tells how many times to use the base as a **factor** in the multiplication. In the example, $7 \cdot 7$ can be written as 7^2 (7 is the base and 2 is the exponent). The exponent 2 means there are two factors.

$7^2 = 7 \cdot 7 = 49$

You can read 7^2 as "seven squared." This is because multiplying a number by itself is called "**squaring** a number." Similarly, raising a number to a power of 3 is called "**cubing** the number." You can read 7^3 as "seven cubed."

You can read 2^5 as "two to the fifth power" or "two to the power of five," or "two raised to the power of five." Read 8^4 as "eight to the fourth power," or "eight to the power of four," or "eight raised to the power of four." This format can be used to read *any* number written in exponential notation. In fact, while 6^3 is most commonly read "six cubed," it can also be read "six to the third power," or "six to the power of three," or "six raised to the power of three."

To find the value of a number written in exponential form, rewrite the number as repeated multiplication and perform the multiplication. Two examples are shown below.

Example

Find the value of 4^2.

Solution

4 is the base. 2 is the exponent.	An exponent means repeated multiplication. The base is 4; 4 is the number being multiplied. The exponent is 2; This means to use two factors of 4 in the multiplication.
$4^2 = 4 \cdot 4$	Rewrite as repeated multiplication.
$4 \cdot 4 = 16$	Multiply.

$4^2 = 16$

Example

Find the value of 2^5.

Solution

$2\cdot2\cdot2\cdot2\cdot2$	Rewrite 2^5 as repeated multiplication. The base is 2, the number being multiplied. The exponent is 5, the number of times to use 2 in the multiplication.
$2\cdot2\cdot2\cdot2\cdot2$ $4\cdot2\cdot2\cdot2$ $8\cdot2\cdot2$ $16\cdot2$ 32	Perform multiplication.

$2^5 = 32$

Exercise

Find the value of 4^3.

A. 12
B. 64
C. 256
D. 43

Answer

A. Incorrect. Multiply $4\cdot4\cdot4$, not $4\cdot3$. The correct answer is 64.
B. Correct. 4 $\cdot4\cdot4 = 64$
C. Incorrect. Use three factors of 4 in the multiplication. The correct answer is 64.
D. Incorrect. 4^3 means $4\cdot4\cdot4$. The correct answer is 64.

Writing Repeated Multiplication Using Exponents

Writing repeated multiplication in exponential notation can save time and space. Consider the example $5\cdot5\cdot5\cdot5$. We can use exponential notation to write this repeated multiplication as 5^4. Since 5 is being multiplied, it is written as the base. Since the base is used 4 times in the multiplication, the exponent is 4. The expression $5\cdot5\cdot5\cdot5$ can be rewritten in shorthand exponential notation as 5^4 and is read, “five to the fourth power” or “five to the power of 4.”

To write repeated multiplication of the same number in exponential notation, first write the number being multiplied as the base. Then count how many times that number is used in the multiplication, and write that number as the exponent. Be sure to count the numbers, *not* the multiplication signs, to determine the exponent.

✓ **Example**

Write $7\cdot7\cdot7$ in exponential notation.

Solution

7 is the base.	The base is the number being multiplied, 7.
Since 7 is used 3 times, 3 is the exponent.	The exponent tells the number of times the base is multiplied.

$7\cdot7\cdot7 = 7^3$ This is read “seven cubed.”

Exercise

Write $10\cdot10\cdot10\cdot10\cdot10$. in exponential notation.

A. 1,000,000
B. 60
C. 10^5

D. 10^6

Answer

A. Incorrect. 1,000,000 is equivalent to $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$, but it is not written in exponential notation. The correct answer is 10^6.

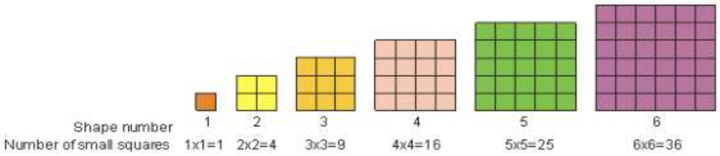
B. Incorrect. This answer is way less than the correct value and is not in exponential notation. You may have thought that six 10s should be written $6 \cdot 10$ and then simplified that to 60. The correct answer is 10^6 because the six 10s are multiplied together.

C.

D.

Understanding and Computing Square Roots

As you saw earlier, 5^2 is called “five squared.” “Five squared” means to multiply five by itself. In mathematics, we call multiplying a number by itself “squaring” the number. We call the result of squaring a whole number a square or a **perfect square**. A perfect square is any number that can be written as a whole number **raised to the power** of 2. For example, 9 is a perfect square because 3^2 is 9. A perfect square number can be represented as a square shape, as shown below. We see that 1, 4, 9, 16, 25, and 36 are examples of perfect squares.



To square a number, multiply the number by itself. 3 squared $= 3^2 = 3 \cdot 3 = 9$.

Below are some more examples of perfect squares.

1 squared	1^2	$1 \cdot 1$	1
2 squared	2^2	$2 \cdot 2$	4
3 squared	3^2	$3 \cdot 3$	9
4 squared	4^2	$4 \cdot 4$	16
5 squared	5^2	$5 \cdot 5$	25
6 squared	6^2	$6 \cdot 6$	36
7 squared	7^2	$7 \cdot 7$	49
8 squared	8^2	$8 \cdot 8$	64
9 squared	9^2	$9 \cdot 9$	81
10 squared	10^2	$10 \cdot 10$	100

The **inverse operation** of squaring a number is called finding the **square root** of a number. Finding a square root is like asking, “what number multiplied by itself will give me this number?” The square root of 25 is 5, because 5 multiplied by itself is equal to 25. Square roots are written with the mathematical symbol, called a **radical sign**, which looks like this: $\sqrt{\ }$. The “square root of 25" is written $\sqrt{25}$.

✓ Example

Find $\sqrt{81}$.

Solution

$\sqrt{81}=9$	Think, what number times itself gives 81? $9\cdot 9=81$

$\sqrt{81}=9$

? Exercise

Find $\sqrt{36}$.

A. 6
B. 18
C. 72
D. 7

Answer

A. Correct. Since $6\cdot 6=36$, $\sqrt{36}=6$.
B. Incorrect. The square root of 36 is the number that you can multiply by itself to get 36. The square root is not the number you multiply by 2 to get 36. The correct answer is 6, because $6\cdot 6=36$.
C. Incorrect. You may have incorrectly added 36 to itself to get 72. The square root of 36 is the number that you can multiply by itself to get 36. The correct answer is 6, because $6\cdot 6=36$.
D. Incorrect. $7\cdot 7=49$, so $\sqrt{49}=7$. The correct answer is 6, because $6\cdot 6=36$. This means $\sqrt{36}=6$.

Summary

Exponential notation is a shorthand way of writing repeated multiplication of the same number. A number written in exponential notation has a base and an exponent, and each of these parts provides information for finding the value of the expression. The base tells what number is being repeatedly multiplied, and the exponent tells how many times the base is used in the multiplication. Exponents 2 and 3 have special names. Raising a base to a power of 2 is called “squaring” a number. Raising a base to a power of 3 is called “cubing” a number. The inverse of squaring a number is finding the square root of a number. To find the square root of a number, ask yourself, “What number can I multiply by itself to get this number?”

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1.5.2: Order of Operations

Learning Objectives

- Use the order of operations to simplify expressions, including those with parentheses.
- Use the order of operations to simplify expressions containing exponents and square roots.

Introduction

People need a common set of rules for performing computation. Many years ago, mathematicians developed a standard **order of operations** that tells you which calculations to make first in an expression with more than one **operation**. Without a standard procedure for making calculations, two people could get two different answers to the same problem. For example, $3+5 \cdot 2$ has only one correct answer. Is it 13 or 16?

The Order of Addition, Subtraction, Multiplication, and Division Operations

First, consider expressions that include one or more of the arithmetic operations: addition, subtraction, multiplication, and division. The order of operations requires that all multiplication and division be performed first, going from left to right in the **expression**. The order in which you compute multiplication and division is determined by which one comes first, reading from left to right.

After multiplication and division has been completed, add or subtract in order from left to right. The order of addition and subtraction is also determined by which one comes first when reading from left to right.

Below are three examples showing the proper order of operations for expressions with addition, subtraction, multiplication, and/or division.

Example

Simplify $3+5 \cdot 2$.

Solution

$3+5 \cdot 2$	Order of operations tells you to perform multiplication before addition.
3+10	Then add.

$3+5 \cdot 2=13$

Example

Simplify $20-16 \div 4$.

Solution

$20-16 \div 4$	Order of operations tells you to perform division before subtraction.
20-4 16	Then subtract.

$20-16 \div 4=16$

Example

Simplify $60-30 \div 3 \cdot 5+7$.

Solution

$60-30\div 3\cdot 5+7$	Order of operations tells you to perform multiplication and division first, working from left to right, before doing addition and subtraction.
$60-10\cdot 5+7$ $60-50+7$	Continue to perform multiplication and division from left to right.
10+7 17	Next, add and subtract from left to right. (Note that addition is not necessarily performed before subtraction.)

$60-30\div 3\cdot 5+7=17$

Grouping Symbols and the Order of Operations

Grouping symbols such as parentheses ( ), brackets [ ], braces { }, and fraction bars can be used to further control the order of the four basic arithmetic operations. The rules of the order of operations require computation within grouping symbols to be completed first, even if you are adding or subtracting within the grouping symbols and you have multiplication outside the grouping symbols. After computing within the grouping symbols, divide or multiply from left to right and then subtract or add from left to right.

✓ Example

Simplify $900\div(6+3\cdot 8)-10$.

Solution

$900\div((6+3\cdot 8))-10$	Order of operations tells you to perform what is inside the parentheses first.
$900\div(6+(3\cdot 8))-10$ $900\div(6+24)-10$	Simplify the expression in the parentheses. Multiply first.
$900\div 30-10$	Then add 6+24.
$900\div 30-10$ $30-10$ 20	Now perform division; then subtract.

$900\div(6+3\cdot 8)-10=20$

When there are grouping symbols within grouping symbols, compute from the inside to the outside. That is, begin simplifying the innermost grouping symbols first. Two examples are shown.

✓ Example

Simplify $4-3[20-3\cdot 4-(2+4)]\div 2$.

Solution

$4-3\div 2$	There are brackets and parentheses in this problem. Compute inside the innermost grouping symbols first.
$4-3[20-3\cdot 4-((2+4)]]\div 2$ $4-3\div 2$	Simplify within parentheses.
$4-3\div 2$ $4-3\div 2$ $4-3\div 2$	Then, simplify within the brackets by multiplying and then subtracting from left to right.
$4-3(2)\div 2$	

$4-3((2)\div 2$ $4-6\div 2$ $4-3$	Multiply and divide from left to right.
$4-3$ 1	Subtract.

$4-3[20-3\cdot 4-(2+4)]\div 2=1$

Remember that parentheses can also be used to show multiplication. In the example that follows, the parentheses are not a grouping symbol; they are a multiplication symbol. In this case, since the problem only has multiplication and division, we compute from left to right. Be careful to determine what parentheses mean in any given problem. Are they a grouping symbol or a multiplication sign?

✓ Example

Simplify $6\div(3)(2)$**.**

Solution

$6\div 3\cdot 2$	This expression has multiplication and division only. The multiplication operation can be shown with a dot.
$6\div 3\cdot 2$ $2\cdot 2$ 4	Since this expression has only division and multiplication, compute from left to right.

$6\div(3)(2)=4$

Consider what happens if braces are added to the problem above: $6\div\{(3)(2)\}$. The parentheses still mean multiplication; the additional braces are a grouping symbol. According to the order of operations, compute what is inside the braces first. This problem is now evaluated as $6\div 6=1$. Notice that the braces caused the answer to change from 1 to 4.

? Exercise

Simplify $40-(4+6)\div 2+3$.

A. 18
B. 38
C. 24
D. 32

Answer

A. Incorrect. Compute the addition in parentheses first. $40-10\div 2+3$. Then, perform division. $40-5+3$. Finally, add and subtract from left to right. The correct answer is 38.
B. Correct. Compute the addition in parentheses first. $40-10\div 2+3$. Then, perform division. $40-5+3$. Finally, add and subtract from left to right.
C. Incorrect. Compute the addition in parentheses first. $40-10\div 2+3$. Then, perform division. $40-5+3$. Finally, add and subtract from left to right. The correct answer is 38.
D. Incorrect. Compute the addition in parentheses first. $40-10\div 2+3$. Then, perform division. $40-5+3$. Finally, with only subtraction and addition left, add and subtract from left to right. The correct answer is 38.

The Order of Operations

1. Perform all operations within grouping symbols first. Grouping symbols include parentheses ( ), braces { }, brackets [ ], and fraction bars.
2. Multiply and Divide, from left to right.

3. Add and Subtract, from left to right.

Performing the Order of Operations with Exponents and Square Roots

So far, our rules allow us to simplify expressions that have multiplication, division, addition, subtraction or grouping symbols in them. What happens if a problem has **exponents** or **square roots** in it? We need to expand our order of operation rules to include exponents and square roots.

If the expression has exponents or square roots, they are to be performed after parentheses and other grouping symbols have been simplified and *before* any multiplication, division, subtraction and addition that are outside the parentheses or other grouping symbols.

Note that you compute from more complex operations to more basic operations. Addition and subtraction are the most basic of the operations. You probably learned these first. Multiplication and division, often thought of as repeated addition and subtraction, are more complex and come before addition and subtraction in the order of operations. Exponents and square roots are repeated multiplication and division, and because they're even more complex, they are performed before multiplication and division. Some examples that show the order of operations involving exponents and square roots are shown below.

✓ Example

Simplify $14+28\div 2^2$**.**

Solution

$14+28\div 2^2$	This problem has addition, division, and exponents in it. Use the order of operations.
$14+28\div 4$	Simplify 2^2
$14+7$	Perform division before addition.
21	Add.

$14+28\div 2^2=21$

✓ Example

Simplify $3^2\cdot 2^3$**.**

Solution

$3^2\cdot 2^3$	This problem has exponents and multiplication in it.
$9\cdot 8$	Simplify 3^2 and 2^3.
72	Perform multiplication.

$3^2\cdot 2^3=72$

✓ Example

Simplify $(3+4)^2+(8)(4)$**.**

Solution

$(3+4)^2+(8)(4)$	This problem has parentheses, exponents, and multiplication in it. The first set of parentheses is a grouping symbol. The second set indicates multiplication. Grouping symbols are handled first.
$7^2+(8)(4)$ $49+(8)(4)$	Add the numbers inside the parentheses that are serving as grouping symbols. Simplify 7^2.

$49+32$	Perform multiplication.
81	Add.

$(3+4)^2+(8)(4)=81$

? Exercise

Simplify $77-(1+4-2)^2$.

A. 68
B. 28
C. 71
D. 156

Answer

A. Correct. $77-(1+4-2)^2=77-(3)^2=77-9=68$
B. Incorrect. Simplify the expression in parentheses first. $77-(1+4-2)^2=77-(3)^2=77-9=68$
C. Incorrect. The exponent of 2 tells you to multiply the number by itself, not by 2; $77-(3)^2=77-9$, not $77-6$. The correct answer is 68.
D. Incorrect. Parentheses are a grouping symbol, and numbers inside them should be computed first. The exponent of 2 tells you to multiply the number by itself, not by $2.77-(1+4-2)^2=77-(3)^2=77-9=68$. The correct answer is 68.

The Order of Operations

1. Perform all operations within grouping symbols first. Grouping symbols include parentheses ( ), braces { }, brackets [ ], and fraction bars.
2. Evaluate exponents and roots of numbers, such as square roots.
3. Multiply and Divide, from left to right.
4. Add and Subtract, from left to right.

Some people use a saying to help them remember the order of operations. This saying is called PEMDAS or “Please Excuse My Dear Aunt Sally.” The first letter of each word begins with the same letter of an arithmetic operation.

The P in Please stands for Parentheses (and other grouping symbols).

The E in Excuse stands for Exponents.

The M and D in My Dear stand for Multiplication and Division (from left to right).

The A and S in Aunt Sally stand for Addition and Subtraction (from left to right).

Note: Even though multiplication comes before division in the saying, division could be performed first. Which is performed first, between multiplication and division, is determined by which comes first when reading from left to right. The same is true of addition and subtraction. Don’t let the saying confuse you about this!

Summary

The order of operations gives us a consistent sequence to use in computation. Without the order of operations, you could come up with different answers to the same computation problem. (Some of the early calculators, and some inexpensive ones, do NOT use the order of operations. In order to use these calculators, the user has to input the numbers in the correct order.)

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CHAPTER OVERVIEW

2: Fractions and Mixed Numbers

- 2.1: Introduction to Fractions and Mixed Numbers
  - 2.1.1: Introduction to Fractions and Mixed Numbers
  - 2.1.2: Proper and Improper Fractions
  - 2.1.3: Factors and Primes
  - 2.1.4: Simplifying Fractions
  - 2.1.5: Comparing Fractions
- 2.2: Multiplying and Dividing Fractions and Mixed Numbers
  - 2.2.1: Multiplying Fractions and Mixed Numbers
  - 2.2.2: Dividing Fractions and Mixed Numbers
- 2.3: Adding and Subtracting Fractions and Mixed Numbers
  - 2.3.1: Adding Fractions and Mixed Numbers
  - 2.3.2: Subtracting Fractions and Mixed Numbers

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SECTION OVERVIEW

2.1: Introduction to Fractions and Mixed Numbers

Topic hierarchy

2.1.1: Introduction to Fractions and Mixed Numbers

2.1.2: Proper and Improper Fractions

2.1.3: Factors and Primes

2.1.4: Simplifying Fractions

2.1.5: Comparing Fractions

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2.1.1: Introduction to Fractions and Mixed Numbers

Learning Objectives

- Identify the numerator and denominator of a fraction.
- Represent a fraction as part of a whole or part of a set.

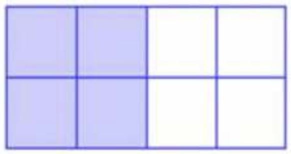
Introduction

Many problems in mathematics deal with whole numbers, which are used to count whole units of things. For example, you can count students in a classroom and the number of dollar bills. You need other kinds of numbers to describe units that are not whole. For example, an aquarium might be partly full. A group may have a meeting, but only some of the members are present.

Fractions are numbers used to refer to a part of a whole. This includes measurements that cannot be written as whole numbers. For example, the width of a piece of notebook paper is more than 8 inches but less than 9 inches. The part longer than 8 inches is written as a fraction. Here, you will investigate how fractions can be written and used to represent quantities that are parts of the whole.

Identifying Numerators and Denominators

A whole can be divided into parts of equal size. In the example below, a rectangle has been divided into eight equal squares. Four of these eight squares are shaded.



The shaded area can be represented by a fraction. A fraction is written vertically as two numbers with a line between them.

The **denominator** (the bottom number) represents the number of equal parts that make up the whole. The **numerator** (the top number) describes the number of parts that you are describing. So returning to the example above, the rectangle has been divided into 8 equal parts, and 4 of them have been shaded. You can use the fraction $\frac{4}{8}$ to describe the shaded part of the whole.

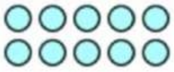
In $\frac{4}{8}$, the 4 is the numerator and tells how many parts are shaded. The 8 is the denominator and tells how many parts are required to make the whole.

Parts of a Set

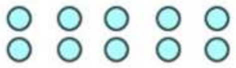
The rectangle model above provides a good, basic introduction to fractions. However, what do you do with situations that cannot be as easily modeled by shading part of a figure? For example, think about the following situation:

Marc works as a Quality Assurance Manager at an automotive plant. Every hour, he inspects 10 cars; $\frac{4}{5}$ of those pass inspection.

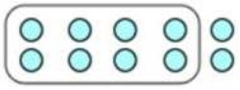
In this case, 10 cars make up the whole group. Each car can be represented as a circle, as shown below.



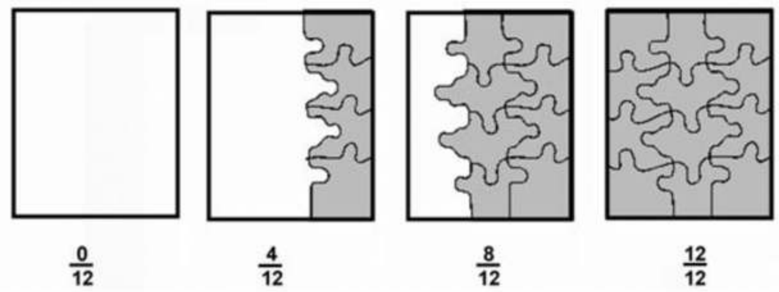
To show $\frac{4}{5}$ of the whole group, you first need to divide the whole group into 5 equal parts. (You know this because the fraction has a denominator of 5.)



To show $\frac{4}{5}$, circle 4 of the equal parts.



Here is another example. Imagine that Aneesh is putting together a puzzle made of 12 pieces. At the beginning, none of the pieces has been put into the puzzle. This means that $\frac{0}{12}$ of the puzzle is complete. Aneesh then puts four pieces together. The puzzle is $\frac{4}{12}$ complete. Soon, he adds four more pieces; 8 out of 12 pieces are now connected. This fraction can be written as $\frac{8}{12}$. Finally, Aneesh adds four more pieces. The puzzle is whole, using all 12 pieces. The fraction can be written as $\frac{12}{12}$.



Note that the number in the denominator cannot be zero. The denominator tells how many parts make up the whole. So if this number is 0, then there are no parts and therefore there can be no whole.

The numerator can be zero, as it tells how many parts you are describing. Notice that in the puzzle example above, you can use the fraction $\frac{0}{12}$ to represent the state of the puzzle when 0 pieces have been placed.

Fractions can also be used to analyze data. In the data table below, 3 out of 5 tosses of a coin came up heads, and 2 out of 5 tosses came up tails. Out of the total number of coin tosses, the portion that was heads can be written as $\frac{3}{5}$. The portion that was tails can be written as $\frac{2}{5}$.

Coin Toss	Result
1	Heads
2	Tails
3	Heads
4	Heads
5	Tails

? Exercise

Sophia, Daphne, and Charlie are all participating in a relay race to raise money for charity. First, Sophia will run 2 miles. Then, Daphne will run 5 miles. Finally, Charlie will end the race by running 3 miles. What fraction of the race will Daphne run?

A. 5 miles
B. $\frac{5}{10}$
C. $\frac{2}{5}$
D. $\frac{5}{3}$

Answer

A. Incorrect. Daphne will run 5 miles, but that does not indicate the fractional part of the race that she will run. To find the fraction, first find the whole length of the race by combining the distances the three people will run: 2+5+3=10. Then consider the distance Daphne will run. The correct answer is $\frac{5}{10}$.
B. Correct. The entire race is 10 miles long, and Daphne will run 5 miles. This means she will run $\frac{5}{10}$ of the race.
C. Incorrect. To find the fraction, first find the whole length of the race by combining the distances the three people will run: 2+5+3=10. Then consider the distance Daphne will run. The correct answer is $\frac{5}{10}$.
D. Incorrect. To find the fraction, first find the whole length of the race by combining the distances the three people will run: 2+5+3=10. Then consider the distance Daphne will run. The correct answer is $\frac{5}{10}$.

Parts of a Whole

The “parts of a whole” concept can be modeled with pizzas and pizza slices. For example, imagine a pizza is cut into 4 pieces, and someone takes 1 piece. Now, $\frac{1}{4}$ of the pizza is gone and $\frac{3}{4}$ remains. Note that both of these fractions have a denominator of 4, which refers to the number of slices the whole pizza has been cut into.

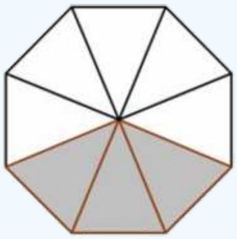
✓ Example

Joaquim bakes a blueberry pie for a potluck dinner. The total pie is cut into 6 equal slices. After everybody eats dessert, only one slice of the pie remains. What fraction of the pie remains?

Solution

$\frac{?}{6}$	The pie was cut into six equal slices, so the denominator of the fraction will be 6.
$\frac{1}{6}$	Only 1 slice remains, so the numerator of the fraction will be 1.
	$\frac{1}{6}$ of the pie remains.

✓ Example



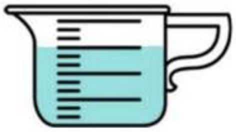
Write a fraction to represent the portion of the octagon that is *not* shaded.

Solution

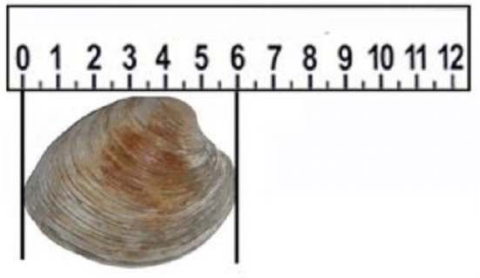
$\frac{?}{8}$	The octagon has eight equal sections, so the denominator of the fraction will be 8.
$\frac{5}{8}$	Five sections are not shaded, so the numerator of the fraction will be 5.
	$\frac{5}{8}$ of the octagon is not shaded.

Measurement Contexts

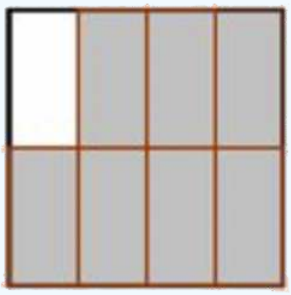
You can use a fraction to represent the quantity in a container. This measuring cup is $\frac{3}{4}$ filled with a liquid. Note that if the cup were $\frac{4}{4}$ full, it would be a whole cup.



You can also use fractions in measuring the length, width, or height of something that is not a full unit. Using a 12-inch ruler, you measure a shell that is 6 inches long. You know that 12 inches equals one foot. So, the length of this shell is $\frac{6}{12}$ of a foot; the 12-inch ruler is the “whole,” and the length of the shell is the “part.”



Exercise



Which fraction represents the portion of the shape that is shaded?

A. $\frac{8}{7}$
B. $\frac{7}{8}$

Answer

A. Incorrect. The total number of parts that make up the whole, 8, is the denominator. That is the number *below* the fraction bar. The number of shaded parts, 7, is the numerator. That is the number *above* the fraction bar. The correct answer is $\frac{7}{8}$.
B. Correct. The total number of parts that make up the whole, 8, is the denominator (below the fraction bar). The number of shaded parts, 7, is the numerator (above the fraction bar).

Summary

Fractions are used to represent parts of a whole. You can use fractions when describing substances, quantities, or diagrams that are not complete. You also use fractions to describe numbers of people or objects that do not make up a complete group. Fractions are written with a numerator and denominator. The numerator (above the fraction bar) tells the number of parts being described, and the denominator (below the fraction bar) tells the number of parts that make up the whole.

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2.1.2: Proper and Improper Fractions

Learning Objectives

- Learn the rules for rounding.
- Round whole numbers to specific place values, including tens, hundreds, and thousands.

Introduction

In some situations, you don’t need an exact answer. In these cases, **rounding** the number to a specific **place value** is possible. For example, if you travelled 973 miles, you might want to round the distance to 1,000 miles, which is easier to think about. Rounding also comes in handy to see if a calculation is reasonable.

Rounding Whole Numbers

These are the rules for rounding **whole numbers**:

First, identify the digit with the place value to which you are rounding. You might circle or highlight the digit so you can focus on it better.

Then, determine the possible numbers that you would obtain by rounding. These possible numbers are close to the number that you’re rounding to, but have zeros in the digits to the right.

If you are rounding 186 to the nearest ten, then 180 and 190 are the two possible numbers to round to, as 186 is between 180 and 190. But how do you know whether to round to 180 or 190?

> Usually, round a number to the number that is closest to the original number.
>
> When a number is halfway between the two possible numbers, round up to the greater number.

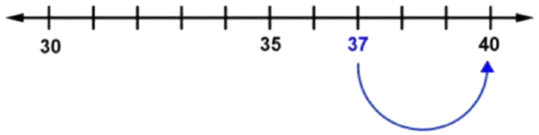
Since 186 is between 180 and 190, and 186 is closer to 190, you round up to 190.

You can use a number line to help you round numbers.

Example

A camera is dropped out of a boat, and sinks to the bottom of a pond that is 37 feet deep. Round 37 to the nearest ten.

Solution

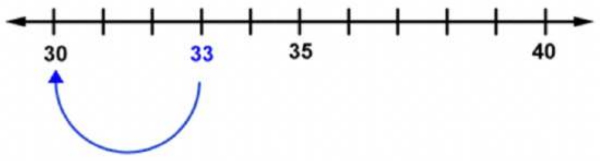
37	The digit you’re rounding to is the tens digit, 3.
30...37 ... 40	37 is between 30 and 40.
	37 is only 3 away from 40, but it’s 7 away from 30. So, 37 is closer to 40.

To the nearest ten, 37 rounds to 40.

Example

Round 33 to the nearest ten.

Solution



33 rounds to 30 because 33 is closer to 30.

To the nearest ten, 33 rounds to 30.

You can determine where to round without using a number line by looking at the digit to the right of the one you're rounding to. If that digit is less than 5, round down. If it's 5 or greater, round up. In the example above, you can see without a number line that 33 is rounded to 30 because the ones digit, 3, is less than 5.

✓ Example

Round 77 to the nearest ten.

Solution
77 rounds to 80 because the ones digit, 7, is 5 or greater.

77 rounded to the nearest ten is 80.

✓ Example

There are 576 jellybeans in a jar. Round this number to the nearest ten.

Solution
576 rounds to 580 because the ones digit, 6, is 5 or greater.

576 rounded to the nearest ten is 580.

In the previous examples, you rounded to the tens place. The rounded numbers had a 0 in the ones place. If you round to the nearest hundred, the rounded number will have zeros in the tens and ones places. The rounded number will resemble 100, 500, or 200.

✓ Example

A runner ran 1,539 meters, but describes the distance he ran with a rounded number. Round 1,539 to the nearest hundred.

Solution
1,539 rounds to 1,500 because the next digit is less than 5.

1,539 rounded to the nearest hundred is 1,500.

If you round to the nearest thousand, the rounded number will have zeros in the hundreds, tens, and ones places. The rounded number will resemble 1,000, 2,000, or 14,000.

✓ Example

A plane's altitude increased by 2,721 feet. Round this number to the nearest thousand.

Solution
2,721 rounds to 3,000 because the next digit, 7, is 5 or greater.

2,721 rounded to the nearest thousand is 3,000.

Now that you know how to round to the nearest ten, hundred, and thousand, try rounding to the nearest ten thousand.

✓ Example

Round 326,749 to the nearest ten thousand.

Solution

326,749 rounds to 330,000 because the next digit, 6, is 5 or greater.

326,749 rounded to the nearest ten thousand is 330,000.

? Exercise

A record number of 23,386 people voted in a city election. Round this number to the nearest hundred.

A. 23,300
B. 23,400
C. 23,000
D. 23,390

Answer

A. Incorrect. The two possible numbers are 23,300 and 23,400, but 23,386 is closer to 23,400. The tens digit, 8, is 5 or greater, so you should round up. The correct answer is 23,400.
B. Correct. The two possible numbers are 23,300 and 23,400, and 23,386 is closer to 23,400. The tens digit, 8, is 5 or greater, so you should round up.
C. Incorrect. This number is rounded to the nearest thousand, not the nearest hundred. The correct answer is 23,400.
D. Incorrect. This number is rounded to the nearest ten, not the nearest hundred. The correct answer is 23,400.

Summary

In situations when you don’t need an exact answer, you can round numbers. When you round numbers, you are always rounding to a particular place value, such as the nearest thousand or the nearest ten. Whether you round up or round down usually depends on which number is closest to your original number. When a number is halfway between the two possible numbers, round up to the larger number.

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2.1.3: Factors and Primes

Learning Objectives

- Recognize (by using the divisibility rule) if a number is divisible by 2, 3, 4, 5, 6, 9, or 10.
- Find the factors of a number.
- Determine whether a number is prime, composite, or neither.
- Find the prime factorization of a number.

Introduction

Natural numbers, also called counting numbers (1, 2, 3, and so on), can be expressed as a product of their **factors**. When working with a fraction, you often need to make the fraction as simple as possible. This means that the **numerator** and the **denominator** have no common factors other than . It will help to find factors, so that later you can simplify and compare fractions.

Tests of Divisibility

When a natural number is expressed as a product of two other natural numbers, those other numbers are factors of the original number. For example, two factors of 12 are 3 and 4, because $3 \cdot 4 = 12$.

When one number can be divided by another number with no remainder, we say the first number is **divisible** by the other number. For example, 20 is divisible by $4 (20 \div 4 = 5)$. If a number is divisible by another number, it is also a **multiple** of that number. For example, 20 is divisible by 4, so 20 is a multiple of 4.

Divisibility tests are rules that let you quickly tell if one number is divisible by another. There are many divisibility tests. Here are some of the most useful and easy to remember:

- A number is divisible by 2 if the last (ones) digit is divisible by 2. That is, the last digit is 0, 2, 4, 6, or 8. (We then say the number is an **even number**.) For example, in the number 236, the last digit is 6. Since 6 is divisible by $2 (6 \div 2 = 3)$, 236 is divisible by 2.
- A number is divisible by 3 if the sum of all the digits is divisible by 3. For example, the sum of the digits of 411 is 4+1+1=6. Since 6 is divisible by $3 (6 \div 3 = 2)$, 411 is divisible by 3.
- A number is divisible by 5 if the last digit is 0 or 5. For example, 275 and 1,340 are divisible by 5 because the last digits are 5 and 0.
- A number is divisible by 10 if the last digit is 0. For example, 520 is divisible by 10 (last digit is 0).

Other useful divisibility tests:

1. : A number is divisible by 4 if the last two digits are divisible by 4.
2. : A number is divisible by 6 if it is divisible by both 2 and 3.
3. : A number is divisible by 9 if the sum of its digits is divisible by 9

Here is a summary of the most commonly used divisibility rules.

A number is divisible by	Example
2 if the last digit is even (0, 2, 4, 6, 8).	426 yes 273 no
3 if the sum of the digits is divisible by 3.	642 yes (6+4+2=12, 12 is divisible by 3) 721 no (7+2+1=10, 10 is not divisible by 3)
4 if the last two digits form a number that is divisible by 4.	164 yes (64 is divisible by 4) 135 no (35 is not divisible by 4)

5 if the last digit is 0 or 5.	685 yes 432 no
6 if the number is divisible by 2 and 3.	324 yes (it is even and 3+2+4=9) 411 no (although divisible by 3, it is not even)
9 if the sum of the digits is divisible by 9.	279 yes (2+7+9=18) 512 no (5+1+2=8)
10 if the last digit is a 0.	620 yes. 238 no

If you need to check for divisibility of a number without a rule, divide (either using a calculator or by hand). If the result is a number without any fractional part or remainder, then the number is divisible by the **divisor**. If you forget a rule, you can also use this strategy.

Exercise

Determine whether 522 is divisible by 2, 3, 4, 5, 6, 9, or 10.

A. 2 and 3 only
B. 4 only
C. 2, 3, 6, and 9 only
D. 4, 5, and 10 only

Answer

A. Incorrect. Although 522 is divisible by 2 (the last digit is even) and 3 (5+2+2=9, which is a multiple of 3), it is also divisible by 6 and 9. The correct answer is 2, 3, 6, and 9 only.
B. Incorrect. The last two digits (22) are not divisible by 4, so 522 is not divisible by 4. 522 has a last digit divisible by 2, so 522 is divisible by 2. The sum of the digits is divisible by 3 (5+2+2=9) and by 9, so 522 is divisible by 3 and 9. Since 522 is divisible by 2 and 3, it is divisible by 6. Since the last digit is not 0 or 5, 522 is not divisible by 5 or 10. The correct answer is 2, 3, 6, and 9 only.
C. Correct. 522 is divisible by 2 (the last digit is even) and 3 (5+2+2=9, which is a multiple of 3). Since it is divisible by 2 and 3, it is also divisible by 6. Also, the sum of the digits is divisible by 9, so 522 is divisible by 9. Since the last digit is not 0 or 5, 522 is not divisible by 5 or 10. The number formed by the last two digits, 22, is not divisible by 4, so 522 is not divisible by 4.
D. Incorrect. The last two digits are not divisible by 4, so 522 is not divisible by 4. The last digit is not 0 or 5 so 522 is not divisible by 5. The last digit is not 0, so 522 is not divisible by 10. However, 522 is divisible by 2 (the last digit is even) and 3 (5+2+2=9, which is a multiple of 3). Since it is divisible by 2 and 3, it is also divisible by 6. Also, the sum of the digits is divisible by 9, so 522 is divisible by 9. The correct answer is 2, 3, 6, and 9 only.

Factoring Numbers

To find all the factors of a number, you need to find all numbers that can divide into the original number without a remainder. The divisibility rules from above will be extremely useful!

Suppose you need to find the factors of 30. Since 30 is a number you are familiar with, and small enough, you should know many of the factors without applying any rules. You can start by listing the factors as they come to mind:

$2 \cdot 15$

$3 \cdot 10$

$5 \cdot 6$

Is that it? Not quite. All natural numbers except 1 also have 1 and the number itself as factors:

$1 \cdot 30$

The factors of 30 are 1,2,3,5,6,10,15, and 30.

When you find one factor of a number, you can easily find another factor: it is the quotient using that first factor as the divisor. For example, once you know 2 is a factor of 30, then $30 \div 2$ is another factor. A pair of factors whose product is a given number is a **factor pair** of the original number. So, 2 and 15 are a factor pair for 30.

What do you do if you need to factor a greater number and you can't easily see its factors? That's where the divisibility rules will come in quite handy. Here is a general set of steps that you may follow:

1. Begin with and check the numbers sequentially, using divisibility rules or division.
2. When you find a factor, find the other number in the factor pair.
3. Keep checking sequentially, until you reach the second number in the last factor pair you found, or until the result of dividing gives a number less than the divisor.

Note that you can stop checking when the result of dividing is less than the number you're checking. This means that you have already found all factor pairs, and continuing the process would find pairs that have been previously found.

If a number has exactly two factors, 1 and itself, the number is a **prime number**. A number that has more factors than itself and 1 is called a **composite number**. The number 1 is considered neither prime nor composite, as its only factor is 1. To determine whether a number is prime, composite, or neither, check factors. Here are some examples.

Number	**Composite, Prime, or Neither?**	**Explanation**
1	Neither	1 does not have two different factors, so it is not prime.
2	Prime	2 has only the factors 2 and 1.
3	Prime	3 has only the factors 3 and 1.
4	Composite	4 has more than two factors: 1, 2, and 4, so it is composite.
5, 7, 11, 13	Prime	Each number has only two factors: 1 and itself.
6, 8, 9, 10, 50, 63	Composite	Each number has more than two factors.

Exercise

Find all the factors of 48.

Answer

1, 2, 3, 4, 6, 8, 12, 16, 24, 48

Prime Factorization

A composite number written as a product of only prime numbers is called the **prime factorization** of the number. One way to find the prime factorization of a number is to begin with the prime numbers 2, 3, 5, 7, 11 and so on, and determine whether the number is divisible by the primes.

For example, if you want to find the prime factorization of 20, start by checking if 20 is divisible by 2. Yes, $2 \cdot 10 = 20$.

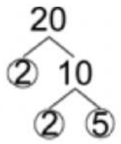
Then factor 10, which is also divisible by $2(2 \cdot 5 = 10)$.

Both of those factors are prime, so you can stop. The prime factorization of 20 is $2 \cdot 2 \cdot 5$, which you can write using **exponential notation** as $2^2 \cdot 5$.

One way to find the prime factorization of a number is to use successive divisions.

10 2\longdiv20	Divide 20 by 2 to get 10. 2 is being used because it is a prime number and a factor of 20. You could also have started with 5.
5 2\longdiv10 2\longdiv20	Then divide 10 by 2 to get 5.
$2 \cdot 2 \cdot 5$	Multiplying these divisors forms the prime factorization of 20.

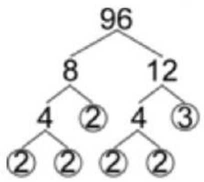
To help you organize the factoring process, you can create a **factor tree**. This is a diagram that shows a factor pair for a composite number. Then, each factor that isn't prime is also shown as a factor pair. You can continue showing factor pairs for composite factors, until you have only prime factors. When a prime number is found as a factor, circle it so you can find it more easily later.

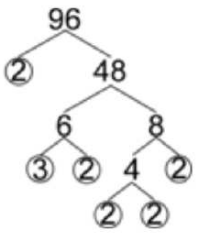


Written using exponential notation, the prime factorization of 20 is again $2^2 \cdot 5$.

Notice that you don't *have* to start checking the number using divisibility of prime numbers. You can factor 20 to $4 \cdot 5$, and then factor 4 to $2 \cdot 2$, giving the same prime factorization: $2 \cdot 2 \cdot 5$.

Now look at a more complicated factorization.





Notice that there are two different trees, but they both produce the same result: five 2s and one 3. Every number will only have one, unique prime factorization. You can use any sets of factor pairs you wish, as long as you keep factoring composite numbers.

When you rewrite the prime factorization of $96(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3)$ in exponential notation, the five 2s can be written as 2^5. So, $96 = 2^5 \cdot 3$.

? Exercise

When finding the prime factorization of 72, Marie began a tree diagram using the two factors 9 and 8. Which of the following statements are true?

1. Marie started the diagram incorrectly and should have started the tree diagram using the factors 2 and 36.
2. Marie's next set of factor pairs could be 3, 3 and 2, 4.
3. Marie's next set of factor pairs could be 3, 3 and 9, 8.
4. Marie didn't have to use a tree diagram.

A. 1 only
B. 2 only
C. 3 and 4 only
D. 2 and 4 only

Answer

A. Incorrect. Marie could have started her tree diagram with the factors 2 and 36, but she does not have to start with those factors. Starting with 9 and 8 is fine. The correct answer is D.
B. Incorrect. Yes, Marie's next set of factor pairs could contain 3, 3, and 2, 4, but statement 4 is also correct. The correct answer is D.

C. Incorrect. Statement 4 is correct: Marie does not have to use a tree diagram, but 3 is not true. When creating a prime factorization, the factors are not combined with the previous composite. The correct answer is D.
D. Correct. Marie’s next set of factor pairs could read 3, 3, and 2, 4, as $3 \cdot 3$ is a factorization of 9 and $2 \cdot 4$ is a factorization of 8. Marie could also find the prime factorization by using successive divisions.

Summary

Finding the factors of a natural number means that you find all the possible numbers that will divide into the given number without a remainder. There are many rules of divisibility to help you to find factors more quickly. A prime number is a number that has exactly two factors. A composite number is a number that has more than two factors. The prime factorization of a number is the product of the number’s prime factors.

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2.1.4: Simplifying Fractions

Learning Objectives

- Find an equivalent fraction with a given denominator.
- Simplify a fraction to lowest terms.

Introduction

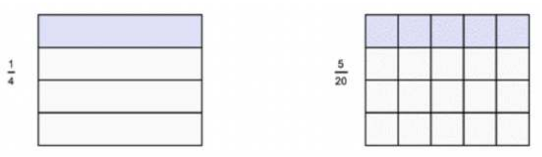
Fractions are used to represent a part of a whole. Fractions that represent the same part of a whole are called **equivalent fractions**. Factoring, multiplication, and division are all helpful tools for working with equivalent fractions.

Equivalent Fractions

We use equivalent fractions every day. Fifty cents can be 2 quarters, and we have $\frac{2}{4}$ of a dollar, because there are 4 quarters in a dollar. Fifty cents is also 50 pennies out of 100 pennies, or $\frac{50}{100}$ of a dollar. Both of these fractions are the same amount of money, but written with a different numerator and denominator.

Think about a box of crackers that contains 3 packets of crackers. Two of these packets are $\frac{2}{3}$ of the box. Suppose each packet has 30 crackers in it. Two packets are also $60(30 \cdot 2)$ crackers out of $90(30 \cdot 3)$ crackers. This is $\frac{60}{90}$ of the box. The fractions $\frac{2}{3}$ and $\frac{60}{90}$ both represent two packets of crackers, so they are equivalent fractions.

Equivalent fractions represent the same part of a whole, even if the numerator and denominator are different. For example, $\frac{1}{4} = \frac{5}{20}$. In these diagrams, both fractions represent one of four rows in the rectangle.



Since $\frac{1}{4}$ and $\frac{5}{20}$ are naming the same part of a whole, they are equivalent.

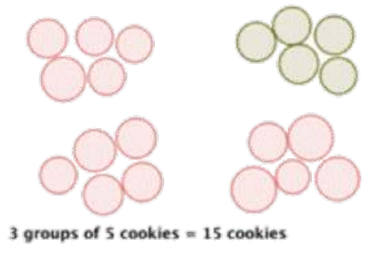
There are many ways to name the same part of a whole using equivalent fractions.

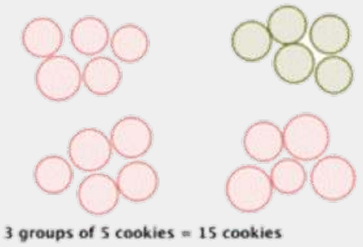
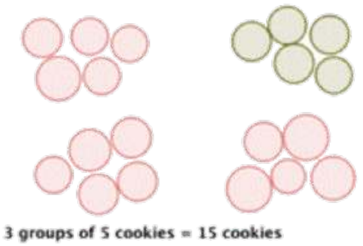
Let's look at an example where you need to find an equivalent fraction.

Example

John is making cookies for a bake sale. He made 20 large cookies, but he wants to give away only $\frac{3}{4}$ of them for the bake sale. What fraction of the cookies does he give away, using 20 as the denominator?

Solution

	Start with 20 cookies.

	Because the denominator of $\frac{3}{4}$ is 4, make 4 groups of cookies, 5 in each group.
	$\frac{3}{4}=\frac{3\cdot5}{4\cdot5}$ because there are 5 cookies in each group. $\frac{3}{4}=\frac{3\cdot5}{4\cdot5}=\frac{15}{20}$

He gives away $\frac{15}{20}$ of the cookies.

When you regroup and reconsider the parts and whole, you are multiplying the numerator and denominator by the same number. In the above example, you multiply 4 by 5 to get the needed denominator of 20, so you also need to multiply the numerator 3 by 5, giving the new numerator of 15.

Finding Equivalent Fractions

To find equivalent fractions, multiply or divide *both* the numerator and the denominator by the same number.

Examples:

$\frac{20}{25}=\frac{20\div5}{25\div5}=\frac{4}{5}$

$\frac{2}{7}=\frac{2\cdot6}{7\cdot6}=\frac{12}{42}$

Exercise

Write an equivalent fraction to $\frac{2}{3}$ that has a denominator of 27.

A. $\frac{26}{27}$
B. $\frac{11}{27}$
C. $\frac{18}{27}$
D. $\frac{12}{18}$

Answer

A. Incorrect. You may have added the difference between the two denominators to the numerator (27-3=24, 24+2=26). Instead, you need to use a multiplying factor of $9(27\div3)$. The correct answer is $\frac{18}{27}$.
B. Incorrect. You may have added the multiplying factor of 9 to the numerator. Instead, multiply the numerator by this factor. The correct answer is $\frac{18}{27}$.
C. Correct. The multiplying factor is 9, so the denominator is $3\cdot9=27$ and the numerator is $2\cdot9=18$.
D. Incorrect. Although this is an equivalent fraction to $\frac{2}{3}$, the denominator is not 27. The correct answer is $\frac{18}{27}$.

Simplifying Fractions

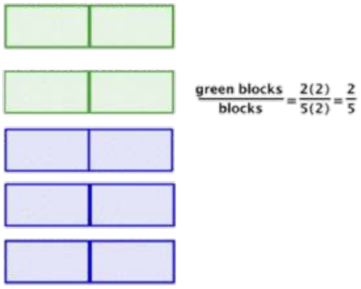
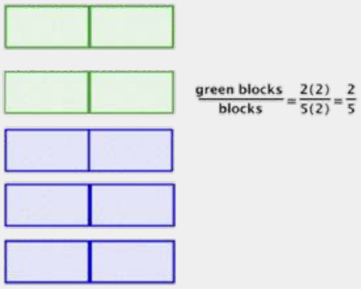
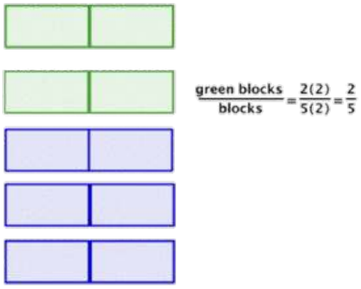
A fraction is in its **simplest form**, or **lowest terms**, when it has the least numerator and the least denominator possible for naming this part of a whole. The numerator and denominator have no common factor other than 1.

Here are 10 blocks, 4 of which are green. So, the fraction that is green is $\frac{4}{10}$. To simplify, you find a common factor and then regroup the blocks by that factor.

✓ Example

Simplify $\frac{4}{10}$.

Solution

	We start with 4 green blocks out of 10 total blocks.
	Group the blocks in twos, since 2 is a common factor. You have 2 groups of green blocks and a total of 5 groups, each group containing 2 blocks.
	Now, consider the groups as the part and you have 2 green groups out of 5 total groups.

$\frac{4}{10} = \frac{2}{5}$ The simplified fraction is $\frac{2}{5}$.

Once you have determined a common factor, you can divide the blocks into the groups by dividing both the numerator and denominator to determine the number of groups that you have.

For example, to simplify $\frac{6}{9}$ you find a common factor of 3, which will divide evenly into both 6 and 9. So, you divide 6 and 9 into groups of 3 to determine how many groups of 3 they contain. This gives $\frac{6 \div 3}{9 \div 3} = \frac{2}{3}$, which means 2 out of 3 groups, and $\frac{2}{3}$ is equivalent to $\frac{6}{9}$.

It may be necessary to group more than one time. Each time, determine a common factor for the numerator and denominator using the tests of divisibility, when possible. If both numbers are even numbers, start with 2. For example:

✓ Example

Simplify $\frac{32}{48}$.

Solution

$\frac{32}{48} = \frac{32 \div 2}{48 \div 2} = \frac{16}{24}$	32 and 48 have a common factor of 2. Divide each by 2.

$\frac{16}{24}=\frac{16\div 2}{24\div 2}=\frac{8}{12}$	16 and 24 have a common factor of 2. Divide each by 2.
$\frac{8}{12}=\frac{8\div 4}{12\div 4}=\frac{2}{3}$	8 and 12 have a common factor of 4. Divide each by 4.
$\frac{32}{48}=\frac{2}{3}$	$\frac{2}{3}$ is the simplified fraction equivalent to $\frac{32}{48}$.

In the example above, 16 is a factor of both 32 and 48, so you could have shortened the solution.

$$\frac{32}{48}=\frac{2\cdot 16}{3\cdot 16}=\frac{2}{3}$$

You can also use **prime factorization** to help regroup the numerator and denominator.

✓ Example

Simplify $\frac{54}{72}$.

Solution

$\frac{54}{72}=\frac{2\cdot 3\cdot 3\cdot 3}{2\cdot 2\cdot 2\cdot 3\cdot 3}$	The prime factorization of 54 is $2\cdot 3\cdot 3\cdot 3$. The prime factorization of 72 is $2\cdot 2\cdot 2\cdot 3\cdot 3$.
$\frac{3\cdot(2\cdot 3\cdot 3)}{2\cdot 2\cdot(2\cdot 3\cdot 3)}$	Rewrite, finding common factors.
$\frac{3}{2\cdot 2}\cdot 1$	$\frac{2\cdot 3\cdot 3}{2\cdot 3\cdot 3}=1$
$\frac{3}{4}$	Multiply: $2\cdot 2$.
$\frac{54}{72}=\frac{3}{4}$	$\frac{3}{4}$ is the simplified fraction equivalent to $\frac{54}{72}$.

Notice that when you *simplify* a fraction, you *divide* the numerator and denominator by the same number, in the same way you *multiply* by the same number to find an *equivalent* fraction with a greater denominator. In the example above, you could have divided the numerator and denominator by 9, a common factor of 54 and 72.

$$\frac{54\div 9}{72\div 9}=\frac{6}{8}$$

Since the numerator (6) and the denominator (8) still have a common factor, the fraction is not yet in lowest terms. So, again divide by the common factor 2.

$$\frac{6\div 2}{8\div 2}=\frac{3}{4}$$

Repeat this process of dividing by a common factor until the only common factor is 1.

Simplifying Fractions to Lowest Terms

To simplify a fraction to lowest terms, divide both the numerator and the denominator by their common factors. Repeat as needed until the only common factor is 1.

? Exercise

Simplify $\frac{36}{72}$.

A. $\frac{3}{6}$
B. $\frac{9}{18}$
C. $\frac{18}{38}$
D. $\frac{1}{2}$

Answer

A. Incorrect. Although $\frac{3}{6}$ is an equivalent fraction to $\frac{36}{72}$, it is not in lowest terms. There is a common factor of 3 in the numerator and denominator. The correct answer is $\frac{1}{2}$.
B. Incorrect. Although $\frac{9}{18}$ is an equivalent fraction to $\frac{36}{72}$, it is not in lowest terms. There are other common factors (9 and 3). The correct answer is $\frac{1}{2}$.

C. Incorrect. You may have divided 72 by 2 and got 38 rather than 36. $\frac{18}{36}$ is an equivalent fraction to $\frac{36}{72}$, but it is not in lowest terms. The correct answer is $\frac{1}{2}$.

D. Correct. $\frac{1}{2}$ is in lowest terms, as 1 is the only common factor of 1 and 2.

Summary

Equivalent fractions do not always have the same numerator and denominator, but they have the same value. A fraction is in lowest terms when the numerator and denominator of the fraction share no common factors other than . A fraction written in lowest terms is called a simplified fraction.

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2.1.5: Comparing Fractions

Learning Objectives

- Determine whether two fractions are equivalent.
- Use > or < to compare fractions.

Introduction

You often need to know when one fraction is greater or less than another fraction. Since a fraction is a part of a whole, to find the greater fraction you need to find the fraction that contains more of the whole. If the two fractions simplify to fractions with a **common denominator**, you can then compare numerators. If the denominators are different, you can find a common denominator first and then compare the numerators.

Determining Equivalent Fractions

Two fractions are **equivalent fractions** when they represent the same part of a whole. Since equivalent fractions do not always have the same numerator and denominator, one way to determine if two fractions are equivalent is to find a common denominator and rewrite each fraction with that denominator. Once the two fractions have the same denominator, you can check to see if the numerators are equal. If they are equal, then the two fractions are equal as well.

One way to find a common denominator is to check to see if one denominator is a factor of the other denominator. If so, the greater denominator can be used as the common denominator.

Example

Are $\frac{2}{6}$ and $\frac{8}{18}$ equivalent fractions?

Solution

Does $\frac{2}{6}=\frac{8}{18}$?	To solve this problem, find a common denominator for the two fractions. This will help you compare the two fractions. Since 6 is a factor of 18, you can write both fractions with 18 as the denominator.
$\frac{2\cdot3}{6\cdot3}=\frac{6}{18}$	Start with the fraction $\frac{2}{6}$. Multiply the denominator, 6, by 3 to get a new denominator of 18. Since you multiply the denominator by 3, you must also multiply the numerator by 3.
$\frac{8}{18}$	The fraction $\frac{8}{18}$ already has a denominator of 18, so you can leave it as is.
$\frac{6}{18}$ does not equal $\frac{8}{18}$	Compare the fractions. Now that both fractions have the same denominator, 18, you can compare numerators.

$\frac{2}{6}$ and $\frac{8}{18}$ are not equivalent fractions.

When one denominator is not a factor of the other denominator, you can find a common denominator by multiplying the denominators together.

Example

Determine whether $\frac{3}{6}$ and $\frac{5}{10}$ are equivalent fractions.

Solution

$6\cdot10=60$	Use 60 as a common denominator.
$\frac{3}{6}=\frac{3\cdot10}{6\cdot10}=\frac{30}{60}$	Multiply the numerator and denominator of $\frac{3}{6}$ by 10 to get 60 in the denominator.

$\frac{5}{10}=\frac{5.6}{10\cdot 6}=\frac{30}{60}$	Multiply the numerator and denominator of $\frac{5}{10}$ by 6.
$\frac{30}{60}=\frac{30}{60}$	Now that the denominators are the same, compare the numerators.
Yes, $\frac{3}{6}$ and $\frac{5}{10}$ are equivalent fractions.	Since 30 is the value of the numerator for both fractions, the two fractions are equal.

Notice in the above example that you can use 30 as the least common denominator since both 6 and 10 are factors of 30. Any common denominator will work.

In some cases, you can simplify one or both of the fractions, which can result in a common denominator.

✓ Example

Determine whether $\frac{2}{3}$ and $\frac{40}{60}$ are equivalent fractions.

Solution

$\frac{40}{60}=\frac{40\div 10}{60\div 10}=\frac{4}{6}$	Simplify $\frac{40}{60}$. Divide the numerator and denominator by the common factor 10.
$\frac{4}{6}=\frac{4\div 2}{6\div 2}=\frac{2}{3}$	$\frac{4}{6}$ is still not in lowest terms, so divide the numerator and the denominator again, this time by the common factor 2.
$\frac{2}{3}=\frac{2}{3}$	Compare the fractions. The numerators and denominators are the same.

Yes, $\frac{2}{3}$ and $\frac{40}{60}$ are equivalent fractions.

Note

In the example above, you could have used the common factor of 20 to simplify $\frac{40}{60}$ directly to $\frac{2}{3}$.

Determining Equivalent Fractions

To determine whether or not two fractions are equivalent:

Step 1: Rewrite one or both of the fractions so that they have common denominators.

Step 2: Compare the numerators to see if they have the same value. If so, then the fractions are equivalent.

? Exercise

Which of the following fraction pairs are equivalent?

A. $\frac{5}{7}$ and $\frac{7}{5}$
B. $\frac{12}{30}$ and $\frac{6}{10}$
C. $\frac{4}{20}$ and $\frac{1}{5}$
D. $\frac{8}{11}$ and $\frac{8}{22}$

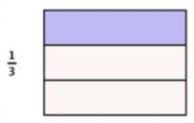
Answer

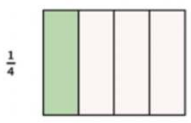
A. Incorrect. Although the same numbers, 5 and 7, are used in each fraction, the numerators and denominators are not equal, so the fractions cannot be equivalent. The correct answer is $\frac{4}{20}$ and $\frac{1}{5}$.
B. Incorrect. 30 is divisible by 10, and 12 is divisible by 6. However, they do not share a common multiple: $6\cdot 2=12$, and $10\cdot 3=30$. This means the fractions are not equivalent. The correct answer is $\frac{4}{20}$ and $\frac{1}{5}$.

C. Correct. Take the fraction $\frac{1}{5}$ and multiply both the numerator and denominator by 4. You are left with the fraction $\frac{4}{20}$. This means that the two fractions are equivalent.
D. Incorrect. The numerators of the two fractions are the same, but the denominators are different. This means the fractions are not equivalent. The correct answer is $\frac{4}{20}$ and $\frac{1}{5}$

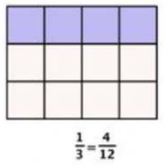
Comparing Fractions Using < and >

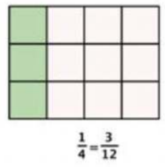
When given two or more fractions, it is often useful to know which fraction is greater than or less than the other. For example, if the discount in one store is $\frac{1}{3}$ off the original price and the discount in another store is $\frac{1}{4}$ off the original price, which store is offering a better deal? To answer this question, and others like it, you can compare fractions.



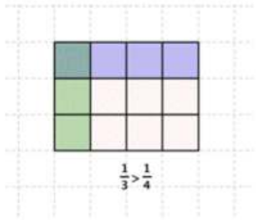


To determine which fraction is greater, you need to find a common denominator. You can then compare the fractions directly. Since 3 and 4 are both factors of 12, you will divide the whole into 12 parts, create equivalent fractions for $\frac{1}{3}$ and $\frac{1}{4}$, and then compare.





Now you see that $\frac{1}{3}$ contains 4 parts of 12, and $\frac{1}{4}$ contains 3 parts of 12. So, $\frac{1}{3}$ is greater than $\frac{1}{4}$.



As long as the denominators are the same, the fraction with the greater numerator is the greater fraction, as it contains more parts of the whole. The fraction with the lesser numerator is the lesser fraction, as it contains fewer parts of the whole.

Recall that the symbol < means “less than”, and the symbol > means “greater than”. These symbols are inequality symbols. So, the true statement 3<8 is read as "3 is less than 8" and the statement 5>3 is read as "5 is greater than 3". One way to help you remember the distinction between the two symbols is to think that the smaller end of the symbol points to the lesser number.

As with comparing whole numbers, the inequality symbols are used to show when one fraction is “greater than” or “less than” another fraction.

Comparing Fractions

To compare two fractions:

Step 1: Compare denominators. If they are different, rewrite one or both fractions with a common denominator.

Step 2: Check the numerators. If the denominators are the same, then the fraction with the greater numerator is the greater fraction. The fraction with the lesser numerator is the lesser fraction. And, as noted above, if the numerators are equal, the fractions are equivalent.

✓ Example

Use < or > to compare the two fractions $\frac{4}{5}$ and $\frac{14}{20}$.

Solution

Is $\frac{4}{5} > \frac{14}{20}$ or is $\frac{4}{5} < \frac{14}{20}$?	You cannot compare the fractions directly because they have different denominators. You need to find a common denominator for the two fractions.
$\frac{4}{5} = \frac{?}{20}$	Since 5 is a factor of 20, you can use 20 as the common denominator.
$\frac{4 \cdot 4}{5 \cdot 4} = \frac{16}{20}$	Multiply the numerator and denominator by 4 to create an equivalent fraction with a denominator of 20.
$\frac{16}{20} > \frac{14}{20}$	Compare the two fractions. $\frac{16}{20}$ is greater than $\frac{14}{20}$.
$\frac{4}{5} > \frac{14}{20}$	If $\frac{16}{20} > \frac{14}{20}$, then $\frac{4}{5} > \frac{14}{20}$, since $\frac{4}{5} = \frac{16}{20}$.

? Exercise

Which of the following is a true statement?

A. $\frac{5}{6} < \frac{24}{30}$
B. $\frac{75}{100} > \frac{9}{12}$
C. $\frac{4}{16} > \frac{1}{3}$
D. $\frac{3}{8} < \frac{20}{40}$

Answer

A. Incorrect. $\frac{5}{6}$ is equivalent to $\frac{25}{30}$, and since 25>24, $\frac{25}{30} > \frac{24}{30}$, which means $\frac{5}{6} > \frac{24}{30}$. The correct answer is $\frac{3}{8} < \frac{20}{40}$.
B. Incorrect. Both fractions simplify to $\frac{3}{4}$, so one isn't greater than or less than the other one. They are equivalent. The correct answer is $\frac{3}{8} < \frac{20}{40}$.
C. Incorrect. Finding a common denominator, you can compare $\frac{12}{48}$ to $\frac{16}{48}$, and see that $\frac{12}{48} < \frac{16}{48}$, which means $\frac{4}{16} < \frac{1}{3}$. The correct answer is $\frac{3}{8} < \frac{20}{40}$.
D. Correct. Simplifying $\frac{20}{40}$, you get the equivalent fraction $\frac{1}{2}$. Since you still don't have a common denominator, write $\frac{1}{2}$ as an equivalent fraction with a denominator of 8: $\frac{1}{2} = \frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$. You find that $\frac{3}{8} < \frac{4}{8}$, so $\frac{3}{8} < \frac{20}{40}$as well.

Summary

You can compare two fractions with like denominators by comparing their numerators. The fraction with the greater numerator is the greater fraction, as it contains more parts of the whole. The fraction with the lesser numerator is the lesser fraction as it contains fewer parts of the whole. If two fractions have the same denominator, then equal numerators indicate equivalent fractions.

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SECTION OVERVIEW

2.2: Multiplying and Dividing Fractions and Mixed Numbers

Topic hierarchy

2.2.1: Multiplying Fractions and Mixed Numbers

2.2.2: Dividing Fractions and Mixed Numbers

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2.2.1: Multiplying Fractions and Mixed Numbers

Learning Objectives

- Multiply two or more fractions.
- Multiply a fraction by a whole number.
- Multiply two or more mixed numbers.
- Solve application problems that require multiplication of fractions or mixed numbers.

Introduction

Just as you add, subtract, multiply, and divide when working with whole numbers, you also use these operations when working with fractions. There are many times when it is necessary to multiply fractions and **mixed numbers**. For example, this recipe will make 4 crumb piecrusts:

5 cups graham crackers

8 tablespoons sugar

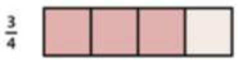
$1\frac{1}{2}$ cups melted butter

$\frac{1}{4}$ teaspoon vanilla

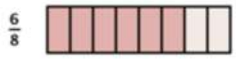
Suppose you only want to make 2 crumb piecrusts. You can multiply all the ingredients by $\frac{1}{2}$, since only half of the number of piecrusts are needed. After learning how to multiply a fraction by another fraction, a whole number or a mixed number, you should be able to calculate the ingredients needed for 2 piecrusts.

Multiplying Fractions

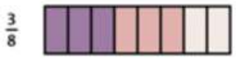
When you multiply a fraction by a fraction, you are finding a "fraction of a fraction." Suppose you have $\frac{3}{4}$ of a candy bar and you want to find $\frac{1}{2}$ of the $\frac{3}{4}$:



By dividing each fourth in half, you can divide the candy bar into eighths.



Then, choose half of those to get $\frac{3}{8}$.



In both of the above cases, to find the answer, you can multiply the numerators together and the denominators together.

Multiplying Two Fractions

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}=\frac{\text{product of the numerators}}{\text{product of the denominators}}$$

Example:

$$\frac{3}{4}\cdot\frac{1}{2}=\frac{3\cdot 1}{4\cdot 2}=\frac{3}{8}$$

Multiplying More Than Two Fractions

$$\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{e}{f}=\frac{a\cdot c\cdot e}{b\cdot d\cdot f}$$

Example:

$$\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}=\frac{1\cdot 2\cdot 3}{3\cdot 4\cdot 5}=\frac{6}{60}$$

Example

$\frac{2}{3} \cdot \frac{4}{5}$	**Multiply.**

Solution

$\frac{2 \cdot 4}{3 \cdot 5}$	Multiply the numerators and multiply the denominators.
$\frac{8}{15}$	Simplify, if possible. This fraction is already in lowest terms.

$\frac{8}{15}$

If the resulting **product** needs to be simplified to lowest terms, divide the numerator and denominator by common factors.

Example

$\frac{2}{3} \cdot \frac{1}{4}$	**Multiply. Simplify the answer.**

Solution

$\frac{2 \cdot 1}{3 \cdot 4}$	Multiply the numerators and multiply the denominators.
$\frac{2}{12}$	Simplify, if possible.
$\frac{2 \div 2}{12 \div 2}$	Simplify by dividing the numerator and denominator by the common factor 2.

$\frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$

You can also simplify the problem before multiplying, by dividing common factors.

Example

$\frac{2}{3} \cdot \frac{1}{4}$	**Multiply. Simplify the answer.**

Solution

$\frac{2 \cdot 1}{3 \cdot 4} = \frac{1 \cdot 2}{3 \cdot 4}$	Reorder the numerators so that you can see a fraction that has a common factor.
$\frac{1 \cdot 1}{3 \cdot 2}$	Simplify. $\frac{2}{4} = \frac{2 \div 2}{4 \div 2} = \frac{1}{2}$

$\frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$

You do not have to use the “simplify first” shortcut, but it could make your work easier because it keeps the numbers in the numerator and denominator smaller while you are working with them.

? Exercise

$\frac{3}{4} \cdot \frac{1}{3}$ Multiply. Simplify the answer.

A. $\frac{3}{12}$
B. $\frac{4}{7}$
C. $\frac{1}{4}$
D. $\frac{36}{144}$

Answer

A. Incorrect. $\frac{3}{12}$ is an equivalent fraction to the correct answer $\frac{1}{4}$, but it is not in lowest terms. You must divide numerator and denominator by the common factor 3. The correct answer is $\frac{1}{4}$.
B. Incorrect. You may have added numerators (3+1) and added denominators (4+3) instead of multiplying. The correct answer is $\frac{1}{4}$.
C. Correct. One way to find this answer is to multiply numerators and denominators, $\frac{3 \cdot 1}{4 \cdot 3} = \frac{3}{12}$, then simplify: $\frac{3 \div 3}{12 \div 3} = \frac{1}{4}$.
D. Incorrect. You probably found a common denominator, multiplied correctly, but then forgot to simplify. Finding a common denominator is not necessary and makes the multiplication harder because you are working with greater than necessary numbers. The correct answer is $\frac{1}{4}$.

Multiplying a Fraction by a Whole Number

When working with both fractions and whole numbers, it is useful to write the whole number as an **improper fraction** (a fraction where the numerator is greater than or equal to the denominator). All whole numbers can be written with a "1" in the denominator. For example: $2 = \frac{2}{1}$, $5 = \frac{5}{1}$, and $100 = \frac{100}{1}$. Remember that the denominator tells how many parts there are in one whole, and the numerator tells how many parts you have.

Multiplying a Fraction and a Whole Number

$a \cdot \frac{b}{c} = \frac{a}{1} \cdot \frac{b}{c}$

Example:

$4 \cdot \frac{2}{3} = \frac{4}{1} \cdot \frac{2}{3} = \frac{8}{3}$

Often when multiplying a whole number and a fraction, the resulting product will be an improper fraction. It is often desirable to write improper fractions as a mixed number for the final answer. You can simplify the fraction before or after rewriting it as a mixed number. See the examples below.

Example

$7 \cdot \frac{3}{5}$	**Multiply. Simplify the answer and write as a mixed number.**

Solution

$\frac{7}{1} \cdot \frac{3}{5}$	Rewrite 7 as the improper fraction $\frac{7}{1}$.
$\frac{7 \cdot 3}{1 \cdot 5} = \frac{21}{5}$	Multiply the numerators and multiply the denominators.
$4\frac{1}{5}$	Rewrite as a mixed number. $21 \div 5 = 4$ with a remainder of 1.

$7 \cdot \frac{3}{5} = 4\frac{1}{5}$

Example

$4 \cdot \frac{3}{4}$	**Multiply. Simplify the answer and write as a mixed number.**

Solution

$\frac{4}{1} \cdot \frac{3}{4}$	Rewrite 4 as the improper fraction $\frac{4}{1}$.
$\frac{4 \cdot 3}{1 \cdot 4}$	Multiply the numerators and multiply the denominators.
$\frac{12}{4} = 3$	Simplify.

$4 \cdot \frac{3}{4} = 3$

Exercise

$3\cdot\frac{5}{6}$ Multiply. Simplify the answer and write it as a mixed number.

A. $1\frac{1}{7}$
B. $2\frac{1}{2}$
C. $\frac{5}{2}$
D. $\frac{8}{6}$

Answer

A. Incorrect. You may have added the numerators and added the denominators to get $\frac{8}{7}$, which is the mixed number $1\frac{1}{7}$. Make sure you multiply numerators and multiply denominators. Multiplying the two numbers gives you $\frac{15}{6}$, and since $15\div 6=2\text{R}3$, the mixed number is $2\frac{3}{6}$. The fractional part simplifies to $\frac{1}{2}$. The correct answer is $2\frac{1}{2}$.
B. Correct. Multiplying the two numbers gives $\frac{15}{6}$, and since $15\div 6=2\text{R}3$, the mixed number is $2\frac{3}{6}$. The fractional part simplifies to $\frac{1}{2}$.
C. Incorrect. Multiplying the numerators and multiplying the denominators results in the improper fraction $\frac{5}{2}$, but you need to express this as a mixed number. The correct answer is $2\frac{1}{2}$.
D. Incorrect. You may have added numerators and placed it over the denominator of 6. Make sure you multiply numerators and multiply denominators. Multiplying the two numbers gives $\frac{15}{6}$, and since $15\div 6=2\text{R}3$, the mixed number is $2\frac{3}{6}$. The fractional part simplifies to $\frac{1}{2}$. The correct answer is $2\frac{1}{2}$.

Multiplying Mixed Numbers

If you want to multiply two mixed numbers, or a fraction and a mixed number, you can again rewrite any mixed number as an improper fraction.

So, to multiply two mixed numbers, rewrite each as an improper fraction and then multiply as usual. Multiply numerators and multiply denominators and simplify. And, as before, when simplifying, if the answer comes out as an improper fraction, then convert the answer to a mixed number.

Example

$2\frac{1}{5}\cdot 4\frac{1}{2}$	**Multiply. Simplify the answer and write as a mixed number.**

Solution

$2\frac{1}{5}=\frac{11}{5}$	Change 2 $\frac{1}{5}$ to an improper fraction. $5\cdot 2+1=11$, and the denominator is 5.
$4\frac{1}{2}=\frac{9}{2}$	Change $4\frac{1}{2}$ to an improper fraction. $2\cdot 4+1=9$, and the denominator is 2.
$\frac{11}{5}\cdot\frac{9}{2}$	Rewrite the multiplication problem, using the improper fractions.
$\frac{11\cdot 9}{5\cdot 2}=\frac{99}{10}$	Multiply numerators and multiply denominators.
$\frac{99}{10}=9\frac{9}{10}$	Write as a mixed number. $99\div 10=9$ with a remainder of 9.

$2\frac{1}{5}\cdot 4\frac{1}{2}=9\frac{9}{10}$

Example

$\frac{1}{2}\cdot 3\frac{1}{3}$	**Multiply. Simplify the answer and write as a mixed number.**

Solution

$3\frac{1}{3}=\frac{10}{3}$	Change $3\frac{1}{3}$ to an improper fraction. $3\cdot 3+1=10$, and the denominator is 3.
$\frac{1}{2}\cdot\frac{10}{3}$	Rewrite the multiplication problem, using the improper fraction in place of the mixed number.
$\frac{1\cdot 10}{2\cdot 3}=\frac{10}{6}$	Multiply numerators and multiply denominators.
$\frac{10}{6}=1\frac{4}{6}$	Rewrite as a mixed number. $10\div 6=1$ with a remainder of 4.
$1\frac{2}{3}$	Simplify the fractional part to lowest terms by dividing the numerator and denominator by the common factor 2.

$\frac{1}{2}\cdot 3\frac{1}{3}=1\frac{2}{3}$

As you saw earlier, sometimes it's helpful to look for common factors in the numerator and denominator before you simplify the products.

✓ Example

$1\frac{3}{5}\cdot 2\frac{1}{4}$	**Multiply. Simplify the answer and write as a mixed number.**

Solution

$1\frac{3}{5}=\frac{8}{5}$	Change $1\frac{3}{5}$ to an improper fraction. $5\cdot 1+3=8$, and the denominator is 5.
$2\frac{1}{4}=\frac{9}{4}$	Change $2\frac{1}{4}$ to an improper fraction. $4\cdot 2+1=9$, and the denominator is 4.
$\frac{8}{5}\cdot\frac{9}{4}$	Rewrite the multiplication problem using the improper fractions.
$\frac{8\cdot 9}{5\cdot 4}=\frac{9\cdot 8}{5\cdot 4}$	Reorder the numerators so that you can see a fraction that has a common factor.
$\frac{9\cdot 8}{5\cdot 4}=\frac{9\cdot 2}{5\cdot 1}$	Simplify. $\frac{8}{4}=\frac{8\div 4}{4\div 4}=\frac{2}{1}$
$\frac{18}{5}$	Multiply.
$\frac{18}{5}=3\frac{3}{5}$	Write as a mixed fraction.

$1\frac{3}{5}\cdot 2\frac{1}{4}=3\frac{3}{5}$

In the last example, the same answer would be found if you multiplied numerators and multiplied denominators without removing the common factor. However, you would get $\frac{72}{20}$, and then you would need to simplify more to get your final answer.

? Exercise

$1\frac{3}{5}\cdot 3\frac{1}{3}$

A. $\frac{80}{15}$
B. $5\frac{5}{15}$
C. $4\frac{14}{15}$
D. $5\frac{1}{3}$

Answer

A. Incorrect. You probably wrote both mixed numbers as improper fractions correctly. You probably also correctly multiplied numerators and denominators. However, this improper fraction still needs to be rewritten as a mixed number and simplified. Dividing $80\div 15=5$ with a remainder of 5 or $5\frac{5}{15}$, then simplifying the fractional part, the correct answer is $5\frac{1}{3}$.

B. Incorrect. You probably wrote both mixed numbers as improper fractions correctly. You probably also correctly multiplied numerators and denominators, and wrote the answer as a mixed number. However, the mixed number is not in lowest terms. $\frac{5}{15}$ can be simplified to $\frac{1}{3}$ by dividing numerator and denominator by the common factor 5. The correct answer is $5\frac{1}{3}$.

C. Incorrect. This is the result of adding the two numbers. To multiply, rewrite each mixed number as an improper fraction: $1\frac{3}{5}=\frac{8}{5}$ and $3\frac{1}{3}=\frac{10}{3}$. Next, multiply numerators and multiply denominators: $\frac{8}{5}\cdot\frac{10}{3}=\frac{80}{15}$. Then, write the resulting improper fraction as a mixed number: $\frac{80}{15}=5\frac{5}{15}$. Finally, simplify the fractional part by dividing both numerator and denominator by the common factor, 5. The correct answer is $5\frac{1}{3}$.

D. Correct. First, rewrite each mixed number as an improper fraction: $1\frac{3}{5}=\frac{8}{5}$ and $3\frac{1}{3}=\frac{10}{3}$. Next, multiply numerators and multiply denominators: $\frac{8}{5}\cdot\frac{10}{3}=\frac{80}{15}$. Then write as a mixed fraction $\frac{80}{15}=5\frac{5}{15}$. Finally, simplify the fractional part by dividing both numerator and denominator by the common factor 5.

Solving Problems by Multiplying Fractions and Mixed Numbers

Now that you know how to multiply a fraction by another fraction, by a whole number, or by a mixed number, you can use this knowledge to solve problems that involve multiplication and fractional amounts. For example, you can now calculate the ingredients needed for the 2 crumb piecrusts.

✓ Example

5 cups graham crackers 8 tablespoons sugar $1\frac{1}{2}$ cups melted butter $\frac{1}{4}$ teaspoon vanilla	**The recipe at left makes 4 piecrusts. Find the ingredients needed to make only 2 piecrusts.**

Solution

	Since the recipe is for 4 piecrusts, you can multiply each of the ingredients by $\frac{1}{2}$ to find the measurements for just 2 piecrusts.
$5\cdot\frac{1}{2}=\frac{5}{1}\cdot\frac{1}{2}=\frac{5}{2}$ $2\frac{1}{2}$ cups of graham crackers are needed.	5 cups graham crackers: Since the result is an improper fraction, rewrite $\frac{5}{2}$ as the improper fraction $2\frac{1}{2}$.
$8\cdot\frac{1}{2}=\frac{8}{1}\cdot\frac{1}{2}=\frac{8}{2}=4$ 4 tablespoons sugar is needed.	8 tablespoons sugar: This is another example of a whole number multiplied by a fraction.
$\frac{3}{2}\cdot\frac{1}{2}=\frac{3}{4}$ $\frac{3}{4}$ cup melted butter is needed.	$1\frac{1}{2}$ cups melted butter: You need to multiply a mixed number by a fraction. So, first rewrite $1\frac{1}{2}$ as the improper fraction $\frac{3}{2}$: $2\cdot1+1$, and the denominator is 2. Then, rewrite the multiplication problem, using the improper fraction in place of the mixed number. Multiply.
$\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$ $\frac{1}{8}$ teaspoon vanilla is needed.	$\frac{1}{4}$ teaspoon vanilla: Here, you multiply a fraction by a fraction.

The ingredients needed for 2 pie crusts are:

$2\frac{1}{2}$ cups graham crackers

4 tablespoons sugar

$\frac{3}{4}$ cup melted butter

$\frac{1}{8}$ teaspoon vanilla

Often, a problem indicates that multiplication by a fraction is needed by using phrases like “half of,” “a third of,” or " $\frac{3}{4}$ of."

✓ Example

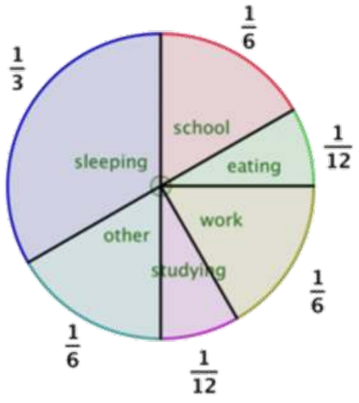
The cost of a vacation is $\$4,500$ and you are required to pay $\frac{1}{5}$ of that amount when you reserve the trip. How much will you have to pay when you reserve the trip?

Solution

$4,500 \cdot \frac{1}{5}$	You need to find $\frac{1}{5}$ of 4,500. “Of” tells you to multiply.
$\frac{4,500}{1} \cdot \frac{1}{5}$	Change 4,500 to an improper fraction by rewriting it with 1 as the denominator.
$\frac{4,500}{5}$	Divide.
900	Simplify.

You will need to pay $\$900$ when you reserve the trip.

✓ Example

	**The pie chart at left represents the fractional part of daily activities.** **Given a 24-hour day, how many hours are spent sleeping? Attending school? Eating? Use the pie chart to determine your answers.**

Solution

$\frac{1}{3} \cdot 24 = \text{number of hours sleeping}$	Sleeping is $\frac{1}{3}$ of the pie, so the number of hours spent sleeping is $\frac{1}{3}$ of 24.
$\frac{1}{3} \cdot \frac{24}{1} = 8$	Rewrite 24 as an improper fraction with a denominator of 1.
$\frac{24}{3} = 8$ 8 **hours sleeping**	Multiply numerators and multiply denominators. Simplify $\frac{24}{3}$ to 8.
$\frac{1}{6} \cdot 24 = \text{number of hours spent at school}$	Attending school is $\frac{1}{6}$ of the pie, so the number of hours spent attending school is $\frac{1}{6}$ of 24.
$\frac{1}{6} \cdot \frac{24}{1}$	Rewrite 24 as an improper fraction with a denominator of 1.
$\frac{24}{6} = 4$ 4 **hours attending school**	Multiply numerators and multiply denominators. Simplify $\frac{24}{6}$ to 4.
$\frac{1}{12} \cdot 24 = \text{number of hours spent eating}$	Eating is $\frac{1}{12}$ of the pie, so the number of hours spent eating is $\frac{1}{12}$ of 24.
$\frac{1}{12} \cdot \frac{24}{1}$	Rewrite 24 as an improper fraction with a denominator of 1.
$\frac{24}{12} = 2$ 2 **hours spent eating**	Multiply numerators and multiply denominators. Simplify $\frac{24}{12}$ to 2.

Hours spent:

sleeping: 8 hours

attending school: 4 hours

eating: 2 hours

? Exercise

Neil bought a dozen (12) eggs. He used $\frac{1}{3}$ of the eggs for breakfast. How many eggs are left?

A. 8
B. 4
C. 9
D. 3

Answer

A. Correct. $\frac{1}{3}$ of 12 is $4\left(\frac{1}{3} \cdot \frac{12}{1} = \frac{12}{3} = 4\right)$, so he used 4 of the eggs. Because $12 - 4 = 8$, there are 8 eggs left.
B. Incorrect. $\frac{1}{3}$ of 12 is 4, but that gives how many eggs Neil used, not how many he had left. You need to subtract 4 from 12 to find the number of remaining eggs. The correct answer is 8.
C. Incorrect. You may have incorrectly found $\frac{1}{3}$ of 12 to be 3. $\frac{1}{3}$ of 12 is 4, and then 12-4 is 8. The correct answer is 8.
D. Incorrect. You need to find $\frac{1}{3}$ of 12, which is 4. Then subtract 4 from 12 to get 8 remaining eggs.

Summary

You multiply two fractions by multiplying the numerators and multiplying the denominators. Often the resulting product will not be in lowest terms, so you must also simplify. If one or both fractions are whole numbers or mixed numbers, first rewrite each as an improper fraction. Then multiply as usual, and simplify.

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2.2.2: Dividing Fractions and Mixed Numbers

Learning Objectives

- Find the reciprocal of a number.
- Divide two fractions.
- Divide two mixed numbers.
- Divide fractions, mixed numbers, and whole numbers.
- Solve application problems that require division of fractions or mixed numbers.

Introduction

There are times when you need to use division to solve a problem. For example, if painting one coat of paint on the walls of a room requires 3 quarts of paint and there are 6 quarts of paint, how many coats of paint can you paint on the walls? You divide 6 by 3 for an answer of 2 coats. There will also be times when you need to divide by a fraction. Suppose painting a closet with one coat only required $\frac{1}{2}$ quart of paint. How many coats could be painted with the 6 quarts of paint? To find the answer, you need to divide 6 by the fraction, $\frac{1}{2}$.

Reciprocals

If the **product** of two numbers is 1, the two numbers are **reciprocals** of each other. Here are some examples:

Original number	**Reciprocal**	**Product**
$\frac{3}{4}$	$\frac{4}{3}$	$\frac{3}{4}\cdot\frac{4}{3}=\frac{3\cdot 4}{4\cdot 3}=\frac{12}{12}=1$
$\frac{1}{2}$	$\frac{2}{1}$	$\frac{1}{2}\cdot\frac{2}{1}=\frac{1\cdot 2}{2\cdot 1}=\frac{2}{2}=1$
$3=\frac{3}{1}$	$\frac{1}{3}$	$\frac{3}{1}\cdot\frac{1}{3}=\frac{3\cdot 1}{1\cdot 3}=\frac{3}{3}=1$
$2\frac{1}{3}=\frac{7}{3}$	$\frac{3}{7}$	$\frac{7}{3}\cdot\frac{3}{7}=\frac{7\cdot 3}{3\cdot 7}=\frac{21}{21}=1$

In each case, the original number, when multiplied by its reciprocal, equals 1.

To create two numbers that multiply together to give an answer of one, the numerator of one is the denominator of the other. You sometimes say one number is the “flip” of the other number: flip $\frac{2}{5}$ to get the reciprocal $\frac{5}{2}$. In order to find the reciprocal of a mixed number, write it first as an **improper fraction** so that it can be “flipped.”

Example

Find the reciprocal of $5\frac{1}{4}$.

Solution

$5\frac{1}{4}=\frac{21}{4}$	Rewrite $5\frac{1}{4}$ as an improper fraction. The numerator is $4\cdot 5+1=21$.
$\frac{4}{21}$	Find the reciprocal by interchanging (“flipping”) the numerator and denominator.

Exercise

What is the reciprocal of $3\frac{2}{5}$?

A. $3\frac{5}{2}$
B. $\frac{17}{5}$
C. $\frac{5}{17}$
D. $\frac{5}{2}$

Answer

A. Incorrect. The fractional parts of this answer and the original mixed number are reciprocals, but in order to find the reciprocal of the entire number, you must write the mixed number as an improper fraction before interchanging numerator and denominator. The correct answer is $\frac{5}{17}$.
B. Incorrect. You found the correct improper fraction that represents $3\frac{2}{5}$ but did not find the reciprocal. The reciprocal of $\frac{17}{5}$ is $\frac{5}{17}$.
C. Correct. First, write $3\frac{2}{5}$ as an improper fraction, $\frac{17}{5}$. The reciprocal of $\frac{17}{5}$ is found by interchanging ("flipping") the numerator and denominator.
D. Incorrect. This is the correct reciprocal for the fractional part of the mixed number, but with a mixed number, you first need to write it as an improper fraction. The mixed number $3\frac{2}{5}$ is written as the improper fraction $\frac{17}{5}$. The reciprocal of $\frac{17}{5}$ is found by interchanging ("flipping") the numerator and denominator.

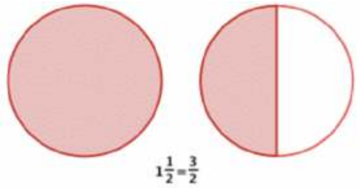
Dividing a Fraction or a Mixed Number by a Whole Number

When you divide by a whole number, you multiply by the reciprocal of the divisor. In the painting example where you need 3 quarts of paint for a coat and have 6 quarts of paint, you can find the total number of coats that can be painted by dividing 6 by 3, $6 \div 3 = 2$. You can also multiply 6 by the reciprocal of 3, which is $\frac{1}{3}$, so the multiplication problem becomes $\frac{6}{1} \cdot \frac{1}{3} = \frac{6}{3} = 2$.

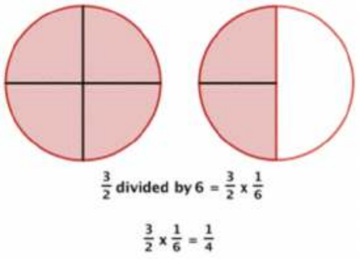
The same idea will work when the divisor is a fraction. If you have $\frac{3}{4}$ of a candy bar and need to divide it among 5 people, each person gets $\frac{1}{5}$ of the available candy: $\frac{1}{5}$ of $\frac{3}{4}$ is $\frac{1}{5} \cdot \frac{3}{4} = \frac{3}{20}$, so each person gets $\frac{3}{20}$ of a whole candy bar.

If you have a recipe that needs to be divided in half, you can divide each ingredient by 2, or you can multiply each ingredient by $\frac{1}{2}$ to find the new amount.

Similarly, with a **mixed number**, you can either divide by the whole number or you can multiply by the reciprocal. Suppose you have $1\frac{1}{2}$ pizzas that you want to divide evenly among 6 people.



Dividing by 6 is the same as multiplying by the reciprocal of, which is $\frac{1}{6}$. Cut the available pizza into six equal-sized pieces.


$\frac{3}{2}$ divided by 6 = $\frac{3}{2} \times \frac{1}{6}$
$\frac{3}{2} \times \frac{1}{6} = \frac{1}{4}$

Each person gets one piece, so each person gets $\frac{1}{4}$ of a pizza.

Dividing a fraction by a whole number is the same as multiplying by the reciprocal, so you can always use multiplication of fractions to solve such division problems.

✓ Example

Find $2\frac{2}{3} \div 4$. Write your answer as a mixed number with any fraction part in lowest terms.

Solution

$2\frac{2}{3} = \frac{8}{3}$	Rewrite $2\frac{2}{3}$ as an improper fraction. The numerator is $2 \cdot 3 + 2$. The denominator is still 3.

$\frac{8}{3} \div 4 = \frac{8}{3} \cdot \frac{1}{4}$	Dividing by 4 or $\frac{4}{1}$ is the same as multiplying by the reciprocal of 4, which is $\frac{1}{4}$.
$\frac{8 \cdot 1}{3 \cdot 4} = \frac{8}{12}$	Multiply numerators and multiply denominators.
$\frac{2}{3}$	Simplify to lowest terms by dividing numerator and denominator by the common factor 4.

$2\frac{2}{3} \div 4 = \frac{2}{3}$

? Exercise

Find $4\frac{3}{5} \div 2$ Simplify the answer and write as a mixed number.

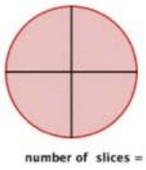
A. $2\frac{3}{10}$
B. $\frac{10}{23}$
C. $\frac{23}{10}$
D. $9\frac{1}{5}$

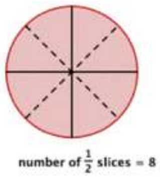
Answer

A. Correct. Write $4\frac{3}{5}$ as the improper fraction $\frac{23}{5}$. Then multiply by $\frac{1}{2}$, the reciprocal of 2. This gives the improper fraction $\frac{23}{10}$, and the mixed number is $23 \div 10 = 2\text{R}3$, which is $2\frac{3}{10}$.
B. Incorrect. After changing the mixed number to an improper fraction, you may have inverted $\frac{23}{5}$ instead of 2. Keep $\frac{23}{5}$, and multiply by the reciprocal of 2, giving you $\frac{23}{5} \cdot \frac{1}{2} = \frac{23}{10}$. Finally, write $\frac{23}{10}$ as a mixed number, $23 \div 10 = 2\text{R}3$, which is $2\frac{3}{10}$.
C. Incorrect. This is the correct improper fraction, but you still need to write the final answer as a mixed number, $23 \div 10 = 2\text{R}3$, which is $2\frac{3}{10}$.
D. Incorrect. You may have forgotten to find the reciprocal of 2 before multiplying. Once you have the improper fraction for $4\frac{3}{5}$, which is $\frac{23}{5}$, multiply by the reciprocal of 2, which is $\frac{1}{2}$, giving you $\frac{23}{5} \cdot \frac{1}{2} = \frac{23}{10}$. Finally, write $\frac{23}{10}$ as a mixed number, $23 \div 10 = 2\text{R}3$, which is $2\frac{3}{10}$.

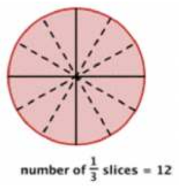
Dividing by a Fraction

Sometimes you need to solve a problem that requires dividing by a fraction. Suppose you have a pizza that is already cut into 4 slices. How many $\frac{1}{2}$ slices are there?





There are 8 slices. You can see that dividing 4 by $\frac{1}{2}$ gives the same result as multiplying 4 by 2. What would happen if you needed to divide each slice into thirds?



You would have 12 slices, which is the same as multiplying 4 by 3.

Dividing with Fractions

Step 1: Find the reciprocal of the number that follows the division symbol.

Step 2: Multiply the first number (the one **before** the division symbol) by the reciprocal of the second number (the one **after** the division symbol).

Examples:

$6 \div \frac{2}{3} = 6 \cdot \frac{3}{2}$ and $\frac{2}{5} \div \frac{1}{3} = \frac{2}{5} \cdot \frac{3}{1}$

An easy way to remember how to divide fractions is the phrase “keep, change, flip”. This means to **KEEP** the first number, **CHANGE** the division sign to multiplication, and then **FLIP** (use the reciprocal) of the second number.

Example

$\frac{2}{3} \div \frac{1}{6}$	**Divide.**

Solution

$\frac{2}{3} \cdot \frac{6}{1}$	Multiply by the reciprocal: Keep $\frac{2}{3}$, change $\div$ to $\cdot$, and flip $\frac{1}{6}$.
$\frac{2 \cdot 6}{3 \cdot 1} = \frac{12}{3}$	Multiply numerators and multiply denominators.
$\frac{12}{3} = 4$	Simplify.

$\frac{2}{3} \div \frac{1}{6} = 4$

Example

$\frac{3}{5} \div \frac{2}{3}$	**Divide.**

Solution

$\frac{3}{5} \cdot \frac{3}{2}$	Multiply by the reciprocal: Keep $\frac{3}{5}$, change $\div$ to $\cdot$, and flip $\frac{2}{3}$.
$\frac{3 \cdot 3}{5 \cdot 2} = \frac{9}{10}$	Multiply numerators and multiply denominators.

$\frac{3}{5} \div \frac{2}{3} = \frac{9}{10}$

When solving a division problem by multiplying by the reciprocal, remember to write all whole numbers and mixed numbers as improper fractions. The final answer should be simplified and written as a mixed number.

Example

$2\frac{1}{4} \div \frac{3}{4}$	**Divide.**

Solution

$\frac{9}{4} \cdot \frac{4}{3}$	Write $2\frac{1}{4}$ as an improper fraction.
$\frac{9}{4} \cdot \frac{4}{3}$	Multiply by the reciprocal: Keep $\frac{9}{4}$, change $\div$ to $\cdot$, and flip $\frac{3}{4}$.
$\frac{9 \cdot 4}{4 \cdot 3} = \frac{36}{12}$	Multiply numerators and multiply denominators.
$\frac{36}{12} = 3$	Simplify.

$2\frac{1}{4} \div \frac{3}{4} = 3$

✓ Example

$3\frac{1}{5} \div 2\frac{1}{10}$	**Divide. Simplify the answer and write as a mixed number.**

Solution

$\frac{16}{5} \div \frac{21}{10}$	Write $3\frac{1}{5}$ and $2\frac{1}{10}$ as improper fractions.
$\frac{16}{5} \cdot \frac{10}{21}$	Multiply by the reciprocal of $\frac{21}{10}$.
$\frac{16 \cdot 10}{5 \cdot 21}$	Multiply numerators, multiply denominators.
$\frac{16 \cdot 10}{21 \cdot 5}$	Regroup.
$\frac{16 \cdot 2}{21 \cdot 1}$	Simplify: $\frac{10}{5} = \frac{2}{1}$
$\frac{16 \cdot 2}{21 \cdot 1} = \frac{32}{21}$	Multiply.
$\frac{32}{21} = 1\frac{11}{21}$ $3\frac{1}{5} \div 2\frac{1}{10} = 1\frac{11}{21}$	Rewrite as a mixed number.

$3\frac{1}{5} \div 2\frac{1}{10} = 1\frac{11}{21}$

? Exercise

Find $5\frac{1}{3} \div \frac{2}{3}$. Simplify the answer and write as a mixed number.

A. $4\frac{1}{2}$
B. $3\frac{5}{9}$
C. $\frac{16}{2}$
D. 8

Answer

A. Incorrect. You may have incorrectly written $5\frac{1}{3}$ as the improper fraction $\frac{9}{3}$. To write $5\frac{1}{3}$ as an improper fraction, multiply 5 times 3 and add 1. The denominator is 3. $5\frac{1}{3} = \frac{16}{3}$. Then change division to multiplication and multiply by the reciprocal of $\frac{2}{3}$, which is $\frac{3}{2}$, giving you $\frac{16}{3} \cdot \frac{3}{2} = \frac{3}{3} \cdot \frac{16}{2} = \frac{16}{2} = 8$.

B. Incorrect. You may have forgotten to use the reciprocal of $\frac{2}{3}$. After finding the improper fraction $\frac{16}{3}$ that represents $5\frac{1}{3}$, change division to multiplication and use the reciprocal of $\frac{2}{3}$, giving you $\frac{16}{3} \cdot \frac{3}{2} = \frac{3}{3} \cdot \frac{16}{2} = \frac{16}{2} = 8$.

C. Incorrect. This is the correct improper fraction, but the final answer needs to be a mixed number. Divide 16 by 2, which is 8 and no remainder. There is no fractional part, so the answer is a whole number, 8.

D. Correct. Write 5 $\frac{1}{3}$ as an improper fraction, $\frac{16}{3}$. Then multiply by the reciprocal of $\frac{2}{3}$, which is $\frac{3}{2}$, giving you $\frac{16}{3} \cdot \frac{3}{2} = \frac{3}{3} \cdot \frac{16}{2} = \frac{16}{2} = 8$.

Dividing Fractions or Mixed Numbers to Solve Problems

Using multiplication by the reciprocal instead of division can be very useful to solve problems that require division and fractions.

✓ Example

A cook has $18\frac{3}{4}$ pounds of ground beef. How many quarter-pound burgers can he make?

Solution

$18\frac{3}{4} \div \frac{1}{4}$	You need to find how many quarter pounds there are $18\frac{3}{4}$, in so use division.
$\frac{75}{4} \div \frac{1}{4}$	Write $18\frac{3}{4}$ as an improper fraction.
$\frac{75}{4} \cdot \frac{4}{1}$	Multiply by the reciprocal.

$\frac{75\cdot 4}{4\cdot 1}$	Multiply numerators and multiply denominators.
$\frac{4}{4}\cdot\frac{75}{1}$	Regroup and simplify $\frac{4}{4}$, which is 1.

75 burgers

✓ Example

A child needs to take $2\frac{1}{2}$ tablespoons of medicine per day in 4 equal doses. How much medicine is in each dose?

Solution

$2\frac{1}{2}\div 4$	You need to make 4 equal doses, so you can use division.
$\frac{5}{2}\div 4$	Write $2\frac{1}{2}$ as an improper fraction.
$\frac{5}{2}\cdot\frac{1}{4}$	Multiply by the reciprocal.
$\frac{5\cdot 1}{2\cdot 4}=\frac{5}{8}$	Multiply numerators and multiply denominators. Simplify, if possible.

$\frac{5}{8}$ tablespoon in each dose.

? Exercise

How many $\frac{2}{5}$-cup salt shakers can be filled from cups of salt?

A. $4\frac{4}{5}$
B. $\frac{60}{2}$
C. 30
D. $\frac{1}{30}$

Answer

A. Incorrect. You probably forgot to find the reciprocal of $\frac{2}{5}$. Find $12\div\frac{2}{5}$. Keep the 12, change division to multiplication, and use the reciprocal ("flip") of $\frac{2}{5}$, giving you: $12\cdot\frac{5}{2}=\frac{12}{1}\cdot\frac{5}{2}=\frac{60}{2}=30$.
B. Incorrect. You still need to find the mixed number that represents $\frac{60}{2}$. In this case, 60 divided by 2 is $30\text{R}0$, so the answer is a whole number, 30.
C. Correct. $12\div\frac{2}{5}$ will show how many salt shakers can be filled. Write 12 as $\frac{12}{1}$ and multiply by the reciprocal ("flip") of $\frac{2}{5}$, giving you $\frac{12}{1}\cdot\frac{5}{2}=\frac{60}{2}=30$.
D. Incorrect. You incorrectly used the expression $\frac{2}{5}\div 12$. This will show how many groups of 12 there are in $\frac{2}{5}$. You need to find how many groups of $\frac{2}{5}$ are in 12, which is $12\div\frac{2}{5}$. Then write 12 as $\frac{12}{1}$ and multiply by the reciprocal ("flip") of $\frac{2}{5}$, giving you $12\cdot\frac{5}{2}=\frac{12}{1}\cdot\frac{5}{2}=\frac{60}{2}=30$.

Summary

Division is the same as multiplying by the reciprocal. When working with fractions, this is the easiest way to divide. Whether you divide by a number or multiply by the reciprocal of the number, the result will be the same. You can use these techniques to help you solve problems that involve division, fractions, and/or mixed numbers.

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SECTION OVERVIEW

2.3: Adding and Subtracting Fractions and Mixed Numbers

Topic hierarchy

2.3.1: Adding Fractions and Mixed Numbers

2.3.2: Subtracting Fractions and Mixed Numbers

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2.3.1: Adding Fractions and Mixed Numbers

Learning Objectives

- Add fractions with like denominators.
- Find the least common multiple (LCM) of two or more numbers.
- Find the common denominator of fractions with unlike denominators.
- Add fractions with unlike denominators.
- Add mixed numbers with like and unlike denominators.
- Solve application problems that require the addition of fractions or mixed numbers.

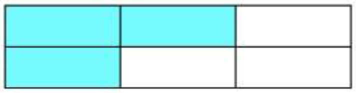
Introduction

Fractions are used in many areas of everyday life: recipes, woodworking, rainfall, timecards, and measurements, to name just a few. Sometimes you have parts of wholes that you need to combine. Just as you can add whole numbers, you can add fractions and mixed numbers. Consider, for example, how to determine the monthly rainfall if you know the daily rainfall in inches. You have to add fractions. Also, consider several painters who are working to paint a house together with multiple cans of paint. They might add the fractions of what remains in each can to determine if there is enough paint to finish the job or if they need to buy more.

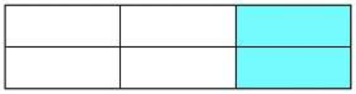
Adding Fractions with Like Denominators

When the pieces are the same size, they can easily be added. Consider the pictures below showing the fractions $\frac{3}{6}$ and $\frac{2}{6}$.

This picture represents $\frac{3}{6}$ shaded because 3 out of 6 blocks are shaded.

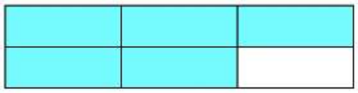


This picture represents $\frac{2}{6}$ shaded because 2 out of 6 blocks are shaded.



If you add these shaded blocks together, you are adding $\frac{3}{6}+\frac{2}{6}$.

You can create a new picture showing 5 shaded blocks in a rectangle containing 6 blocks.



So, $\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$.

Without drawing rectangles and shading boxes, you can get this answer simply by adding the numerators, 3+2, and keeping the denominator, 6, the same. This procedure works for adding any fractions that have the same denominator, called **like denominators**.

Example

$\frac{3}{5}+\frac{1}{5}$	**Add.**

Solution

$\frac{3+1}{5}$	Since the denominator of each fraction is 5, these fractions have like denominators.
$\frac{4}{5}$	So, add the numerators and write the sum over the denominator, 5.

$\frac{3}{5}+\frac{1}{5}=\frac{4}{5}$

Example

$\frac{3}{8}+\frac{5}{8}$	**Add. Simplify the answer.**

Solution

$\frac{3}{8}+\frac{5}{8}=\frac{3+5}{8}=\frac{8}{8}$	The denominators are alike, so add the numerators.
$\frac{8}{8}=1$	Simplify the fraction.

$\frac{3}{8}+\frac{5}{8}=1$

Example

$\frac{11}{12}+\frac{5}{12}$	**Add. Simplify the answer and write as a mixed number.**

Solution

$\frac{11}{12}+\frac{5}{12}=\frac{11+5}{12}=\frac{16}{12}$	The denominators are alike, so add the numerators.
$\frac{16}{12}=\frac{16\div4}{12\div4}=\frac{4}{3}$	Simplify the fraction. 16 and 12 have a common factor of 4.
$\frac{4}{3}=1\frac{1}{3}$	Write the improper fraction as a mixed number, by dividing: $4\div3=1$ with a remainder of 1.

$\frac{11}{12}+\frac{5}{12}=1\frac{1}{3}$

In the previous example, the fraction was simplified and then converted to a mixed number. You could just as easily have first converted the improper fraction to a mixed number and then simplified the fraction in the mixed number. Notice that the same answer is reached with both methods.

$\frac{16}{12}=1\frac{4}{12}$

The fraction $\frac{4}{12}$ can be simplified. $\frac{4}{12}=\frac{4\div4}{12\div4}=\frac{1}{3}$

But, don't forget about the 1 that is part of the mixed number! The final answer is $1\frac{1}{3}$.

Adding Fractions with Like Denominators

1. Add the numerators (the number in the top of each fraction).
2. Keep the denominator (the bottom number) the same.
3. Simplify to lowest terms.

Exercise

$\frac{7}{10}+\frac{8}{10}$ Add. Simplify the answer and write as a mixed number.

A. $\frac{15}{20}$
B. $\frac{9}{10}$
C. $\frac{3}{2}$
D. $1\frac{1}{2}$

Answer

A. Incorrect. When you add fractions with like denominators, only add the numerators, not the denominators. Keep the denominator the same. The correct answer is $1\frac{1}{2}$.
B. Incorrect. To add two fractions with the same denominator, add the two numerators together. Then simplify the fraction. The correct answer is $1\frac{1}{2}$.

C. Incorrect. The answer is to be expressed as a simplified mixed number, not as an improper fraction. The correct answer is $1\frac{1}{2}$.
D. Correct. $\frac{7+8}{10}=\frac{15}{10}=1\frac{5}{10}=1\frac{1}{2}$.

Finding Least Common Multiples

Sometimes fractions do not have the same denominator. They have **unlike denominators**. Think about the example of the house painters. If one painter has $\frac{2}{3}$ can of paint and his painting partner has $\frac{1}{2}$ can of paint, how much do they have in total? How can you add these fractions when they do not have like denominators?

The answer is that you can rewrite one or both of the fractions so that they have the same denominator. This is called finding a **common denominator**. While any common denominator will do, it is helpful to find the **least common multiple** of the two numbers in the denominator because this will save having to simplify at the end. The least common multiple is the least number that is a multiple of two or more numbers. Least common multiple is sometimes abbreviated LCM.

There are several ways to find common multiples, some of which you used when comparing fractions. To find the least common multiple (LCM), you can list the multiples of each number and determine which multiples they have in common. The least of these numbers will be the least common multiple. Consider the numbers 4 and 6. Some of their multiples are shown below. You can see that they have several common multiples, and the *least* of these is 12.

4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64
6	12	18	24	30	36	42	48	54	60	66	68				

✓ Example

Find the least common multiple of 30 and 50.

Solution

30, 60, 90, 120, 150, 180, 210, 240	List some multiples of 30.
50, 100, 150, 200, 250	List some multiples of 50.
150 is found on both lists of multiples.	Look for the least number found on both lists.

The least common multiple of 30 and 50 is 150.

The other method for finding the least common multiple is to use **prime factorization**. This is the method you need for working with rational expressions. The following shows how the factor method works with the numeric example, 4 and 6.

Start by finding the prime factorization of each denominator:

$4=2\cdot 2$

$6=3\cdot 2$

Identify the greatest number of times any factor appears in either factorization and multiply those factors to get the least common multiple. For 4 and 6, it would be:

$3\cdot 2\cdot 2=12$

Notice that 2 is included twice, because it appears twice in the prime factorization of 4. 12 is the least common multiple of 4 and 6.

The next example also shows how to use prime factorization.

✓ Example

Find the least common multiple of 28 and 40.

Solution

$28=2\cdot 2\cdot 7$	Write the prime factorization of 28.

$40 = 2 \cdot 2 \cdot 2 \cdot 5$	Write the prime factorization of 40.
$2 \cdot 2 \cdot 2 \cdot 5 \cdot 7 = 280$	Write the factors the greatest number of times they appear in either factorization and multiply.

The least common multiple of 28 and 40 is 280.

Exercise

Find the least common multiple of 12 and 80.

A. 240
B. 120
C. 960
D. 480

Answer

A. Correct. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 240$
B. Incorrect. 120 is not a multiple of 80. The correct answer is 240.
C. Incorrect. 960 is a common multiple, but it is not the least common multiple. The correct answer is 240.
D. Incorrect. 480 is a common multiple but it is not the least common multiple. The correct answer is 240.

Finding Least Common Denominators

You can use the least common multiple of two denominators as the **least common denominator** for those fractions. Then you rewrite each fraction using the same denominator.

The example below shows how to use the least common multiple as the least common denominator.

Example

Rewrite the fractions $\frac{2}{3}$ and $\frac{1}{2}$ as fractions with a least common denominator.

Solution

Multiples of 3 include 3, 6, 9, 12 Multiples of 2 include 2, 4, 6 6 is the least common denominator.	Find the least common multiple of the denominators. This is the least common denominator.
$\frac{2}{3} \cdot \frac{2}{2} = \frac{4}{6}$	Rewrite $\frac{2}{3}$ with a denominator of 6.
$\frac{1}{2} \cdot \frac{3}{3} = \frac{3}{6}$	Rewrite $\frac{1}{2}$ with a denominator of 6.

The fraction $\frac{2}{3}$ can be rewritten as $\frac{4}{6}$.

The fraction $\frac{1}{2}$ can be rewritten as $\frac{3}{6}$.

Exercise

Find the least common denominator. Then express each fraction using the least common denominator: $\frac{3}{4}$ and $\frac{1}{6}$.

A. $\frac{18}{24}, \frac{4}{24}$
B. $\frac{3}{12}, \frac{1}{12}$
C. 24
D. $\frac{9}{12}, \frac{2}{12}$

Answer

A. Incorrect. 24 is a common denominator, but not the *least* common denominator for the fractions. The correct answer is $\frac{9}{12}, \frac{2}{12}$.
B. Incorrect. 12 is the least common denominator, but these fractions are not equivalent to the original fractions. The numerators are incorrect. The correct answer is $\frac{9}{12}, \frac{2}{12}$.
C. Incorrect. This is a multiple of the denominators, but the task was to rewrite the fractions with a common denominator. The correct answer is $\frac{9}{12}, \frac{2}{12}$.
D. Correct. $\frac{3}{4} \cdot \frac{3}{3} = \frac{9}{12}, \frac{1}{6} \cdot \frac{2}{2} = \frac{2}{12}$.

Adding Fractions with Unlike Denominators

To add fractions with unlike denominators, first rewrite them with like denominators. Then, you know what to do! The steps are shown below.

Adding Fractions with Unlike Denominators

1. Find a common denominator.
2. Rewrite each fraction using the common denominator.
3. Now that the fractions have a common denominator, you can add the numerators.
4. Simplify to lowest terms, expressing improper fractions as mixed numbers.

You can always find a common denominator by multiplying the two denominators together. See the example below.

Example

$\frac{2}{3} + \frac{1}{5}$	**Add. Simplify the answer.**

Solution

$3 \cdot 5 = 15$	Since the denominators are not alike, find a common denominator by multiplying the denominators.
$\frac{2}{3} \cdot \frac{5}{5} = \frac{10}{15}$ $\frac{1}{5} \cdot \frac{3}{3} = \frac{3}{15}$	Rewrite each fraction with a denominator of 15.
$\frac{10}{15} + \frac{3}{15} = \frac{13}{15}$	Add the fractions by adding the numerators and keeping the denominator the same. Make sure the fraction cannot be simplified.

$\frac{2}{3} + \frac{1}{5} = \frac{13}{15}$

You can find a common denominator by finding the common multiples of the denominators. The least common multiple is the easiest to use.

Example

$\frac{3}{7} + \frac{2}{21}$	**Add. Simplify the answer.**

Solution

Multiples of 7 include 7, 14, 21 Multiples of 21 include **21**	Since the denominators are not alike, find the least common denominator by finding the least common multiple (LCM) of 7 and 21.
$\frac{3}{7} \cdot \frac{3}{3} = \frac{9}{21}$ $\frac{2}{21}$	Rewrite each fraction with a denominator of 21.

$\frac{9}{21}+\frac{2}{21}=\frac{11}{21}$	Add the fractions by adding the numerators and keeping the denominator the same. Make sure the fraction cannot be simplified.

$\frac{3}{7}+\frac{2}{21}=\frac{11}{21}$

You can also add more than two fractions as long as you first find a common denominator for all of them. An example of a sum of three fractions is shown below. In this example, you will use the prime factorization method to find the LCM.

✓ **Example**

$\frac{3}{4}+\frac{1}{6}+\frac{5}{8}$	**Add. Simplify the answer and write as a mixed number.**

Solution

$4=2\cdot 2$ $6=3\cdot 2$ $8=2\cdot 2\cdot 2$ LCM: $2\cdot 2\cdot 2\cdot 3=24$	Since the denominators are not alike, find the least common denominator by finding the least common multiple (LCM) of 4, 6, and 8.
$\frac{3}{4}\cdot\frac{6}{6}=\frac{18}{24}$ $\frac{1}{6}\cdot\frac{4}{4}=\frac{4}{24}$ $\frac{5}{8}\cdot\frac{3}{3}=\frac{15}{24}$	Rewrite each fraction with a denominator of 24.
$\frac{18}{24}+\frac{4}{24}+\frac{15}{24}=\frac{37}{24}$	Add the fractions by adding the numerators and keeping the denominator the same.
$\frac{37}{24}=1\frac{13}{24}$	Write the improper fraction as a mixed number and simplify the fraction.

$\frac{3}{4}+\frac{1}{6}+\frac{5}{8}=1\frac{13}{24}$

? **Exercise**

$\frac{2}{3}+\frac{4}{5}+\frac{1}{12}$ Add. Simplify the answer and write as a mixed number.

A. $1\frac{33}{60}$
B. $1\frac{11}{20}$
C. $\frac{31}{20}$
D. $\frac{7}{20}$

Answer

A. Incorrect. The fraction in the mixed number can be simplified. The correct answer is $1\frac{11}{20}$.
B. Correct. $\frac{40}{60}+\frac{48}{60}+\frac{5}{60}=\frac{93}{60}=1\frac{33}{60}=1\frac{11}{20}$.
C. Incorrect. Express the improper fraction as a mixed number. The correct answer is $1\frac{11}{20}$.
D. Incorrect. Find a common denominator; express each fraction using the common denominator; add the numerators and simplify. The correct answer is $1\frac{11}{20}$.

Adding Mixed Numbers

Just as you can add whole numbers and proper fractions, you can also add mixed numbers. To add mixed numbers, add the whole numbers together and the fraction parts of the mixed numbers together and then recombine to express the value as a mixed number. The steps for adding two mixed numbers are shown in the examples below.

You can keep the whole numbers and the fractions together using a vertical method for adding mixed numbers as shown below.

Example

$2\frac{1}{8}+3\frac{3}{8}$	**Add. Simplify the answer and write as a mixed number.**

Solution

$\begin{array}{r} 2\frac{1}{8} \\ +3\frac{3}{8} \\ \hline \end{array}$	Arrange the mixed numbers vertically so the whole numbers align and the fractions align.
$\begin{array}{r} 2\frac{1}{8} \\ +3\frac{3}{8} \\ \hline 5\frac{4}{8} \end{array}$	Add whole numbers. Add fractions.
$5\frac{4}{8}=5\frac{1}{2}$	Simplify the fraction.

$2\frac{1}{8}+3\frac{3}{8}=5\frac{1}{2}$

When adding mixed numbers, you may also need to find a common denominator first. Consider the example below.

Example

$8\frac{5}{6}+7\frac{4}{9}$	**Add. Simplify the answer and write as a mixed number.**

Solution

Multiples of 6 include 6, 12, 18 Multiples of 9 include 9, 18	Find a least common denominator for the fractions.
$\frac{5}{6}\cdot\frac{3}{3}=\frac{15}{18}$ $\frac{4}{9}\cdot\frac{2}{2}=\frac{8}{18}$	Express each fraction with a denominator of 18.
$\begin{array}{r} 8\frac{15}{18} \\ +7\frac{8}{18} \\ \hline \end{array}$	Arrange the mixed numbers vertically so the whole numbers align and the fractions align.
$\begin{array}{r} 8\frac{15}{18} \\ +7\frac{8}{18} \\ \hline 15\frac{23}{18} \end{array}$	Add whole numbers. Add fractions.
$\frac{23}{18}=1\frac{5}{18}$	Write the improper fraction as a mixed number.
$15+1+\frac{5}{18}$ $16\frac{5}{18}$	Combine whole numbers and fraction to write a mixed number.

$8\frac{5}{6}+7\frac{4}{9}=16\frac{5}{18}$

Exercise

$3\frac{3}{5}+1\frac{4}{9}$ Add. Simplify the answer and write as a mixed number.

A. $\frac{2}{45}$
B. $5\frac{2}{45}$
C. $4\frac{1}{2}$
D. $4\frac{47}{45}$

Answer

A. Incorrect. The whole numbers must be added also. The correct answer is $5\frac{2}{45}$.
B. Correct. $3+1+\frac{3}{5}+\frac{4}{9}$, $4+\frac{27}{45}+\frac{20}{45}=4+\frac{47}{45}=4+1\frac{2}{45}=5\frac{2}{45}$.
C. Incorrect. To add fractions with unlike denominators, you must first find a common denominator. The correct answer is $5\frac{2}{45}$.

D. Incorrect. A simplified mixed number is a whole number and a proper fraction. The correct answer is $5\frac{2}{45}$.

Adding Fractions to Solve Problems

Knowing how to add fractions is useful in a variety of situations. When reading problems, look for phrases that help you know you want to add the fractions.

✓ Example

A stack of pamphlets is placed on top of a book. If the stack of pamphlets is $3\frac{1}{4}$ inches thick and the book is $5\frac{3}{4}$ inches thick, how high is the pile?

Solution

$3\frac{1}{4}+5\frac{3}{4}$	Find the total height of the pile by adding the thicknesses of the stack of pamphlets and the book.
$3+5+\frac{1}{4}+\frac{3}{4}$	Group the whole numbers and fractions to make adding easier.
$8+\frac{1}{4}+\frac{3}{4}$	Add whole numbers.
$\frac{1}{4}+\frac{3}{4}=\frac{4}{4}=1$	Add fractions.
$8+1=9$	Combine whole number and fraction.

The pile is 9 inches high.

✓ Example

A cake recipe requires $2\frac{1}{4}$ cups of milk and $1\frac{1}{2}$ cups of melted butter. If these are the only liquids, how much liquid is in the recipe?

Solution

$2\frac{1}{4}+1\frac{1}{2}$	Find the total amount of liquid by adding the quantities.
$2+1+\frac{1}{4}+\frac{1}{2}$	Group the whole numbers and fractions to make adding easier.
$3+\frac{1}{4}+\frac{1}{2}$	Add whole numbers.
$3+\frac{1}{4}+\frac{2}{4}=3+\frac{3}{4}$	Add fractions. Recall that $\frac{1}{2}=\frac{2}{4}$.
$3\frac{3}{4}$	Combine whole number and fraction.

There are $3\frac{3}{4}$ cups of liquid in the recipe.

? Exercise

What is the total rainfall in a three-day period if it rains $3\frac{1}{4}$ inches the first day, $\frac{3}{8}$ inch the second day, and $2\frac{1}{2}$ inches on the third day?

A. 6 inches
B. $6\frac{1}{8}$ inches
C. $5\frac{1}{8}$ inches
D. $5\frac{9}{8}$ inches

Answer

A. Incorrect. The answer is not a whole number because the fractions have a sum of $\frac{9}{8}$. The correct answer is $6\frac{1}{8}$ inches.
B. Correct. $5\frac{9}{8}=6\frac{1}{8}$ inches.

C. Incorrect. When you add the fractions you get $5+\frac{9}{8}$. The $5+\frac{9}{8}$ still contains an improper fraction. The $\frac{9}{8}$ needs to be changed to $1\frac{1}{8}$ and then recombined with the 5. You need to combine the 1 from the $1\frac{1}{8}$ with the 5 to get $6\frac{1}{8}$, not $5\frac{1}{8}$. The correct answer is $6\frac{1}{8}$.

D. Incorrect. $5\frac{9}{8}$ is not the correct mixed number; the fractional part of the number is still improper. The correct answer is $6\frac{1}{8}$.

Summary

Adding fractions with like denominators involves adding the numerators and keeping the denominator the same. Always simplify the answer. To add fractions with unlike denominators, first find a common denominator. The least common denominator is easiest to use. The least common multiple can be used as the least common denominator. Adding mixed numbers involves adding the fractional parts, adding the whole numbers, and then recombining them as a mixed number.

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2.3.2: Subtracting Fractions and Mixed Numbers

Learning Objectives

- Subtract fractions with like and unlike denominators.
- Subtract mixed numbers without regrouping.
- Subtract mixed numbers with regrouping.
- Solve application problems that require the subtraction of fractions or mixed numbers.

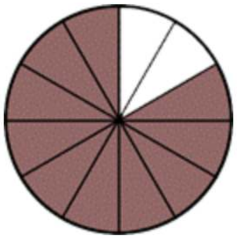
Introduction

Sometimes subtraction, rather than addition, is required to solve problems that involve fractions. Suppose you are making pancakes and need $4\frac{1}{2}$ cups of flour but you only have $2\frac{3}{4}$ cups. How many additional cups will you have to get to make the pancakes? You can solve this problem by subtracting the mixed numbers.

Subtracting Fractions

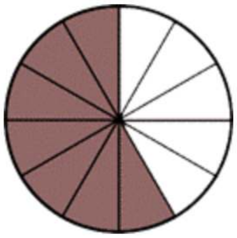
The most simple fraction subtraction problems are those that have two proper fractions with a **common denominator**. That is, each denominator is the same. The process is just as it is for addition of fractions with **like denominators**, except you subtract! You subtract the second numerator from the first and keep the denominator the same.

Imagine that you have a cake with equal-sized pieces. Some of the cake has already been eaten, so you have a fraction of the cake remaining. You could represent the cake pieces with the picture below.



The cake is cut into 12 equal pieces to start. Two are eaten, so the remaining cake can be represented with the fraction $\frac{10}{12}$. If 3 more pieces of cake are eaten, what fraction of the cake is left? You can represent that problem with the expression $\frac{10}{12}-\frac{3}{12}$.

If you subtract 3 pieces, you can see below that $\frac{7}{12}$ of the cake remains.



You can solve this problem without the picture by subtracting the numerators and keeping the denominator the same:

$\frac{10}{12}-\frac{3}{12}=\frac{7}{12}$

Subtracting Fractions with Like Denominators

If the denominators (bottoms) of the fractions are the same, subtract the numerators (tops) and keep the denominator the same. Remember to simplify the resulting fraction, if possible.

Example

$\frac{6}{7}-\frac{1}{7}$	**Subtract.**

Solution

$\frac{6-1}{7}=\frac{5}{7}$	Both fractions have a denominator of 7, so subtract the numerators and keep the same denominator.

$\frac{6}{7}-\frac{1}{7}=\frac{5}{7}$

Example

$\frac{5}{9}-\frac{2}{9}$	**Subtract. Simplify the answer.**

Solution

$\frac{5}{9}-\frac{2}{9}=\frac{3}{9}$	The fractions have a **like denominator**, also known as a common denominator, so subtract the numerators.
$\frac{3\div 3}{9\div 3}=\frac{1}{3}$	Simplify the fraction.

$\frac{5}{9}-\frac{2}{9}=\frac{1}{3}$

If the denominators are not the same (they have **unlike denominators**), you must first rewrite the fractions with a common denominator. The **least common denominator**, which is the least common multiple of the denominators, is the most efficient choice, but any common denominator will do. Be sure to check your answer to be sure that it is in simplest form. You can use prime factorization to find the **least common multiple** (LCM), which will be the least common denominator (LCD). See the example below.

Example

$\frac{1}{5}-\frac{1}{6}$	**Subtract. Simplify the answer.**

Solution

$5\cdot 6=30$	The fractions have unlike denominators, so you need to find a common denominator. Recall that a common denominator can be found by multiplying the two denominators together.
$\frac{1}{5}\cdot\frac{6}{6}=\frac{6}{30}$ $\frac{1}{6}\cdot\frac{5}{5}=\frac{5}{30}$	Rewrite each fraction as an equivalent fraction with a denominator of 30.
$\frac{6}{30}-\frac{5}{30}=\frac{1}{30}$	Subtract the numerators. Simplify the answer if needed.

$\frac{1}{5}-\frac{1}{6}=\frac{1}{30}$

The example below shows using multiples to find the least common multiple, which will be the least common denominator.

Example

$\frac{5}{6}-\frac{1}{4}$	**Subtract. Simplify the answer.**

Solution

Multiples of 6 include 6, **12**, 18, 24 Multiples of 4 include 4, 8, **12**, 16, 20	Find the least common multiple of the denominators.
12 is the least common multiple of 6 and 4.	This is the least common denominator.

$\frac{5}{6}\cdot\frac{2}{2}=\frac{10}{12}$ $\frac{1}{4}\cdot\frac{3}{3}=\frac{3}{12}$	Rewrite each fraction with a denominator of 12.
$\frac{10}{12}-\frac{3}{12}=\frac{7}{12}$	Subtract the fractions. Simplify the answer if needed.

$\frac{5}{6}-\frac{1}{4}=\frac{7}{12}$

? Exercise

$\frac{2}{3}-\frac{1}{6}$ Subtract and simplify the answer.

A. $\frac{1}{3}$
B. $\frac{3}{6}$
C. $\frac{5}{6}$
D. $\frac{1}{2}$

Answer

A. Incorrect. Find a least common denominator and subtract; then simplify. The correct answer is $\frac{1}{2}$.
B. Incorrect. Simplify the fraction. The correct answer is $\frac{1}{2}$.
C. Incorrect. Subtract, don't add, the fractions. The correct answer is $\frac{1}{2}$
D. Correct. $\frac{4}{6}-\frac{1}{6}=\frac{3}{6}=\frac{1}{2}$

Subtracting Mixed Numbers

Subtracting mixed numbers works much the same way as adding mixed numbers. To subtract mixed numbers, subtract the whole number parts of the mixed numbers and then subtract the fraction parts in the mixed numbers. Finally, combine the whole number answer and the fraction answer to express the answer as a mixed number.

✓ Example

$6\frac{4}{5}-3\frac{1}{5}$	**Subtract. Simplify the answer and write as a mixed number.**

Solution

$6-3=3$ $\frac{4}{5}-\frac{1}{5}=\frac{3}{5}$	Subtract the whole numbers and subtract the fractions.
$3\frac{3}{5}$	Combine the fraction and the whole number. Make sure the fraction in the mixed number is simplified.

$6\frac{4}{5}-3\frac{1}{5}=3\frac{3}{5}$

Sometimes it might be easier to express the mixed number as an improper fraction first and then solve. Consider the example below.

✓ Example

$8\frac{1}{3}-4\frac{2}{3}$	**Subtract. Simplify the answer and write as a mixed number.**

Solution

$8\frac{1}{3}=\frac{8\cdot3+1}{3}=\frac{24+1}{3}=\frac{25}{3}$ $4\frac{2}{3}=\frac{4\cdot3+2}{3}=\frac{12+2}{3}=\frac{14}{3}$	Write each mixed number as an improper fraction.
$\frac{25}{3}-\frac{14}{3}=\frac{11}{3}$	Since the fractions have a like denominator, subtract the numerators.

$\frac{11}{3}=3\frac{2}{3}$	Write the answer as a mixed number. Divide 11 by 3 to get 3 with a remainder of 2.

$8\frac{1}{3}-4\frac{2}{3}=3\frac{2}{3}$

Since addition is the inverse operation of subtraction, you can check your answer to a subtraction problem with addition. In the example above, if you add $4\frac{2}{3}$ to your answer of $3\frac{2}{3}$, you should get $8\frac{1}{3}$.

$$\begin{aligned} 4\tfrac{2}{3}+3\tfrac{2}{3} \\ 4+3+\tfrac{2}{3}+\tfrac{2}{3} \\ 7+\tfrac{4}{3} \\ 7+1\tfrac{1}{3} \\ 8\tfrac{1}{3} \end{aligned}$$

Sometimes you have to find a common denominator in order to solve a mixed number subtraction problem.

✓ Example

$7\frac{1}{2}-2\frac{1}{3}$	**Subtract. Simplify the answer and write as a mixed number.**

Solution

$2\cdot 3=6$	Recall that a common denominator can easily be found by multiplying the denominators together.
$\frac{1}{2}\cdot\frac{3}{3}=\frac{3}{6}$ $\frac{1}{3}\cdot\frac{2}{2}=\frac{2}{6}$	Rewrite each fraction using the common denominator 6.
$\frac{3}{6}-\frac{2}{6}=\frac{1}{6}$	Subtract the fractions.
$7-2=5$	Subtract the whole numbers.
$5\frac{1}{6}$	Combine the whole number and the fraction.

$7\frac{1}{2}-2\frac{1}{3}=5\frac{1}{6}$

? Exercise

$9\frac{4}{5}-4\frac{2}{3}$

Subtract. Simplify the answer and write it as a mixed number.

A. $\frac{2}{15}$
B. $5\frac{2}{15}$
C. $4\frac{7}{15}$
D. 5

Answer

A. Incorrect. Subtract the whole numbers, too. The correct answer is $5\frac{2}{15}$.
B. Correct. $9-4=5$; $\frac{4}{5}-\frac{2}{3}=\frac{12}{15}-\frac{10}{15}=\frac{2}{15}$. Combining them gives $5\frac{2}{15}$.
C. Incorrect. Subtract, don't add, the fractions. The correct answer is $5\frac{2}{15}$.
D. Incorrect. Subtract the fractions as well as the whole numbers. The correct answer is $5\frac{2}{15}$.

Subtracting Mixed Numbers with Regrouping

Sometimes when subtracting mixed numbers, the fraction part of the second mixed number is larger than the fraction part of the first number. Consider the problem: $7\frac{1}{6}-3\frac{5}{6}$. The standard procedure would be to subtract the fractions, but $\frac{1}{6}-\frac{5}{6}$ would result in a negative number. You don't want that! You can regroup one of the whole numbers from the first number, writing the first mixed number in a different way:

$$7\frac{1}{6} = 7 + \frac{1}{6} = 6 + 1 + \frac{1}{6}$$
$$6 + \frac{6}{6} + \frac{1}{6} = 6 + \frac{7}{6} = 6\frac{7}{6}$$

Now, you can write an equivalent problem to the original:

$$6\frac{7}{6} - 3\frac{5}{6}$$

Then, you just subtract like you normally subtract mixed numbers:

$$6 - 3 = 3$$

$$\frac{7}{6} - \frac{5}{6} = \frac{2}{6} = \frac{1}{3}$$

So, the answer is $3\frac{1}{3}$.

As with many fraction problems, you may need to find a common denominator. Remember that a key part of adding and subtracting fractions and mixed numbers is making sure to have a common denominator as a first step! In the example below, the original fractions do not have a like denominator. You need to find one before proceeding with the next steps.

✓ Example

$7\frac{1}{5} - 3\frac{1}{4}$	**Subtract. Simplify the answer and write as a mixed number.**

Solution

Multiples of 5 include 5, 10, 15, **20**, 25 Multiples of 4 include 4, 8, 12, 16, **20**, 24	Find a least common denominator. 20 is the least common multiple, so use it for the least common denominator.
$\frac{1}{5} \cdot \frac{4}{4} = \frac{4}{20}$ $\frac{1}{4} \cdot \frac{5}{5} = \frac{5}{20}$	Rewrite each fraction using the common denominator.
$7\frac{4}{20} - 3\frac{5}{20}$	Write the expression using the mixed numbers with the like denominator.
$7\frac{4}{20} = 6 + 1 + \frac{4}{20}$ $6 + \frac{20}{20} + \frac{4}{20}$ $6 + \frac{24}{20}$ $6\frac{24}{20}$	Since the second fraction part, $\frac{5}{20}$, is larger than the first fraction part, $\frac{4}{20}$, regroup one of the whole numbers and write it as $\frac{20}{20}$.
$7\frac{4}{20} - 3\frac{5}{20}$ $6\frac{24}{20} - 3\frac{5}{20}$	Rewrite the subtraction expression using the equivalent fractions.
$6 - 3 = 3$ $\frac{24}{20} - \frac{5}{20} = \frac{19}{20}$	Subtract the whole numbers, subtract the fractions.
$3\frac{19}{20}$	Combine the whole number and the fraction.

$$7\frac{1}{5} - 3\frac{1}{4} = 3\frac{19}{20}$$

Sometimes a mixed number is subtracted from a whole number. In this case, you can also rewrite the whole number as a mixed number in order to perform the subtraction. You use an equivalent mixed number that has the same denominator as the fraction in the other mixed number.

✓ Example

$8 - 4\frac{2}{5}$	**Subtract. Simplify the answer and write as a mixed number.**

Solution

$8 = 7 + 1$ $7 + \frac{5}{5}$ $7\frac{5}{5}$	Regroup one from the whole number and write it as $\frac{5}{5}$.
$7\frac{5}{5} - 4\frac{2}{5}$	Rewrite the subtraction expression using the equivalent fractions.
$7 - 4 = 3$ $\frac{5}{5} - \frac{2}{5} = \frac{3}{5}$	Subtract the whole numbers, subtract the fractions.
$3\frac{3}{5}$	Combine the whole number and the fraction.

$8 - 4\frac{2}{5} = 3\frac{3}{5}$

Subtracting Mixed Numbers

If the fractional part of the mixed number being subtracted is larger than the fractional part of the mixed number from which it is being subtracted, or if a mixed number is being subtracted from a whole number, follow these steps:

1. Subtract 1 from the whole number part of the mixed number being subtracted.
2. Add that 1 to the fraction part to make an improper fraction. For example: $7\frac{2}{3} = 6 + \frac{3}{3} + \frac{2}{3} = 6\frac{5}{3}$
3. Then, subtract as with any other mixed numbers.

Alternatively, you can change both numbers to improper fractions and then subtract.

? Exercise

$15 - 13\frac{1}{4}$ Subtract. Simplify the answer and write as a mixed number.

A. $2\frac{1}{4}$
B. $28\frac{1}{4}$
C. $1\frac{3}{4}$
D. $2\frac{3}{4}$

Answer

A. Incorrect. This is the answer to $15\frac{1}{4} - 13$. The fraction has to be subtracted from the 15. The correct answer is $1\frac{3}{4}$.
B. Incorrect. Subtract, don’t add, the quantities. The correct answer is $1\frac{3}{4}$.
C. Correct. $14\frac{4}{4} - 13\frac{1}{4} = 1\frac{3}{4}$
D. Incorrect. Subtract 1 from the whole number when rewriting it as a mixed number. The correct answer is $1\frac{3}{4}$.

Subtracting Fractions and Mixed Numbers to Solve Problems

Knowing how to subtract fractions and mixed numbers is useful in a variety of situations. When reading problems, look for key words that indicate that the problem can be solved using subtraction.

✓ Example

Sherry loves to quilt, and she frequently buys fabric she likes when she sees it. She had purchased 5 yards of blue print fabric and decided to use $2\frac{3}{8}$ yards of it in a quilt. How much of the blue print fabric will she have left over after making the quilt?

Solution

$5 - 2\frac{3}{8}$	Write an expression using subtraction to describe the situation.
$4\frac{8}{8} - 2\frac{3}{8}$	Rewrite the whole number as a mixed number.
$4\frac{8}{8} - 2\frac{3}{8} = 2\frac{5}{8}$	Subtract. Check that the mixed number is simplified.

Sherry has $2\frac{5}{8}$ yards of blue print fabric left over.

Example

Pilar and Farouk are training for a marathon. On a recent Sunday, they both completed a run. Farouk ran $12\frac{7}{8}$ miles and Pilar ran $14\frac{3}{4}$ miles. How many more miles did Pilar run than Farouk?

Solution

$14\frac{3}{4}-12\frac{7}{8}$	Write an expression using subtraction to describe the situation.
$14\frac{6}{8}-12\frac{7}{8}$	Rewrite the mixed numbers using the least common denominator.
$14\frac{6}{8}$ $13+1+\frac{6}{8}$ $13+\frac{8}{8}+\frac{6}{8}$ $13\frac{14}{8}$	Since the fraction part of the second mixed number is larger than the fraction part of the first mixed number, regroup one as a fraction and rewrite the first mixed number.
$13\frac{14}{8}-12\frac{7}{8}$	Write the subtraction expression in its new form.
$1\frac{7}{8}$	Subtract.

Pilar ran $1\frac{7}{8}$ miles more than Farouk.

Example

Mike and Jose are painting a room. Jose used $\frac{2}{3}$ of a can of paint and Mike used $\frac{1}{2}$ of a can of paint. How much more paint did Jose use? Write the answer as a fraction of a can.

Solution

$\frac{2}{3}-\frac{1}{2}$	Write an expression using subtraction to describe the situation.
$\frac{2\cdot 2}{3\cdot 2}=\frac{4}{6}$ $\frac{1\cdot 3}{2\cdot 3}=\frac{3}{6}$	Rewrite the fractions using a common denominator.
$\frac{4}{6}-\frac{3}{6}=\frac{1}{6}$	Subtract. Check that the fraction is simplified.

Jose used $\frac{1}{6}$ of a can more paint than Mike.

Exercise

Mariah's sunflower plant grew $18\frac{2}{3}$ inches in one week. Her tulip plant grew $3\frac{3}{4}$ inches in one week. How many more inches did the sunflower grow in a week than the tulip?

A. $22\frac{5}{12}$ inches
B. $15\frac{1}{12}$ inches
C. 15 inches
D. $14\frac{11}{12}$ inches

Answer

A. Incorrect. Subtract, don't add, the fractions. The correct answer is $14\frac{11}{12}$ inches.
B. Incorrect. Subtract $\frac{2}{3}-\frac{3}{4}$, not $\frac{3}{4}-\frac{2}{3}$. The correct answer is $14\frac{11}{12}$ inches.
C. Incorrect. Subtract the fractions as well as the whole numbers in the mixed numbers. The correct answer is $14\frac{11}{12}$ inches.
D. Correct. $17\frac{20}{12}-3\frac{9}{12}=14\frac{11}{12}$

Summary

Subtracting fractions and mixed numbers combines some of the same skills as adding whole numbers and adding fractions and mixed numbers. When subtracting fractions and mixed numbers, first find a common denominator if the denominators are not alike, rewrite each fraction using the common denominator, and then subtract the numerators. When subtracting mixed numbers, if the fraction in the second mixed number is larger than the fraction in the first mixed number, rewrite the first mixed number by regrouping one whole as a fraction. Alternatively, rewrite all fractions as improper fractions and then subtract. This process is also used when subtracting a mixed number from a whole number.

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CHAPTER OVERVIEW

3: Decimals

3.1: Introduction to Decimals
  3.1.1: Decimals and Fractions
  3.1.2: Ordering and Rounding Decimals
3.2: Decimal Operations
  3.2.1: Adding and Subtracting Decimals
  3.2.2: Multiplying and Dividing Decimals
  3.2.3: Estimation with Decimals

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SECTION OVERVIEW

3.1: Introduction to Decimals

Topic hierarchy

3.1.1: Decimals and Fractions

3.1.2: Ordering and Rounding Decimals

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3.1.1: Decimals and Fractions

Learning Objectives

- Read and write numbers in decimal notation.
- Write decimals as fractions.
- Write fractions as decimals.

Introduction

In addition to fraction notation, decimal notation is another way to write numbers between 0 and 1. Decimals can also be used to write numbers between any two whole numbers. For example, you may have to write a check for $2,003.38. Or, in measuring the length of a room, you may find that the length is between two whole numbers, such as 35.24 feet. In this topic, you will focus on reading and writing decimal numbers, and rewriting them in fraction notation.

To read or write numbers written in decimal notation, you need to know the **place value** of each digit, that is, the value of a digit based on its position within a number. With decimal numbers, the position of a numeral in relation to the decimal point determines its place value. For example, the place value of the 4 in 45.6 is in the *tens* place, while the place value of 6 in 45.6 is in the *tenths* place.

Decimal Notation

Decimal numbers are numbers whose place values are based on 10s. Whole numbers are actually decimal numbers that are greater than or equal to zero. The place-value chart can be extended to include numbers less than one, which are sometimes called **decimal fractions**. A decimal point is used to separate the whole number part of the number and the fraction part of the number.

Let’s say you are measuring the length of a driveway and find that it is 745 feet. You would say this number as seven hundred forty-five. Then, a more accurate measurement shows that it is 745.36 feet. Let’s place this number in a place-value chart.

What you want to examine now are the place values of the decimal part, which are the numbers 3 and 6 in the chart below.

Decimal Numbers	Hundreds	Tens	Ones	Decimal Point	Tenths	Hundredths
745.36	7	4	5	.	**3**	**6**

Notice how the place-value names start from the decimal point. To the left of the decimal point are the ones, tens, and hundreds places, where you put digits that represent whole numbers that are greater than or equal to zero. To the right of the decimal point are the tenths and hundredths, where you put digits that represent numbers that are fractional parts of one, numbers that are more than zero and less than one.

Again, the place value of a number depends on how far away it is from the decimal point. This is evident in the chart below, where each number has the digit "4" occupying a different place value.

Decimal Numbers	Thousands	Hundreds	Tens	Ones	Decimal Point	Tenths	Hundredths	Thousandths
0.004				0	.	0	0	4
0.04				0	.	0	4	
0.4				0	.	4		
4				4	.			
40			4	0	.			
400		4	0	0	.			
4000	4	0	0	0	.			

Imagine that as a large balloon deflates, the volume of air inside it goes from 1,000 liters, to 100 liters, to 10 liters, to 1 liter. Notice that you're dividing a place value by ten as you go to the right. You divide 100 by 10 to get to the tens place. This is because there are 10 tens in 100. Then, you divide 10 by 10 to get to the ones place, because there are 10 ones in 10.

Now, suppose the balloon continues to lose volume, going from 1 liter, to 0.1 liters, to 0.01 liters, and then to 0.001 liters. Notice that you continue to divide by 10 when moving to decimals. You divide 1 by $10\left(\frac{1}{10}\right)$ to get to the tenths place, which is basically breaking one into 10 pieces. And to get to the hundreds place, you break the tenth into ten more pieces, which results in the fraction $\frac{1}{100}$. The relationship between decimal places and fractions is captured in the table below.

Word Form	Decimal Notation	Fraction Notation
one thousand	1,000	$\frac{1,000}{1}$
one hundred	100	$\frac{100}{1}$
ten	10	$\frac{10}{1}$
one	1	$\frac{1}{1}$
one tenth	0.1	$\frac{1}{10}$
one hundredth	0.01	$\frac{1}{100}$
one thousandth	0.001	$\frac{1}{1,000}$

Consider a number with more digits. Suppose a fisherman has a net full of fish that weighs 1,357.924 kilograms. To write this number, you need to use the thousands place, which is made up of 10 hundreds. You also use the thousandths place, which is $\frac{1}{10}$ of a hundredth. In other words, there are ten thousandths in one hundredth.

Decimal Numbers	no *th* side					*th* side		
	Thousands	Hundreds	Tens	Ones	Decimal point	Tenths	Hundredths	Thousandths
1,357.924	1	3	5	7	.	**9**	**2**	**4**

As you can see, moving from the decimal point to the left is ones, tens, hundreds, thousands, etc. This is the "no *th* side," which are the numbers greater than or equal to one. Moving from the decimal point to the right is tenths, hundredths, thousandths. This is the "*th* side," which are the numbers less than 1.

1,357.924

no *th* side *th* side

> The pattern going to the right or the left from the decimal point is the same, but there are two big differences:
>
> 1. The place values to the right of the decimal point all end in "th".
> 2. There is no such thing as "one-ths." From your work with fractions, you know that 5 and $\frac{5}{1}$ are the same.

✓ Example

What is the place value of 8 in 4,279.386?

Solution

Write the number in a place-value chart. Read the value of the 8 from the chart.

Decimal Numbers	no *th* side					*th* side		
	Thousands	Hundreds	Tens	Ones	Decimal point	Tenths	Hundredths	Thousandths
4,279.386	4	2	7	9	.	3	**8**	6

In the number 4,279.386, the 8 is in the hundredths place.

? Exercise

What is the place value of the 7 in 324.2671?

A. thousands
B. thousandths
C. hundreds
D. hundredths

Answer

A. Incorrect. The digit 7 is to the right of the decimal point, which means that it is less than one and on the *th* side. The correct answer is thousandths.
B. Correct. The digit 7 is three decimal places to the right of the decimal point, which means that it is in the thousandths place.
C. Incorrect. The digit 7 is three decimal places to the right of the decimal point, which means that it is in the thousandths place.
D. Incorrect. The digit 7 is three decimal places to the right of the decimal point, which means that it is in the thousandths place.

Reading Decimals

The easiest way to read a decimal number is to read the decimal fraction part as a fraction. (Don't simplify the fraction though.) Suppose you have 0.4 grams of yogurt in a cup. You would say, "4 tenths of a gram of yogurt," as the 4 is in the tenths place.

Note that the denominator of the fraction written in fraction form is always a power of ten, and the number of zeros in the denominator is the same as the number of decimal places to the right of the decimal point. See the examples in the table below for further guidance.

Decimal Notation	Fraction Notation	Word Form
0.5	$\frac{5}{10}$	five tenths
0.34	$\frac{34}{100}$	thirty-four hundredths
0.896	$\frac{896}{1,000}$	eight hundred ninety-six thousandths

Notice that 0.5 has *one* decimal place. Its equivalent fraction, $\frac{5}{10}$, has a denominator of 10—which is 1 followed by *one* zero. In general, when you are converting decimals to fractions, the denominator is always 1, followed by the number of zeros that correspond to the number of decimal places in the original number.

Another way to determine which number to place in the denominator is to use the place value of the last digit without the "ths" part. For example, if the number is 1.458, the 8 is in the thousandths place. Take away the "ths" and you have a thousand, so the number is written as $1\frac{458}{1000}$.

✓ Example

Write 0.68 in word form.

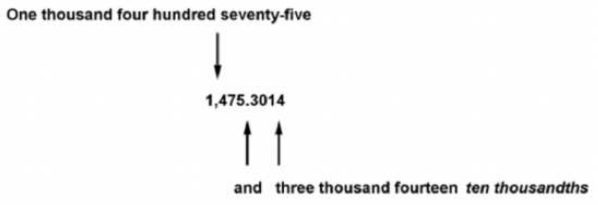
Solution

$0.68 = \frac{68}{100}$=sixty-eight hundredths	Note that the number is read as a fraction. Also note that the denominator has 2 zeros, the same as the number of decimal places in the original number.

The number 0.68 in word form is sixty-eight hundredths.

Recall that a **mixed number** is a combination of a whole number and a fraction. In the case of a decimal, a mixed number is also a combination of a whole number and a fraction, where the fraction is written as a decimal fraction.

To read mixed numbers, say the whole number part, the word “and” (representing the decimal point), and the number to the right of the decimal point, followed by the name and the place value of the last digit. You can see this demonstrated in the diagram below, in which the last digit is in the ten thousandths place.


One thousand four hundred seventy-five
1,475.3014
and
three thousand fourteen *ten thousandths*

Another way to think about this is with money. Suppose you pay $ 15,264.25 for a car. You would read this as *fifteen thousand, two hundred sixty-four dollars and twenty-five cents*. In this case, the “cents” means “hundredths of a dollar,” so this is the same as saying *fifteen thousand, two hundred sixty-four and twenty-five hundredths*. A few more examples are shown in the table below.

Decimal Notation	Fraction Notation	Word Form
9.4	$9\frac{4}{10}$	Nine and four tenths
87.49	$87\frac{49}{100}$	Eighty-seven and forty-nine hundredths
594.236	$594\frac{236}{1000}$	Five hundred ninety-four and two hundred thirty-six thousandths

✓ Example

Write 4.379 in word form.

Solution

$4.379 = 4\frac{379}{1,000}$= four and three hundred seventy-nine thousandths	The decimal fraction is read as a fraction. Note that the denominator has 3 zeros, the same as the number of decimal places in the original number.

The number 4.379 in word form is four and three hundred seventy-nine thousandths.

? Exercise

Write 2.364 in word form.

A. two and three hundred sixty-four hundredths
B. two and three hundred sixty-four thousandths
C. two thousand three hundred sixty-four
D. three hundred sixty-four tenths and two

Answer

A. two and three hundred sixty-four hundredths

Incorrect. You indicated the wrong decimal place in your answer. The correct answer is two and three hundred sixty-four thousandths.

B. Correct. 2.364 is the same as $2\frac{364}{1,000}$, so in addition to the whole number 2, you have three hundred sixty-four thousandths.

C. Incorrect. You ignored the decimal point. The correct answer is a decimal; in this case, two and three hundred sixty-four thousandths.

D. Incorrect. You indicated the wrong decimal place in your answer, and the whole number part should be mentioned before the decimal part. The correct answer is two and three hundred sixty-four thousandths.

Writing Decimals as Simplified Fractions

As you have seen above, every decimal can be written as a fraction. To convert a decimal to a fraction, place the number after the decimal point in the numerator of the fraction and place the number 10,100, or 1,000, or another power of 10 in the denominator. For example, 0.5 would be written as $\frac{5}{10}$. You'll notice that this fraction can be further simplified, as $\frac{5}{10}$ reduces to $\frac{1}{2}$, which is the final answer.

Let's get more familiar with this relationship between decimal places and zeros in the denominator by looking at several examples. Notice that in each example, the number of decimal places is different.

✓ Example

Write 0.6 as a simplified fraction.

Solution

$0.6=\frac{6}{10}$	The last decimal place is tenths, so use 10 for your denominator. The number of zeros in the denominator is always the same as the number of decimal places in the original decimal.
$\frac{6\div 2}{10\div 2}=\frac{3}{5}$	Simplify the fraction.

$0.6=\frac{3}{5}$

Let's look at an example in which a number with two decimal places is written as a fraction.

✓ Example

Write 0.64 as a simplified fraction.

Solution

$0.64=\frac{64}{100}$	The last decimal place is hundredths, so use 100 for your denominator. The number of zeros in the denominator is always the same as the number of decimal places in the original decimal.
$\frac{64\div 4}{100\div 4}=\frac{16}{25}$	Simplify the fraction.

$0.64=\frac{16}{25}$

Now, examine how this is done in the example below using a decimal with digits in three decimal places.

✓ Example

Write 0.645 as a simplified fraction.

Solution

$0.645=\frac{645}{1{,}000}$	Note that there are 3 zeros in the denominator, which is the same as the number of decimal places in the original decimal.
$\frac{645\div 5}{1{,}000\div 5}=\frac{129}{200}$	Simplify the fraction.

$0.645=\frac{129}{200}$

You can write a fraction as a decimal even when there are zeros to the right of the decimal point. Here is an example in which the only digit greater than zero is in the thousandths place.

✓ Example

Write 0.007 as a simplified fraction.

Solution

$0.007=\frac{7}{1,000}$	Note that 7 is in the thousandths place, so you write 1,000 in the denominator. The number of zeros in the denominator is always the same as the number of decimal places in the original decimal. The fraction cannot be simplified further.

$0.007=\frac{7}{1,000}$

When writing decimals greater than 1, you only need to change the decimal part to a fraction and keep the whole number part. For example, 6.35 can be written as $6\frac{35}{100}$.

✓ Example

Write 8.65 as a simplified mixed fraction.

Solution

$8.65=8\frac{65}{100}=8\frac{13}{20}$	Rewrite 0.65 as $\frac{65}{100}$. Note that the number of zeros in the denominator is two, which is the same as the number of decimal places in the original decimal. Then simplify $\frac{65}{100}$ by dividing numerator and denominator by 5.

$8.65=8\frac{13}{20}$

? Exercise

Write 0.25 as a fraction.

A. $\frac{2}{5}$
B. $\frac{1}{4}$
C. $\frac{4}{1}$
D. $\frac{5}{2}$

Answer

A. Incorrect. You may have put the digit from the tenths place in the numerator, and the digit from the hundredths place in the denominator. The correct answer is $\frac{1}{4}$.
B. Correct. The number 0.25 can be written as $\frac{25}{100}$, which reduces to $\frac{1}{4}$.
C. Incorrect. You probably confused the numerator and the denominator. The correct answer is $\frac{1}{4}$.
D. Incorrect. You may have put the digit from the tenths place in the denominator, and the digit from the hundredths place in the numerator. The correct answer is $\frac{1}{4}$.

Writing Fractions as Decimals

Just as you can write a decimal as a fraction, every fraction can be written as a decimal. To write a fraction as a decimal, divide the numerator (top) of the fraction by the denominator (bottom) of the fraction. Use long division, if necessary, and note where to place the decimal point in your answer. For example, to write $\frac{3}{5}$ as a decimal, divide 3 by 5, which will result in 0.6.

✓ Example

Write $\frac{1}{2}$ as a decimal.

Solution

0.5 2\longdiv$1.0$ -1.0 0	Using long division, you can see that dividing 1 by 2 results in 0.5.

$\frac{1}{2}=0.5$

Note that you could also have thought about the problem like this: $\frac{1}{2}=\frac{?}{10}$, and then solved for?. One way to think about this problem is that 10 is five times greater than 2, so will have to be five times greater than 1. What number is five times greater than 1? Five is, so the solution is $\frac{1}{2}=\frac{5}{10}$.

Now look at a more complex example, where the final digit of the answer is in the thousandths place.

✓ Example

Write $\frac{3}{8}$ as a decimal.

Solution

0.375 8\longdiv$3.000$ -24 60 -56 40 -40 0	Using long division, you can see that dividing 3 by 8 results in 0.375.

$\frac{3}{8}=0.375$

Converting from fractions to decimals sometimes results in answers with decimal numbers that begin to repeat. For example, $\frac{2}{3}$ converts to 0.666, a repeating decimal, in which the 6 repeats infinitely. You would write this as $0.\overline{6}$, with a bar over the first decimal digit to indicate that the 6 repeats. Look at this example of a problem in which two consecutive digits in the answer repeat.

✓ Example

Convert $\frac{4}{11}$ to a decimal.

Solution

0.3636 11\longdiv$4.0000$ -33 70 -66 40 -33 70 -66 4	Using long division, you can see that dividing 4 by 11 results in 0.36 repeating. As a result, this is written with a line over it as $0.\overline{36}$.

$\frac{4}{11}=0.\overline{36}$

With numbers greater than 1, keep the whole number part of the mixed number as the whole number in the decimal. Then use long division to convert the fraction part to a decimal. For example, $2\frac{3}{20}$ can be written as 2.15.

✓ Example

Convert $2\frac{1}{4}$ to a decimal.

Solution

$\begin{array}{r} 0.25 \\ 4\backslash\text{longdiv}1.00 \\ -8 \\ \hline 20 \\ -20 \\ \hline 0 \end{array}$	Knowing that the whole number 2 will remain the same during the conversion, focus only on the decimal part. Using long division, you can see that dividing 1 by 4 results in 0.25.
$2+0.25=2.25$	Now bring back the whole number 2, and the resulting fraction is 2.25.

$2\frac{1}{4}=2.25$

Tips on Converting Fractions to Decimals

To write a fraction as a decimal, divide the numerator (top) of the fraction by the denominator (bottom) of the fraction.

In the case of repeating decimals, write the repeating digit or digits with a line over it. For example, 0.333 repeating would be written as $0.\overline{3}$.

Summary

Decimal notation is another way to write numbers that are less than 1 or that combine whole numbers with decimal fractions, sometimes called mixed numbers. When you write numbers in decimal notation, you can use an extended place-value chart that includes positions for numbers less than one. You can write numbers written in fraction notation (fractions) in decimal notation (decimals), and you can write decimals as fractions. You can always convert between fractional notation and decimal notation.

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3.1.2: Ordering and Rounding Decimals

Learning Objectives

- Use a number line to assist with comparing decimals.
- Compare decimals, beginning with their digits from left to right.
- Use < or > to compare decimals.
- Round a given decimal to a specified place.

Introduction

Decimal numbers are a combination of whole numbers and numbers between whole numbers. It is sometimes important to be able to compare decimals to know which is greater. For example, if someone ran the 100-meter dash in 10.57 seconds, and someone else ran the same race in 10.67 seconds, you can compare the decimals to determine which time is faster. Knowing how to compare decimals requires an understanding of decimal place value, and is similar to comparing whole numbers.

When working with decimals, there are times when a precise number isn't needed. When that's true, rounding decimal numbers is helpful. For example, if the pump at the gas station shows that you filled a friend's car with 16.478 gallons of gasoline, you may want to round the number and just tell her that you filled it with 16.5 gallons.

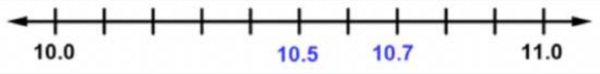
Comparing Decimals

You can use a number line to compare decimals. The number that is further to the right is greater. Examine this method in the example below.

Example

Use < or > for [] to write a true sentence: 10.5 [] 10.7.

Solution



Here, the digits in the *tenths* places differ. The numbers are both plotted on a number line ranging from 10.0 to 11.0. Because 10.5 is to the left of 10.7 on the number line, 10.5 is less than (<) 10.7.

10.5 < 10.7

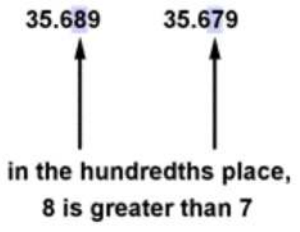
Another approach to comparing decimals is to compare the digits in each number, beginning with the greatest place value, which is on the left. When one digit in a decimal number is greater than the corresponding digit in the other number, then that decimal number is greater.

For example, first compare the tenths digits. If they are equal, move to the hundredths place. If these digits are not equal, the decimal with the greater digit is the greater decimal number. Observe how this is done in the examples below.

Example

Use < or > for [] to write a true sentence: 35.689 [] 35.679.

Solution



Here, the numbers in the tens, ones, and tenths places, 35.6, are the same. However, the digits in the hundredths places differ. Because 8 is greater than 7, 35.689 is greater than 35.679.

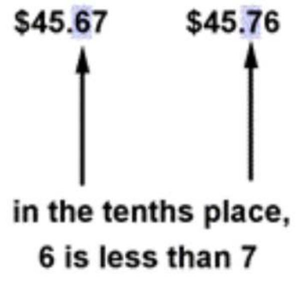
35.689 > 35.679

If more than two digits in the two numbers differ, focus on the digit in the greatest place value. Look at this example in which two sums of money are compared.

✓ Example

Use < or > for [] to write a true sentence: $ 45.67 [] $ 45.76.

Solution



Here, the place value that determines which amount is greater is not the hundredths place, but the tenths place. Because 6 tenths is less than 7 tenths, $45.67 is less than $ 45.76.

$ 45.67 < $ 45.76

If one number has more decimal places than another, you may use 0s as placeholders in the number with fewer decimal places to help you compare. For example, if you are comparing 4.75 and 4.7, you may find it helpful to write 4.7 as 4.70 so that each number has three digits. Note that adding this extra 0 does not change the value of the decimal; you are adding 0 hundredths to the number. You can add placeholder 0s as long as you remember to add the zeros at the end of the number, to the right of the decimal point.

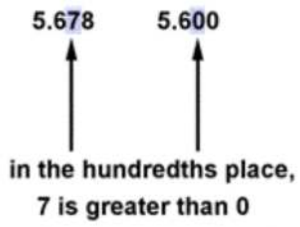
This is demonstrated in the example below.

✓ Example

Use < or > for [] to write a true sentence: 5.678 [] 5.6.

Solution

Here, you can insert two zeros in the hundredths and thousandths places of 5.6 so that you have the same number of digits for each. Then, you can compare digits. Look for the greatest place value that has different values in each number. Here, it is the hundredths place.



Because 7 is greater than 0, 5.678 is greater than 5.600.

5.678 > 5.6

Strategies for Comparing Decimals

- Use a number line to assist with comparing decimals, as you did with whole numbers.
- Compare decimals beginning with their digits from left to right. When two digits are not equal, the one with the greater digit is the larger number.

? Exercise

Use < or > for [] to write a true sentence: 45.675 [] 45.645.

A. 45.675 < 45.645
B. 45.675 > 45.645

Answer

A. Incorrect. Comparing the digits in the hundredths places shows that 7 is greater than 4. As a result, 45.675 > 45.645.
B. Correct. 45.675 > 45.645, because the 7 in 45.675 is greater than the corresponding 4 in 45.645.

Rounding Decimals

Rounding with decimals is like rounding with whole numbers. As with whole numbers, you round a number to a given place value. Everything to the right of the given place value becomes a zero, and the digit in the given place value either stays the same or increases by one.

With decimals, you can "drop off" the zeros at the end of a number without changing its value. For example, 0.20=0.2, as $\frac{20}{100}$ simplifies to $\frac{2}{10}$. Of course, you cannot drop zeros before the decimal point: $200 \neq 20$.

The zeros that occur at the end of a decimal number are called **trailing zeros**.

Dropping Zeros with Whole Numbers and Decimals

Dropping zeros at the end of a whole number changes the value of a number.	$7200 \neq 72$ $200,000 \neq 2$
Dropping zeros at the end of a decimal does not change the number's value.	$36.00 = 36$ $1.00000 = 1$

One way to think of it is to consider the number "thirty-six dollars." This amount can be written equally well one of two ways:

$\$36 = \$36.00$

Any zero at the very end of a decimal number can be dropped:

18.25000=18.2500=18.250=18.25

✓ Example

A sprinter ran a race in 7.354 seconds. What was the sprinter's time, rounded to the nearest tenth of a second?

Solution

7.354	Look at the first digit to the right of the tenths digit.
$7.354 \rightarrow 7.400$	Since 5=5, round the 3 up to 4.
7.400	Change all digits to the right of the given place value into zeros. This is an intermediate step that you don't actually write down.
7.4	Since 0.400=0.4, the zeros are not needed and should be dropped.

7.354 rounded to the nearest tenth is 7.4

In the example above, the digit next to the selected place value is 5, so you round up. Let's look at a case in which the digit next to the selected place value is less than 5.

✓ Example

Round 7.354 to the nearest hundredth.

Solution

7.354	Look at the first digit to the right of the hundredths digit.
$7.354 \rightarrow 7.350$	Since 4<5, leave 5 as is.
7.35	The zeros to the right of the given place value are not needed and should be dropped.

7.354 rounded to the nearest hundredth is 7.35.

Sometimes you're asked to round a decimal number to a place value that is in the whole number part. Remember that you may not drop zeros to the left of the decimal point.

✓ Example

Round 1,294.6374 to the nearest hundred.

Solution

$1,294.6374$	Look at the first digit to the right of the hundreds place.
$1,294.6374 \rightarrow 1,300.000$	9 is greater than 5, so round the 2 up to 3.
$1,300$	Zeros to the left of the decimal must be included. Zeros to the right of the decimal can be dropped and should be dropped.

1,294.6374 rounded to the nearest hundred is 1,300.

Tip on Rounding Decimals

When you round, everything to the right of the given place value becomes a zero, and the digit in the given place value either stays the same or rounds up one. Trailing zeros after the decimal point should be dropped.

? Exercise

Round 10.473 to the nearest tenth.

A. 10.47
B. 10.4
C. 10.473
D. 10.5

Answer

A. Incorrect. You rounded to the nearest hundredth, when you should have rounded to the nearest tenth. The correct answer is 10.5.
B. Incorrect. You rounded down when you should have rounded up. The correct answer is 10.5.
C. Incorrect. You did not round this number at all. If you had rounded to the nearest tenth, you would have found the correct answer, 10.5.
D. Correct. You rounded to the nearest tenth, and were right to round up, because the 7 in 10.473 is greater than 5.

Summary

Ordering, comparing, and rounding decimals are important skills. You can figure out the relative sizes of two decimal numbers by using number lines and by comparing the digits in the same place value of the two numbers. To round a decimal to a specific place value, change all digits to the right of the given place value to zero, and then round the digit in the given place value either up or down. Trailing zeros after the decimal point should be dropped.

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SECTION OVERVIEW

3.2: Decimal Operations

Topic hierarchy

3.2.1: Adding and Subtracting Decimals

3.2.2: Multiplying and Dividing Decimals

3.2.3: Estimation with Decimals

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3.2.1: Adding and Subtracting Decimals

Learning Objectives

- Add two or more decimals.
- Subtract two or more decimals, with and without regrouping.
- Solve application problems that require decimal addition or subtraction.

Introduction

Since dollars and cents are typically written as decimals, you often need to work with decimals. Knowing how to add and subtract **decimal numbers** is essential when you deposit money to (and withdraw money from) your bank account; perform an incorrect calculation, and you may be costing yourself some cash!

When adding or subtracting decimals, it is essential that you pay attention to the **place value** of the digits in the numbers you are adding or subtracting. This will be the key idea in the discussion that follows. Let's begin with an everyday example that illustrates this idea before moving into more general techniques.

Adding Decimals

Suppose Celia needs \$0.80 to ride the bus from home to her office. She reaches into her purse and pulls out the following coins: 3 quarters, 1 dime and 2 pennies. Does she have enough money to ride the bus?

Take a moment to think about this problem. Does she have enough money? Some people may solve it like this: "I know each quarter is 25¢, so three quarters is 75¢. Adding a dime brings me to 85¢, and then another two pennies is 87¢. So, Celia does have enough money to ride the bus."

This problem provides a good starting point for our conversation because you can use your knowledge about pocket change to understand the basics about how to add decimals. The coins you use every day can all be represented as whole cent values, as shown above. But they can also be represented as decimal numbers, too, because quarters, dimes, nickels, and pennies are each worth less than one whole dollar.

Currency	**Value (cents)**	**Value (dollars)**
Dollar bill	100¢	\$1
Quarter	25¢	\$0.25
Dime	10¢	\$0.10
Nickel	5¢	\$0.05
Penny	1¢	\$0.01

Celia has 87¢. You can also write this amount in terms of the number of dollars she has: \$0.87. The table below shows a step-by-step approach to adding the coins in terms of cents and also as dollars. As you review the table, pay attention to the place values.

Coin Combination	**Value (cents)**	**Value (dollars)**
Quarter	25¢	\$0.25
Quarter	25¢	\$0.25
Quarter	25¢	\$0.25
Dime	10¢	\$0.10
Penny	1¢	\$0.01
+Penny	1¢	\$0.01
Eighty-seven cents	87¢	\$0.87

When you add whole numbers, as shown in the **Value (cents)** column above, you line up the numbers so that the digits in the ones place-value column are aligned.

In order to keep the numbers in the proper place-value column when adding decimals, align the decimal points. This will keep the numbers aligned; ones to ones, tenths to tenths, hundredths to hundredths, and so on. Look at the column titled **Value (dollars)**. You will see that place value is maintained, and that the decimal points align from top to bottom.

Adding Decimals

To add decimals:

- Align the decimal points, which will allow all the digits to be aligned according to their place values.
- Add just as you would add whole numbers, beginning on the right and progressing to the left.
- Write the decimal point in the sum, aligned with the decimal points in the numbers being added.

Example

Add. 0.23+4.5+20.32

Solution

$\begin{array}{r} 0.23 \\ 4.5 \\ +20.32 \\ \hline \end{array}$	Write the numbers so that the decimal points are aligned.
$\begin{array}{r} 0.23 \\ 4.50 \\ +20.32 \\ \hline \end{array}$	Optional: Write an extra 0 at the end of 4.5 to keep the numbers in the correct position. (Adding this zero does not change the value of the decimal or the sum of the three numbers.)
$\begin{array}{r} 0.23 \\ 4.50 \\ +20.32 \\ \hline 25.05 \end{array}$	Add. Begin at the right and move left. Align the decimal point in the sum with the decimal points in the numbers being added.

0.23+4.5+20.32=25.05

Example

Add. 4.041+8+510.042

Solution

$\begin{array}{r} 4.041 \\ 8. \\ +510.042 \\ \hline \end{array}$	Write the numbers so that the decimal points are aligned.
$\begin{array}{r} 4.041 \\ 8.000 \\ +510.042 \\ \hline \end{array}$	As presented in the problem, the number 8 does not have a decimal point. You can rewrite this number as 8.0, 8.00, or 8.000 without changing the value of the number. Using 8.000 will allow you to align it with the other two numbers.
$\begin{array}{r} 4.041 \\ 8.000 \\ +510.042 \\ \hline 522.083 \end{array}$	Add. Begin at the right and move left. Align the decimal point in the sum with the decimal points in the numbers being added.

4.041+8+510.042=522.083

Exercise

Add: 0.08+0.156

A. 0.956
B. 0.236
C. 0.164
D. 0.1568

Answer

A. Incorrect. Pay attention to the locations of the decimal points. An answer of 0.956 would have been correct for the problem 0.8+0.156. The correct answer is 0.236.
B. Correct. Line up the decimal points and then add. The correct answer is 0.236.
C. Incorrect. Pay attention to the locations of the decimal points. An answer of 0.164 would have been correct for the problem 0.008+0.156. You can only add zeros at the end of the number. The correct answer is 0.236.
D. Incorrect. Pay attention to the locations of the decimal points. An answer of 0.1568 would have been correct for the problem 0.0008+0.156. You can only add zeros at the end of the number. The correct answer is 0.236.

Subtracting Decimals

Subtracting decimals uses the same setup as adding decimals: line up the decimal points, and then subtract.

In cases where you are subtracting two decimals that extend to different place values, it often makes sense to add extra zeros to make the two numbers line up—this makes the subtraction a bit easier to follow.

Subtracting Decimals

To subtract decimals:

- Align the decimal points, which will allow all of the digits to be aligned according to their place values.
- Subtract just as you would subtract whole numbers, beginning on the right and progressing to the left.
- Align the decimal point in the difference directly below the decimal points in the numbers that were subtracted.

Example

Subtract. 39.672-5.431

Solution

$\begin{array}{r} 39.672 \\ -\quad 5.431 \\ \hline \end{array}$	Write the numbers so that the decimal points are aligned.
$\begin{array}{r} 39.672 \\ -\quad 5.431 \\ \hline 34.241 \end{array}$	Subtract. Begin at the right and move left. Align the decimal point in the difference with the decimal points in the numbers being subtracted.

39.672-5.431=34.241

Example

Subtract. 0.9-0.027

Solution

$\begin{array}{l} \;\;0.9 \\ -0.027 \\ \hline \end{array}$	Write the numbers so that the decimal points are aligned.
$\begin{array}{r} 0.900 \\ -0.027 \\ \hline \end{array}$	Optional: Write two extra 0s after 9. This will help you line up the numbers and perform the subtraction.

$\begin{array}{r} 0.900 \\ -0.027 \\ \hline 0.873 \end{array}$	Subtract. Regroup as needed.

0.9-0.027=0.873

Exercise

Subtract. 43.21-8.1

A. 35.11
B. 42.40
C. 37.79
D. 35.2

Answer

A. Correct. Line up the two numbers so that the decimal points are aligned, and then subtract. The difference is 35.11.
B. Incorrect. Pay attention to the location of the decimal point. An answer of 42.40 would have been correct for the problem 43.21-0.81. The correct answer is 35.11.
C. Incorrect. To subtract, you need to align the decimal points first. The correct answer is 35.11.
D. Incorrect. Pay attention to the location of the decimal point. An answer of 35.2 would have been correct for the problem 43.21-8.01. The correct answer is 35.11.

Solving Problems

In adding and subtracting decimals, you may have noticed that as long as you line up the decimal points in the numbers you are adding or subtracting, you can operate upon them as you would whole numbers.

Determining whether you need to add or subtract in a given situation is also straightforward. If two quantities are being combined, then add them. If one is being withdrawn from the other, then subtract them.

Example

Javier has a balance of $1,800.50 in his personal checking account. He pays two bills out of this account: a $50.23 electric bill, and a $70.80 cell phone bill.

How much money is left in Javier's checking account after he pays these bills?

Solution

$\begin{array}{r} 1800.50 \\ -\quad 50.23 \\ \hline \end{array}$	Since Javier is paying out money, you will subtract, starting with the electric bill.
$\begin{array}{r} 1800.50 \\ -\quad 50.23 \\ \hline 1750.27 \end{array}$	Align the decimals and subtract, regrouping as needed.
$\begin{array}{r} 1750.27 \\ -\quad 70.80 \\ \hline \end{array}$	Javier has $1,750.27 remaining after he pays his electric bill. Next, subtract his cell phone bill, $70.80, from this new amount.
$\begin{array}{r} 1750.27 \\ -\quad 70.80 \\ \hline 1679.47 \end{array}$	Align the decimals and subtract, regrouping as needed.

Javier has $1,679.47 left in his checking account after paying his bills.

Exercise

Helene ran the 100-meter dash twice on Saturday. The difference between her two times was 0.3 seconds. Which pair of numbers below could have been her individual race times?

A. 14.22 and 14.25 seconds
B. 14.22 and 17.22 seconds
C. 14.22 and 14.58 seconds
D. 14.22 and 13.92 seconds

Answer

A. Incorrect. The difference between these times is 0.03 seconds, not 0.3 seconds. The correct answer is 14.22 and 13.92 seconds.
B. Incorrect. The difference between these times is 3 seconds, not 0.3 seconds. The correct answer is 14.22 and 13.92 seconds.
C. Incorrect. The difference between these times is 0.36 seconds, not 0.3 seconds. The correct answer is 14.22 and 13.92 seconds.
D. Correct. 14.22-13.92=0.3; the difference between Helene's two race times is 0.3 seconds.

Summary

When adding or subtracting decimals, you must always align the decimal points, which will allow the place-value positions to fall in place. Then add or subtract as you do with whole numbers, regrouping as necessary. You can use these operations to solve real-world problems involving decimals, especially those with money.

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3.2.2: Multiplying and Dividing Decimals

Learning Objectives

- Multiply two or more decimals.
- Multiply a decimal by a power of 10.
- Divide by a decimal.
- Divide a decimal by a power of 10.
- Solve application problems that require decimal multiplication or division.

Introduction

As with whole numbers, sometimes you run into situations where you need to multiply or divide decimals. And just as there is a correct way to multiply and divide whole numbers, so, too, there is a correct way to multiply and divide decimals.

Imagine that a couple eats dinner at a Japanese steakhouse. The bill for the meal is $58.32 —which includes a tax of $4.64. To calculate the tip, they can double the tax. So if they know how to multiply $4.64 by 2, the couple can figure out how much they should leave for the tip.

Here's another problem. Andy just sold his van that averaged 20 miles per gallon of gasoline. He bought a new pickup truck and took it on a trip of 614.25 miles. He used 31.5 gallons of gas to make it that far. Did Andy get better gas mileage with the new truck?

Both of these problems can be solved by multiplying or dividing decimals. Here's how to do it.

Multiplying Decimals

Multiplying decimals is the same as multiplying whole numbers except for the placement of the decimal point in the answer. When you multiply decimals, the decimal point is placed in the **product** so that the number of decimal places in the product is the sum of the decimal places in the **factors**.

Let's compare two multiplication problems that look similar: $214 \cdot 36$ and $21.4 \cdot 3.6$.

$$\begin{array}{r r}
214 & 21.4 \\
\times \quad 36 & \times \quad 3.6 \\
\hline
1284 & 1284 \\
6420 & 6420 \\
\hline
\mathbf{7,704} & \mathbf{77.04}
\end{array}$$

Notice how the digits in the two solutions are exactly the same: 7, 7, 0, and 4. The multiplication does not change at all. The difference lies in the placement of the decimal point in the final answers: $214 \cdot 36 = 7,704$, and $21.4 \cdot 3.6 = 77.04$.

To find out where to put the decimal point in a decimal multiplication problem, count the total number of decimal places in each of the factors.

21.4 the first factor has one decimal place

3.6 the second factor has one decimal place

77.04 the product will have 1+1=2 decimal places

Note that the decimal points do not have to be aligned as for addition and subtraction.

✓ Example

$3.04 \cdot 6.1 = ?$

Solution

$\begin{array}{r} 3.04 \\ \times \quad 6.1 \\ \hline 304 \\ 18240 \\ \hline 18544 \end{array}$	Set up the problem. Multiply 3.04 by 6.1. Add 304 and 18240.
	Count the total number of decimal places in the factors and insert the decimal point in the product.
$\begin{array}{r} 3.04 \\ \times \quad 6.1 \\ \hline 304 \\ 18240 \\ \hline 18.544 \end{array}$	← 2 decimal places in 3.04. ← 1 decimal place in 6.1. ← 3 total decimal places.

$3.04 \cdot 6.1 = 18.544$

Sometimes you may need to insert zeros in front of the product so that you have the right number of decimal places. See the final answer in the example below:

Example

0.037 \cdot 0.08=?

Solution

$\begin{array}{r} 0.037 \\ \times \quad 0.08 \\ \hline 296 \end{array}$	Set up the problem. Multiply 0.037 by 0.08. Count the total number of decimal places in the factors and insert the decimal point in the product.
$\begin{array}{r} 0.037 \\ \times \quad 0.08 \\ \hline 0.00296 \end{array}$	3 decimal places in 0.037. 2 decimal places in 0.08. 5 decimal places in the product.
$0.037 \cdot 0.08 = 0.00296$	Note that you needed to add zeros before 296 to get the 5 decimal places.

If one or more zeros occur on the right in the product, they are not dropped until after the decimal point is inserted.

Example

$2.04 \cdot 1.95 \ =?$

Solution

$\begin{array}{r} 2.04 \\ \times \quad 1.95 \\ \hline 1020 \\ 18360 \\ 20400 \\ \hline 39780 \end{array}$	Set up the problem. Multiply 2.04 by 1.95. Add 1020,18360, and 20400.
$\begin{array}{r} 2.04 \\ \times \quad 1.95 \\ \hline 1020 \\ 18360 \\ 20400 \\ \hline 3.9780 \end{array}$	← 2 decimal places in 2.04 ← 2 decimal places 1.95 ← 4 decimal places.
$2.04 \cdot 1.95 = 3.978$	Answer can omit the final trailing 0.

Multiplying Decimals

To multiply decimals:

- Set up and multiply the numbers as you do with whole numbers.
- Count the total number of decimal places in both of the factors.
- Place the decimal point in the product so that the number of decimal places in the product is the sum of the decimal places in the factors.
- Keep all zeros in the product when you place the decimal point. You can drop the zeros on the right once the decimal point has been placed in the product. If the number of decimal places is greater than the number of digits in the product, you can insert zeros in front of the product.

Exercise

Multiply. $51.2 \cdot 3.08$

A. 15769.6
B. 1576.96
C. 157.696
D. 15.7696

Answer

A. Incorrect. Pay attention to the placement of the decimal point. The correct answer is 157.696.
B. Incorrect. Pay attention to the placement of the decimal point. The correct answer is 157.696.
C. Correct. To find the product, multiply $512 \cdot 308 = 157696$. Count the total number of decimal places in the factors, 3, and then place a decimal point in the product so that the product has three decimal places as well. The answer is 157.696.
D. Incorrect. Pay attention to the placement of the decimal point. The correct answer is 157.696.

Multiplying by Tens

Take a moment to multiply 4.469 by 10. Now do $4.469 \cdot 100$. Finally, do $4.469 \cdot 1,000$. Notice any patterns in your products?

$$\begin{array}{r} 4.469 \\ \times \quad 10 \\ \hline 44.690 \end{array} \qquad \begin{array}{r} 4.469 \\ \times \quad 100 \\ \hline 446.900 \end{array} \qquad \begin{array}{r} 4.469 \\ \times \quad 1000 \\ \hline 4469.000 \end{array}$$

Notice that the products keep getting greater by one place value as the multiplier (10, 100, and 1,000) increases. In fact, the decimal point moves to the right by the same number of zeros in the power of ten multiplier.

$$4.469 \cdot 10 = 44.69 \qquad 4.469 \cdot 100 = 446.9 \qquad 4.469 \cdot 1,000 = 4469$$

You can use this observation to help you quickly multiply any decimal by a power of ten (10, 100, 1,000, etc).

Example

$0.03 \cdot 100 = ?$

Solution

$0.03 \cdot 100 = ?$	100 has two zeros.
$0.03 \cdot 100 = 3$	Move the decimal point two places to the right to find the product.

$0.03 \cdot 100 = 3$

Multiplying a Decimal by a Power of Ten

To multiply a decimal number by a power of ten (such as 10, 100, 1,000, etc.), count the number of zeros in the power of ten. Then move the decimal point that number of places to the right.

For example, $0.054 \cdot 100 = 5.4$. The multiplier 100 has two zeros, so you move the decimal point in 0.054 two places to the right—for a product of 5.4.

Dividing Decimals

To divide decimals, you will once again apply the methods you use for dividing whole numbers. Look at the two problems below. How are the methods similar?

$$\begin{array}{r} 867 \\ 3\longdiv 2601 \\ -24 \\ \hline 20 \\ -18 \\ \hline 21 \\ -21 \\ \hline 0 \end{array} \qquad \begin{array}{r} 8.67 \\ 3\longdiv 26.01 \\ -24 \\ \hline 2\ 0 \\ -1\ 8 \\ \hline 21 \\ -21 \\ \hline 0 \end{array}$$

Notice that the division occurs in the same way—the only difference is the placement of the decimal point in the **quotient**.

Example

$18.32 \div 8 = ?$

Solution

$8\longdiv 18.32$	Set up the problem.
$\begin{array}{r} 2.29 \\ 8\longdiv 18.32 \\ -16 \\ \hline 23 \\ -16 \\ \hline 72 \\ -72 \\ \hline 0 \end{array}$	Divide.
\(\ \begin{array}{r} 2.29\ \ 8\longdiv{18.32}\ \end{array}\)	Place the decimal point in the quotient. It should be placed directly above the decimal point in the dividend.

$18.32 \div 8 = 2.29$

But what about a case where you are dividing by a decimal, as in the problem below?

$0.3\longdiv 260.1$

In cases like this, you can use powers of 10 to help create an easier problem to solve. In this case, you can multiply the **divisor**, 0.3, by 10 to move the decimal point 1 place to the right. If you multiply the divisor by 10, then you also have to multiply the **dividend** by 10 to keep the quotient the same. The new problem, with its solution, is shown below.

Example

$260.1 \div 0.3 = ?$

Solution

0.3\longdiv260.1	Set up the problem.
3.\longdiv2601.	Multiply divisor and dividend by 10 to create a whole number divisor.
867 3\longdiv2601 −24 20 −18 21 −21 0	Divide.

$260.1 \div 0.3 = 867$

Often, the dividend will still be a decimal after multiplying by a power of 10. In this case, the placement of the decimal point must align with the decimal point in the dividend.

✓ Example

$15.275 \div 3.25 \ =?$

Solution

3.25\longdiv15.275	Set up the problem.
325.\longdiv1527.5	Multiply divisor and dividend by 100 to create a whole number divisor.
4.7 325.\longdiv1527.5 −1300 2275 −2275 0	Divide. 325 goes into 1527 four times, so the number 4 is placed above the digit 7. The decimal point in the quotient is placed directly above the decimal point in the dividend.

$15.275 \div 3.25 \ = 4.7$

Dividing Decimals

Dividing Decimals by Whole Numbers

Divide as you would with whole numbers. Then place the decimal point in the quotient directly above the decimal point in the dividend.

Dividing by Decimals

To divide by a decimal, multiply the divisor by a power of ten to make the divisor a whole number. Then multiply the dividend by the same power of ten. You can think of this as moving the decimal point in the dividend the same number of places to the right as you move the decimal point in the divisor.

Then place the decimal point in the quotient directly over the decimal point in the dividend. Finally, divide as you would with whole numbers.

? Exercise

Divide: $25.095 \div 0.5$.

A. 5,019
B. 501.9

C. 50.19
D. 0.5019

Answer

A. Incorrect. Multiply both the divisor and the dividend by 10 (this will change 0.5 into a whole number), and then divide. Then place the decimal point in the quotient directly over the decimal point in the dividend. The correct answer is 50.19.
B. Incorrect. Multiply both the divisor and the dividend by 10 (this will change 0.5 into a whole number), and then divide. Then place the decimal point in the quotient directly over the decimal point in the dividend. The correct answer is 50.19.
C. Correct. This problem can be set up as $250.95 \div 5$; the quotient is 50.19.
D. Incorrect. Remember that when you divide, you do not count the total number of decimal places in the divisor and dividend. You change the divisor to a whole number, then move the decimal point in the dividend the same number of places and divide. Finally, place the decimal point in the quotient directly over the decimal point in the dividend. The correct answer is 50.19.

Dividing by Tens

Recall that when you multiply a decimal by a power of ten (10, 100, 1,000, etc.), the placement of the decimal point in the product will move to the right according to the number of zeros in the power of ten. For instance, $4.12 \cdot 10 = 41.2$.

Multiplication and division are inverse operations, so you can expect that if you divide a decimal by a power of ten, the decimal point in the quotient will also correspond to the number of zeros in the power of ten. The difference is that the decimal point moves to the right when you multiply; it moves to the left when you divide.

$$\begin{array}{r} .4469 \\ 10\text{\longdiv}4.4690 \\ -4\ 0 \\ \hline 46 \\ -40 \\ \hline 69 \\ -60 \\ \hline 90 \\ -90 \\ \hline 0 \end{array} \quad \begin{array}{r} .04469 \\ 100\text{\longdiv}4.46900 \\ -4\ 00 \\ \hline 469 \\ -400 \\ \hline 690 \\ -600 \\ \hline 900 \\ -900 \\ \hline 0 \end{array} \quad \begin{array}{r} .004469 \\ 1000\text{\longdiv}4.469000 \\ -4\ 000 \\ \hline 4690 \\ -4000 \\ \hline 6900 \\ -6000 \\ \hline 9000 \\ -9000 \\ \hline 0 \end{array}$$

In the examples above, notice that each quotient still contains the digits 4469—but as another 0 is added to the end of each power of ten in the divisor, the decimal point moves an additional place to the left in the quotient.

Dividing by Powers of Ten

To divide a decimal by a power of ten (10, 100, 1,000, ect.), count the number of zeros in the divisor. Then move the decimal point in the dividend that number of decimal places to the left; this will be your quotient.

Example

$31.05 \div 10 = ?$

Solution

$31.05 \div 10 = ?$	10 has one zero.
$31.05 \div 10 = 3.105$	Move the decimal point one place to the left in the dividend; this is the quotient.

$31.05 \div 10 = 3.105$

Exercise

Divide. $0.045 \div 100$

A. 0.00045
B. 0.045
C. 4.5
D. 4,500

Answer

A. Correct. There are two zeros in the divisor, 100, so to find the quotient, take the dividend, 0.045, and move the decimal point two places to the left. The quotient is 0.00045.
B. Incorrect. 0.045 is the dividend in the problem; it cannot be the quotient unless the divisor is 1. The correct answer is 0.00045.
C. Incorrect. 4.5 would be the correct answer if you *multiplied* 0.045 by 100, not divided it by 100. The correct answer is 0.00045.
D. Incorrect. 4,500 would be the correct answer if you *multiplied* 0.045 by 100,000, not divided it by 100. The correct answer is 0.00045.

Solving Problems by Multiplying or Dividing Decimals

Now let’s return to the two problems from the beginning of this section. You know how to multiply and divide with decimals now. Let’s put that knowledge to the test.

Example

A couple eats dinner at a Japanese steakhouse. The bill for the meal totals $58.32—which includes a tax of $4.64. To calculate the tip, they can double the tax. How much tip should the couple leave?

Solution

$\begin{array}{r} 4.64 \\ \times \quad 2 \\ \hline \end{array}$	Set up a multiplication problem.
$\begin{array}{r} 4.64 \\ \times \quad 2 \\ \hline 928 \end{array}$	Multiply 4.64 by 2.
$\begin{array}{r} 4.64 \\ \times \quad 2 \\ \hline 9.28 \end{array}$	Count the number of decimal places in the two factors, and place the decimal point accordingly.

The couple should leave a tip of $9.28.

Example

Andy just sold his van that averaged 20 miles per gallon of gasoline. He bought a new pickup truck and took it on a trip of 614.25 miles. He used 31.5 gallons of gas for the trip. Did Andy get better gas mileage with the new truck?

Solution

31.5\longdiv614.25	Set up a division problem.
315.\longdiv6142.5	Make the divisor a whole number by multiplying by 10; do the same to the dividend.

$\begin{array}{r} 19.5 \\ 315.\backslash\text{longdiv}6142.5 \\ \underline{-315} \\ 2992 \\ \underline{-2835} \\ 1575 \\ \underline{-1575} \\ 0 \end{array}$	Divide. Insert a decimal point in the quotient so that it is directly above the decimal point in the dividend.

Andy gets 19.5 miles per gallon now. He used to get 20 miles per gallon. He does not get better gas mileage with the new truck.

Summary

Learning to multiply and divide with decimals is an important skill. In both cases, you work with the decimals as you have worked with whole numbers, but you have to figure out where the decimal point goes. When multiplying decimals, the number of decimal places in the product is the sum of the decimal places in the factors. When dividing by decimals, move the decimal point in the dividend the same number of places to the right as you move the decimal point in the divisor. Then place the decimal point in the quotient above the decimal point in the dividend.

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3.2.3: Estimation with Decimals

Learning Objectives

- Estimate the answer to a decimal problem.

Introduction

Being able to **estimate** your answer is a very useful skill. Not only will it help you decide if your answer is reasonable when doing homework problems or answering test questions, it can prove to be very helpful in everyday life. When shopping, you can estimate how much money you have spent, the tip for a restaurant bill, or the price of an item on sale. By **rounding** and then doing a quick calculation, you will at least know if you are close to the exact answer.

Estimating with Decimals

Consider this problem. Stewart wanted to buy a DVD home theater system that cost \$345.23. He also wanted a universal remote priced at \$32.90. He used a calculator to add the costs and the sum that he got was \$674.23. He was surprised!

Whether Stewart can afford to spend \$674.23 or not is not really the problem here. Rather, the problem is whether the system and remote would total \$674.23. Round both item costs to the tens place: \$345.23 is about \$350, and \$32.90 is about \$30. Is \$350+\$30 close to \$ 674.23? Of course not!

Estimating an answer is a good skill to have. Even when using a calculator, you can get an incorrect answer by accidentally pressing a wrong button. When decimals are involved, it's very easy to put the decimal point in the wrong place, and then your numbers can be drastically wrong.

Example

Hakim wrote checks for \$64.20, \$47.89, and \$95.80. Estimate the total of all three checks.

Solution

	To estimate the total, first round each of the check values. You want to round to the nearest \$10 in this example.
$64.20 \rightarrow 60$ $47.89 \rightarrow 50$ $95.80 \rightarrow 100$	Since 4<5, round 64.20 to 60. Since 7>5, round 47.89 up to 50. Since 5=5, round 95.80 up to 100.
$60 + 50 + 100 = 210$	Add the estimates to find the estimated total.

The total estimate for the three checks is \$210.

You can use estimation to see if you have enough money for a purchase. In this case, it is best to round all the numbers up to make sure that you have enough money.

Example

Sherry has \$50 and wants to buy CDs that cost \$11.50 each. About how many CDs can she buy?

Solution

$11.50 \rightarrow 12.00$	Round \$11.50 to the nearest whole number. Since you want to make sure that Sherry has enough money, round up to 12.00 or 12.
$50 \div 12 = 4R2$	Divide 50 by 12. The amount of the remainder is not important.

Sherry can buy about 4 CDs with \$50.

You will generally estimate when you compute the amount of tip to leave when you eat at a restaurant. Recall that an easy way to compute the tip is to double the tax. You can probably do this in your head, if you estimate this product by rounding to the nearest $1. You can round up if the service is good or round down if not.

✓ Example

After a delicious meal at a restaurant, the bill for two is $45.36, which includes tax of $3.74. The service was very good. How much tip would you leave if you follow the rule to double the amount of tax?

Solution

$3.74 \rightarrow 4.00$	Round up to $4.00, or $4, as the service was good.
$4 \cdot 2 = 8$	Multiply the rounded number by 2 to double this number.

The tip for good service would be $8.

When using rounding in addition and subtraction problems, you usually round all numbers to the same place value—this makes adding or subtracting a bit easier. However, when you use rounding to help you multiply or divide numbers, it's usually better to round the numbers so they each have only one or two digits that are not 0. This is shown in the example below.

✓ Example

Jin is building a model ship based on a real one. Each length of the model is 0.017 times the actual length of the ship. The real ship is 132 feet long. Estimate the length of the model, then use a calculator to find the actual length of the model.

Solution

$0.017 \rightarrow 0.02$	The scale of the model is 0.017. It wouldn't make sense to round this value to a whole number or even tenths, but you can round it to hundredths. Since 7>5, round up to 0.020 or just 0.02.
$132 \rightarrow 130$	You will round 132 to the tens to make it easier to work with. Since 2<5, round to 130.
$130 \cdot 0.02$ $130 \cdot 2 = 260$ $130 \cdot 0.02 = 2.60$	Multiply $130 \cdot 2 = 260$. Then place the decimal point to get 2.60. Use a calculator to find the exact length. The estimate is fairly accurate!

The model length is about 2.6ft, or exactly 2.244ft.

? Exercise

Evelyn is purchasing 287 ceramic tiles for her new kitchen. Each one costs $0.21. Which of the following is the most accurate estimate for the cost of purchasing the tiles for her kitchen?

A. $90
B. $60
C. $50
D. $40

Answer

A. Incorrect. You may have rounded 287 to 300 and then 0.21 to 0.30, to arrive at $300 \cdot 0.30 = 90$, or $90. This is too high; 0.21 rounds down to 0.20, not up to 0.30. The correct answer is $60.
B. Correct. 287 rounds to 300, and 0.21 rounds to 0.20, to arrive at $300 \cdot 0.20 = 60$ or $60.

C. Incorrect. You may have rounded 287 to 250 and then 0.21 to 0.20, to arrive at $250 \cdot 0.20 = 50$, or \$50. This is too low; to the tens place, 287 rounds up to 290, not down to 250. Then $290 \cdot 0.20 = \$58$, the closest answer to this is the \$60. The correct answer is \$60.

D. Incorrect. You may have rounded 287 to 200 and then 0.21 to 0.20, to arrive at $200 \cdot 0.20 = 400$, or \$40. This is too low; to the hundreds place, 287 rounds up to 300, not down to 200. The correct answer is \$60.

Summary

Estimation is useful when you don’t need an exact answer. It also lets you check to be sure an exact answer is close to being correct. Estimating with decimals works just the same as estimating with whole numbers. When rounding the values to be added, subtracted, multiplied, or divided, it helps to round to numbers that are easy to work with.

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CHAPTER OVERVIEW

4: Ratios, Rates, and Proportions

4.1: Ratio and Rates

  4.1.1: Simplifying Ratios and Rates

4.2: Proportions

  4.2.1: Understanding Proportions

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SECTION OVERVIEW

4.1: Ratio and Rates

Topic hierarchy

4.1.1: Simplifying Ratios and Rates

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4.1.1: Simplifying Ratios and Rates

Learning Objectives

- Write ratios and rates as fractions in simplest form.
- Find unit rates.
- Find unit prices.

Introduction

Ratios are used to compare amounts or quantities or describe a relationship between two amounts or quantities. For example, a ratio might be used to describe the cost of a month's rent as compared to the income earned in one month. You may also use a ratio to compare the number of elephants to the total number of animals in a zoo, or the amount of calories per serving in two different brands of ice cream.

Rates are a special type of ratio used to describe a relationship between different units of measure, such as speed, wages, or prices. A car can be described as traveling 60 miles per hour; a landscaper might earn $35 per lawn mowed; gas may be sold at $3 per gallon.

Ratios

Ratios compare quantities using division. This means that you can set up a ratio between two quantities as a division expression between those same two quantities.

Here is an example. If you have a platter containing 10 sugar cookies and 20 chocolate chip cookies, you can compare the cookies using a ratio.

The ratio of sugar cookies to chocolate chip cookies is:

$$\frac{\text{sugar cookies}}{\text{chocolate chip cookies}}=\frac{10}{20}$$

The ratio of chocolate chip cookies to sugar cookies is:

$$\frac{\text{chocolate chip cookies}}{\text{sugar cookies}}=\frac{20}{10}$$

You can write the ratio using words, a fraction, and also using a colon as shown below.

$$\begin{array}{c}\text{ratio of sugar cookies to}\\ \text{chocolate chip cookies}\\ 10 \text{ to } 20\\ \frac{10}{20}\\ 10:20\end{array}$$

Some people think about this ratio as: "For every 10 sugar cookies I have, I have 20 chocolate chip cookies."

You can also simplify the ratio just as you simplify a fraction.

$$\frac{10}{20}=\frac{10\div 10}{20\div 10}=\frac{1}{2}$$

So we can also say that:

$$\begin{array}{c}\text{ratio of sugar cookies to}\\ \text{chocolate chip cookies}\\ 1 \text{ to } 2\\ \frac{1}{2}\\ 1:2\end{array}$$

How to Write a Ratio

A ratio can be written in three different ways:

- with the word "to": 3 to 4
- as a fraction: $\frac{3}{4}$

- with a colon: 3:4

A ratio is simplified if it is equivalent to a fraction that has been simplified.

Below are two more examples that illustrate how to compare quantities using a ratio, and how to express the ratio in simplified form.

✓ Example

A basketball player takes 50 jump shots during a practice. She makes 28 of them. What is the ratio of shots made to shots taken? Simplify the ratio.

Solution

$\frac{\text{shots made}}{\text{shots taken}}$	Identify the relationship.
$\frac{28}{50}$	Express the two quantities in fraction form.
$\frac{28 \div 2}{50 \div 2} = \frac{14}{25}$	Simplify the fraction to express the ratio in simplest form.
$14 : 25$ $14 \text{ to } 25$	Consider the two other ways to write a ratio. You’ll want to express your answer in a particular format if required.

The ratio of shots made to shots taken is $\frac{14}{25}$, 14:15, or 14 to 15.

Often, one quantity in the ratio is greater than the second quantity. You do not have to write the ratio so that the lesser quantity comes first; the important thing is to keep the relationship consistent.

✓ Example

Paul is comparing the amount of calories in a large order of French fries from his two favorite fast food restaurants. Fast Foodz advertises that an order of fries has 450 calories, and Beef Stop states that its fries have 300 calories. Write a ratio that represents the amount of calories in the Fast Foodz fries compared to the calories in Beef Stop fries.

Solution

$\frac{\text{calories in Fast Foodz fries}}{\text{calories in Beef Stop fries}}$	Identify the relationship.
$\frac{450}{300}$	Write a ratio comparing calories.
$\frac{450 \div 150}{300 \div 150} = \frac{3}{2}$	Simplify the ratio. 450 and 300 have a common factor of 150.

The ratio of calories in Fast Foodz fries to Beef Stop fries is \frac{3}{2}, 3:2, or 3 to 2.

Ratios can compare a *part to a part* or a *part to a whole*. Consider the example below that describes guests at a party.

✓ Example

Luisa invites a group of friends to a party. Including Luisa, there are a total of 22 people, 10 of whom are women.

Which is greater: the ratio of women to men at the party, or the ratio of women to the total number of people present?

Solution

$\frac{\text{number of women}}{\text{number of men}}$	Identify the first relationship.
$\frac{10}{12}$	Write a ratio comparing women to men. Since there are 22 people and 10 are women, 12 must be men.
$\frac{10 \div 2}{12 \div 2} = \frac{5}{6}$	Simplify the ratio. 10 and 12 have a common factor of 2; the ratio of women to men at the party is $\frac{5}{6}$.

$\frac{\text{number of women}}{\text{number of people}}$	Identify the next relationship.
$\frac{10}{22}$	Write a ratio comparing the number of women to the total number of people at the party.
$\frac{10\div 2}{22\div 2}=\frac{5}{11}$	Simplify the ratio. 10 and 22 have a common factor of 2.
$\frac{5\cdot 11}{6\cdot 11}=\frac{55}{66}$ $\frac{5\cdot 6}{11\cdot 6}=\frac{30}{66}$	Rewrite each fraction with a common denominator, 66.
$\frac{5}{6}>\frac{5}{11}$	Since $\frac{55}{66}>\frac{30}{66}$.

The ratio of women to men at the party, $\frac{5}{6}$, is greater than the ratio of women to the total number of people, $\frac{5}{11}$.

? Exercise

A poll at Forrester University found that 4,000 out of 6,000 students are unmarried. Find the ratio of unmarried to married students. Express as a simplified ratio.

A. 3 to 2
B. 1 to 3
C. 2 to 1
D. 2 to 3

Answer

A. Incorrect. The ratio 3 to 2 compares the total number of students to the number of unmarried students. The correct answer is 2 to 1.
B. Incorrect. The ratio 1 to 3 compares the number of married students to the total number of students. The correct answer is 2 to 1.
C. Correct. If 4,000 students out of 6,000 are unmarried, then 2,000 must be married. The ratio of unmarried to married students can be represented as 4,000 to 2,000, or simply 2 to 1.
D. Incorrect. The ratio 2 to 3 compares the number of unmarried students to the total number of students. The correct answer is 2 to 1.

Rates

A rate is a ratio that compares two different quantities that have different units of measure. A rate is a comparison that provides information such as dollars per hour, feet per second, miles per hour, and dollars per quart, for example. The word "per" usually indicates you are dealing with a rate. Rates can be written using words, using a colon, or as a fraction. It is important that you know which quantities are being compared.

For example, an employer wants to rent 6 buses to transport a group of 300 people on a company outing. The rate to describe the relationship can be written using words, using a colon, or as a fraction; and you must include the units.

$$\text{six buses per 300 people}$$
$$6\text{ buses}:300\text{ people}$$
$$\frac{6\text{ buses}}{300\text{ people}}$$

As with ratios, this rate can be expressed in simplest form by simplifying the fraction.

$$\frac{6\text{ buses}\div 6}{300\text{ people}\div 6}=\frac{1\text{ bus}}{50\text{ people}}$$

This fraction means that the rate of buses to people is 6 to 300 or, simplified, 1 bus for every 50 people.

✓ Example

Write the rate as a simplified fraction: 8 phone lines for 36 employees.

Solution

$\frac{8 \text{ phone lines}}{36 \text{ employees}}$	Write as a fraction.
$\frac{8 \text{ phone lines} \div 4}{36 \text{ employees} \div 4} = \frac{2 \text{ phone lines}}{9 \text{ employees}}$	Simplify the fraction by using the common factor of 4.

The rate of phone lines for employees can be expressed as $\frac{2 \text{ phone lines}}{9 \text{ employees}}$.

✓ Example

Write the rate as a simplified fraction: 6 flight attendants for 200 passengers.

Solution

$\frac{6 \text{ flight attendants}}{200 \text{ passengers}}$	Write as a fraction.
$\frac{6 \text{ flight attendants} \div 2}{200 \text{ passengers} \div 2} = \frac{3 \text{ flight attendants}}{100 \text{ passengers}}$	Simplify the fraction by using the common factor of 2.

The rate of flight attendants to passengers is $\frac{3 \text{ flight attendants}}{100 \text{ passengers}}$.

? Exercise

Anyla rides her bike 18 blocks in 20 minutes. Express her rate as a simplified fraction.

A. 18 : 20
B. $\frac{9 \quad \text{blocks}}{10 \quad \text{minutes}}$
C. $\frac{9 \text{ minutes}}{10 \text{ blocks}}$
D. $\frac{18 \text{ blocks}}{20 \text{ minutes}}$

Answer

A. 18 : 20

Incorrect. Anyla’s trip compares quantities with different units, so it can be described as a rate. Since rates compare two quantities measured in different units of measurement, they must include their units. The correct answer is $\frac{9 \text{ blocks}}{10 \text{ minutes}}$.

B. $\frac{9 \quad \text{blocks}}{10 \quad \text{minutes}}$

Correct. Anyla’s trip compares quantities with different units (blocks and minutes), so it is a rate and can be written $\frac{18 \quad \text{blocks}}{20 \quad \text{minutes}}$. This fraction can be simplified by dividing both the numerator and the denominator by 2.

C. $\frac{9 \text{ minutes}}{10 \text{ blocks}}$

Incorrect. 18 blocks in 20 minutes is not equivalent to 10 blocks in 9 minutes. Check the units again in your answer. The correct answer is $\frac{9 \text{ blocks}}{10 \text{ minutes}}$.

D. $\frac{18 \text{ blocks}}{20 \text{ minutes}}$

Incorrect. Anyla’s trip compares quantities with different units, so it can be described as a rate. This is a correct representation and includes the units, but the fraction can be simplified. The correct answer is $\frac{9 \text{ blocks}}{10 \text{ minutes}}$.

Finding Unit Rates

A **unit rate** compares a quantity to one unit of measure. You often see the speed at which an object is traveling in terms of its unit rate.

For example, if you wanted to describe the speed of a boy riding his bike—and you had the measurement of the distance he traveled in miles in 2 hours—you would most likely express the speed by describing the distance traveled in *one* hour. This is a unit rate; it gives the distance traveled per one hour. The denominator of a unit rate will always be one.

Consider the example of a car that travels 300 miles in 5 hours. To find the unit rate, you find the number of miles traveled in ***one*** hour.

$$\frac{300\text{ miles}\div 5}{5\text{ hours}\div 5}=\frac{60\text{ miles}}{1\text{ hour}}$$

A common way to write this unit rate is 60 miles per hour.

✓ Example

A crowded subway train has 375 passengers distributed evenly among 5 cars. What is the unit rate of passengers per subway car?

Solution

$\frac{\text{passengers}}{\text{subway cars}}$	Identify the relationship.
$\frac{375\text{ passengers}}{5\text{ subway cars}}$	Write the rate as a fraction.
$\frac{375\text{ passengers }\div 5}{5\text{ subway cars }\div 5}=\frac{75\text{ passengers}}{1\text{ subway car}}$	Express the fraction with 1 in the denominator to find the number of passengers in one subway car.

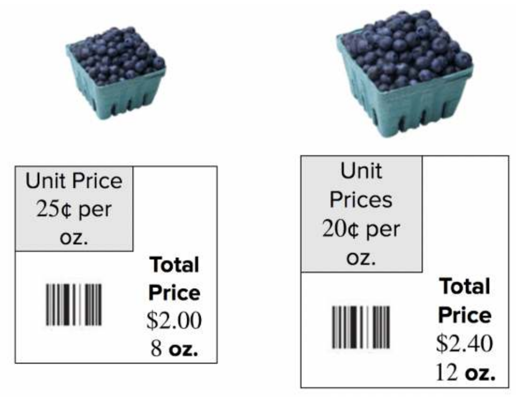
The unit rate of the subway car is 75 riders per subway car.

Finding Unit Prices

A **unit price** is a unit rate that expresses the price of something. The unit price always describes the price of *one* unit, so that you can easily compare prices.

You may have noticed that grocery shelves are marked with the unit price (as well as the total price) of each product. This unit price makes it easy for shoppers to compare the prices of competing brands and different package sizes.

Consider the two containers of blueberries shown below. It might be difficult to decide which is the better buy just by looking at the prices; the container on the left is cheaper, but you also get fewer blueberries. A better indicator of value is the price *per single ounce* of blueberries for each container.



Look at the unit prices—the container on the right is actually a better deal, since the price per ounce is *lower* than the unit price of the container on the left. You pay more money for the larger container of blueberries, but you also get more blueberries than you would with the smaller container. Put simply, the container on the right is a better value than the container on the left.

So, how do you find the unit price?

Imagine a shopper wanted to use unit prices to compare a 3-pack of tissue for $4.98 to a single box of tissue priced at $1.60. Which is the better deal?

Find the unit price of the 3-pack: $\frac{\$4.98}{3\text{ boxes}}$

Since the price given is for 3 boxes, divide both the numerator and the denominator by 3 to get the price of 1 box, the unit price. The unit price is $1.66 per box.

The unit price of the 3-pack is \$1.66 per box; compare this to the price of a single box at \$1.60. Surprisingly, the 3-pack has a higher unit price! Purchasing the single box is the better value.

Like rates, unit prices are often described with the word “per.” Sometimes, a slanted line / is used to mean “per.” The price of the tissue might be written \$1.60/box, which is read "\$1.60 per box."

Example

3 **pounds of sirloin tips cost \$21. What is the unit price per pound?**

Solution

$\frac{\$21.00}{3 \text{ pounds}}$	Write a rate to represent the cost per number of pounds.
$\frac{\$21.00 \div 3}{3 \text{ pounds} \div 3} = \frac{\$7.00}{1 \text{ pound}}$	Express the fraction with 1 in the denominator by dividing both the numerator and the denominator by 3.

The unit price of the sirloin tips is \$7.00/pound.

The following example shows how to use unit price to compare two products and determine which has the lower price.

Example

Sami is trying to decide between two brands of crackers. Which brand has the lower unit price?

Brand A: \$1.12 for 8 ounces

Brand B: \$1.56 for 12 ounces

Solution

Brand A	
$\frac{\$1.12}{8 \text{ ounces}}$	Write a rate to represent the cost per ounce for Brand A.
$\frac{\$1.12 \div 8}{8 \text{ ounces} \div 8} = \frac{\$0.14}{1 \text{ ounce}}$	Find the unit price of Brand A by dividing both the numerator and the denominator by 8.
Brand B	
$\frac{\$1.56}{12 \text{ ounces}}$	Write a rate to represent the cost per ounce for Brand B.
$\frac{\$1.56 \div 12}{12 \text{ ounces } \div 12} = \frac{\$0.13}{1 \text{ ounce}}$	Find the unit price of Brand B by dividing both the numerator and the denominator by 12.
$\frac{\$0.14}{1 \text{ ounce}} > \frac{\$0.13}{1 \text{ ounce}}$	Compare unit prices.

The unit price of Brand A crackers is 14 cents/ounce and the unit price of Brand B is 13 cents/ounce. Brand B has a lower unit price and represents the better value.

? Exercise

A shopper is comparing two packages of rice at the grocery store. A 10-pound package costs \$9.89 and a 2-pound package costs \$1.90. Which package has the lower unit price to the nearest cent? What is its unit price?

A. The 2-pound bag has a lower unit price of \$.95/pound.
B. The 10-pound bag has a lower unit price of \$0.99/pound.
C. The 10-pound bag has a lower unit price of \$.95/pound.
D. The 2-pound bag has a lower price of \$1.89/2pound.

Answer

A. Correct. The unit price per pound for the 2-pound bag is $\$1.90 \div 2 = \$0.95$. The unit price per pound for the 10-pound bag is $\$9.89 \div 10 = \$0.989$, which rounds to \$0.99.
B. Incorrect. $\$9.89 \div 10 = \$0.989$, which rounds to \$0.99. $\$1.90 \div 2 = \$0.95$. The 2-pound bag has a lower unit price. The correct answer is A.
C. Incorrect. $\$9.89 \div 10 = \$0.989$, which rounds to \$0.99. The correct answer is A.
D. Incorrect. A unit price is the price for one unit; in this case, you need to find the cost of one pound, not two pounds. The correct answer is A.

Summary

Ratios and rates are used to compare quantities and express relationships between quantities measured in the same units of measure and in different units of measure. They both can be written as a fraction, using a colon, or using the words “to” or “per”. Since rates compare two quantities measured in different units of measurement, such as dollars per hour or sick days per year, they must include their units. A unit rate or unit price is a rate that describes the rate or price for one unit of measure.

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SECTION OVERVIEW

4.2: Proportions

Topic hierarchy

4.2.1: Understanding Proportions

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4.2.1: Understanding Proportions

Learning Objectives

- Determine whether a proportion is true or false.
- Find an unknown in a proportion.
- Solve application problems using proportions.

Introduction

A true **proportion** is an equation that states that two **ratios** are equal. If you know one ratio in a proportion, you can use that information to find values in the other equivalent ratio. Using proportions can help you solve problems such as increasing a recipe to feed a larger crowd of people, creating a design with certain consistent features, or enlarging or reducing an image to scale.

For example, imagine you want to enlarge a 5-inch by 8-inch photograph to fit a wood frame that you purchased. If you want the shorter edge of the enlarged photo to measure 10 inches, how long does the photo have to be for the image to scale correctly? You can set up a proportion to determine the length of the enlarged photo.

Determining Whether a Proportion is True or False

A proportion is usually written as two equivalent fractions. For example:

$$\frac{12\text{ inches}}{1\text{ foot}}=\frac{36\text{ inches}}{3\text{ feet}}$$

Notice that the equation has a ratio on each side of the equal sign. Each ratio compares the same units, inches and feet, and the ratios are equivalent because the units are consistent, and $\frac{12}{1}$ is equivalent to $\frac{36}{3}$.

Proportions might also compare two ratios with the same units. For example, Juanita has two different-sized containers of lemonade mix. She wants to compare them. She could set up a proportion to compare the number of ounces in each container to the number of servings of lemonade that can be made from each container.

$$\frac{40\text{ ounces}}{84\text{ ounces}}=\frac{10\text{ servings}}{21\text{ servings}}$$

Since the units for each ratio are the same, you can express the proportion without the units:

$$\frac{40}{84}=\frac{10}{21}$$

When using this type of proportion, it is important that the numerators represent the same situation. In the example above they represent 40 ounces for 10 servings. The denominators should also represent the same situation, which is ounces for servings.

Juanita could also have set up the proportion to compare the ratios of the container sizes to the number of servings of each container.

$$\frac{40\text{ ounces}}{10\text{ servings}}=\frac{84\text{ ounces}}{21\text{ servings}}$$

Sometimes you will need to figure out whether two ratios are, in fact, a true or false proportion. Below is an example that shows the steps of determining whether a proportion is true or false.

Example

Is the proportion true or false?

$$\frac{100\text{ miles}}{4\text{ gallons}}=\frac{50\text{ miles}}{2\text{ gallons}}$$

Solution

miles (check)	The units are consistent across the numerators.
gallons (check)	The units are consistent across the denominators. Write each ratio in simplest form.
$\frac{100\div4}{4\div4}=\frac{25}{1}$ $\frac{50\div2}{2\div2}=\frac{25}{1}$ $\frac{25}{1}=\frac{25}{1}$	Since the simplified fractions are equivalent, the proportion is true.

The proportion is true.

Identifying True Proportions

To determine if a proportion compares equal ratios or not, you can follow these steps.

1. Check to make sure that the units in the individual ratios are consistent either vertically or horizontally. For example, $\frac{\text{miles}}{\text{hour}} = \frac{\text{miles}}{\text{hour}}$ or $\frac{\text{miles}}{\text{miles}} = \frac{\text{hour}}{\text{hour}}$ are valid setups for a proportion.
2. Express each ratio as a simplified fraction.
3. If the simplified fractions are the same, the proportion is *true*; if the fractions are different, the proportion is *false*.

Sometimes you need to create a proportion before determining whether it is true or not. An example is shown below.

Example

One office has 3 printers for 18 computers. Another office has 20 printers for 105 computers. Is the ratio of printers to computers the same in these two offices?

Solution

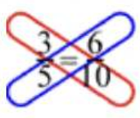
$\frac{\text{printers}}{\text{computers}} = \frac{\text{printers}}{\text{computers}}$	Identify the relationship.
$\frac{3\text{ printers}}{18\text{ computers}} = \frac{20\text{ printers}}{105\text{ computers}}$	Write ratios that describe each situation, and set them equal to each other.
Printers (check)	Check that the units in the numerators match.
Computers (check)	Check that the units in the denominators match. Simplify each fraction and determine if they are equivalent.
$\frac{3\div 3}{18\div 3} = \frac{1}{6}$ $\frac{20\div 5}{105\div 5} = \frac{4}{21}$ $\frac{1}{6} \neq \frac{4}{21}$	Since the simplified fractions are not equal (designated by the $\neq$ sign), the proportion is not true.

The ratio of printers to computers is **not** the same in these two offices.

There is another way to determine whether a proportion is true or false. This method is called "finding the cross product" or "cross multiplying."

To cross multiply, you multiply the numerator of the first ratio in the proportion by the denominator of the other ratio. Then multiply the denominator of the first ratio by the numerator of the second ratio in the proportion. If these products are equal, the proportion is true; if these products are not equal, the proportion is not true.

This strategy for determining whether a proportion is true is called cross-multiplying because the pattern of the multiplication looks like an "x" or a criss-cross. Below is an example of finding a cross product, or cross multiplying.



In this example, you multiply $3 \cdot 10 = 30$, and then multiply $5 \cdot 6 = 30$. Both products are equal, so the proportion is true.

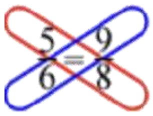
Below is another example of determining if a proportion is true or false by using cross products.

Example

Is the proportion true or false?

$\frac{5}{6} = \frac{9}{8}$

Solution

	Identify the cross product relationship.
$5 \cdot 8 = 40$ $6 \cdot 9 = 54$	Use cross products to determine if the proportion is true or false.
$40 \neq 54$	Since the products are not equal, the proportion is false.

The proportion is false.

Exercise

Is the proportion $\frac{3}{5} = \frac{24}{40}$ true or false?

A. True
B. False

Answer

A. Correct. Using cross products, you find that $3 \cdot 40 = 120$ and $5 \cdot 24 = 120$, so the cross products are equal and the proportion is true.
B. Incorrect. The cross products are equal, so the proportion is true. The correct answer is true.

Finding an Unknown Quantity in a Proportion

If you know that the relationship between quantities is proportional, you can use proportions to find missing quantities. Below is an example.

Example

Solve for the unknown quantity, n.

$\frac{n}{4} = \frac{25}{20}$

Solution

$20 \cdot n = 4 \cdot 25$	Cross multiply.
$20n = 100$	You are looking for a number that when you multiply it by 20 you get 100.
5 20\longdiv100	You can find this value by dividing 100 by 20.

$n = 5$

Now back to the original example. Imagine you want to enlarge a 5-inch by 8-inch photograph to make the length 10 inches and keep the proportion of the width to length the same. You can set up a proportion to determine the width of the enlarged photo.

5
inches
8 inches
10
inches
? inches

Example

Find the length of a photograph whose width is 10 inches and whose proportions are the same as a 5-inch by 8-inch photograph.

Solution

$\frac{\text{width}}{\text{length}}$	Determine the relationship.
Original photo: $\frac{5\text{ inches wide}}{8\text{ inches long}}$ Enlarged photo: $\frac{10\text{ inches wide}}{n\text{ inches long}}$	Write a ratio that compares the length to the width of each photograph. Use a letter to represent the quantity that is not known (the width of the enlarged photo).
$\frac{5}{8}=\frac{10}{n}$	Write a proportion that states that the two ratios are equal.
$5\cdot n=8\cdot 10$	Cross multiply.
$5n=80$	You are looking for a number that, when it is multiplied by 5, will give you 80.
$\frac{5n}{5}=\frac{80}{5}$ $n=\frac{80}{5}$	Divide both sides by 5 to isolate the variable.
$n=16$	16 5\longdiv80

The length of the enlarged photograph is 16 inches.

Solving Application Problems Using Proportions

Setting up and solving a proportion is a helpful strategy for solving a variety of proportional reasoning problems. In these problems, it is always important to determine what the unknown value is, and then identify a proportional relationship that you can use to solve for the unknown value. Below are some examples.

Example

Among a species of tropical birds, 30 out of every 50 birds are female. If a certain bird sanctuary has a population of 1,150 of these birds, how many of them would you expect to be female?

Solution

Let x=the number of female birds in the sanctuary.	Determine the unknown item: the number of female birds in the sanctuary. Assign a letter to this unknown quantity.
$\frac{30\text{ female birds}}{50\text{ birds}}=\frac{x\text{ female birds in sanctuary}}{1{,}150\text{ birds in sanctuary}}$	Set up a proportion, setting the ratios equal.
$\frac{30\div 10}{50\div 10}=\frac{3}{5}$ $\frac{3}{5}=\frac{x}{1150}$	Simplify the ratio on the left to make the upcoming cross multiplication easier.
$3\cdot 1150=5\cdot x$ $3,450=5x$	Cross multiply.
690 5\longdiv3, 450	What number, when multiplied by 5, gives a product of 3,450? You can find this value by dividing 3,450 by 5.
$x=690$ birds	

You would expect 690 birds in the sanctuary to be female.

Example

It takes Sandra 1 hour to word process 4 pages. At this rate, how long will she take to complete 27 pages?

Solution

$\frac{4\text{ pages}}{1\text{ hour}} = \frac{27\text{ pages}}{x\text{ hours}}$	Set up a proportion comparing the pages she types and the time it takes to type them.
$4 \cdot x = 1 \cdot 27$	Cross multiply.
$4x = 27$	You are looking for a number that, when it is multiplied by 4, will give you 27.
6.75 4\longdiv27.00	You can find this value by dividing 27 by 4.
$x = 6.75$ hours	

It will take Sandra 6.75 hours to complete 27 pages.

Exercise

A map uses a scale where 2 inches represents 5 miles. If the distance between two cities is shown on a map as 20 inches, how many miles apart are the two cities?

A. 50 inches
B. 50 miles
C. 8 miles
D. 100 miles

Answer

A. Incorrect. The distance between the cities is measured in inches on the map, but in miles in actuality. The correct answer is 50 miles.
B. Correct. Setting up the proportion $\frac{2\text{ inches}}{5\text{ miles}} = \frac{20\text{ inches}}{x}$, you find that $x = 50$ miles.
C. Incorrect. When you solve a proportion, you cross multiply: $\frac{2\text{ inches}}{5\text{ miles}} = \frac{20\text{ inches}}{x}$, $2x = 5 \cdot 20$. The correct answer is 50 miles.
D. Incorrect. When you solve a proportion, you cross multiply: $\frac{2\text{ inches}}{5\text{ miles}} = \frac{20\text{ inches}}{x}$, $2x = 5 \cdot 20$. The correct answer is 50 miles.

Summary

A proportion is an equation comparing two ratios. If the ratios are equivalent, the proportion is true. If not, the proportion is false. Finding a cross product is another method for determining whether a proportion is true or false. Cross multiplying is also helpful for finding an unknown quantity in a proportional relationship. Setting up and solving proportions is a skill that is useful for solving a variety of problems.

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CHAPTER OVERVIEW

5: Percents

- 5.1: Introduction to Percents
  - 5.1.1: Convert Percents, Decimals, and Fractions
- 5.2: Solving Percent Problems
  - 5.2.1: Solving Percent Problems

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SECTION OVERVIEW

5.1: Introduction to Percents

Topic hierarchy

5.1.1: Convert Percents, Decimals, and Fractions

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5.1.1: Convert Percents, Decimals, and Fractions

Learning Objectives

- Describe the meaning of percent.
- Represent a number as a decimal, percent, and fraction.

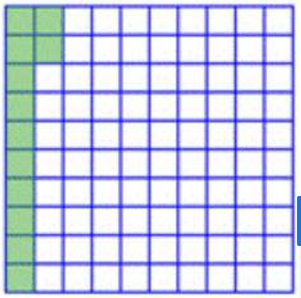
Introduction

Three common formats for numbers are fractions, decimals, and percents. Percents are often used to communicate a *relative* amount. You have probably seen them used for discounts, where the percent of discount can apply to different prices. Percents are also used when discussing taxes and interest rates on savings and loans.

The Meaning of Percent

A **percent** is a ratio of a number to 100. *Per cent* means "per 100," or "how many out of 100." You use the symbol % after a number to indicate percent.

Notice that 12 of the 100 squares in the grid below have been shaded green. This represents 12 percent (12 per 100).

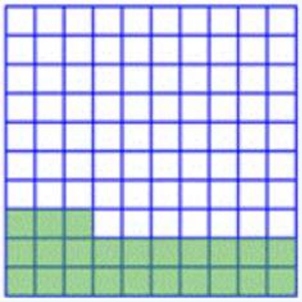


$$12\% = 12 \text{ percent } = 12 \text{ parts out of } 100 = \frac{12}{100}$$

How many of the squares in the grid above are unshaded? Since 12 are shaded and there are a total of 100 squares, 88 are unshaded. The unshaded portion of the whole grid is 88 parts out of 100, or 88% of the grid. Notice that the shaded and unshaded portions together make 100% of the grid (100 out of 100 squares).

Example

What percent of the grid is shaded?



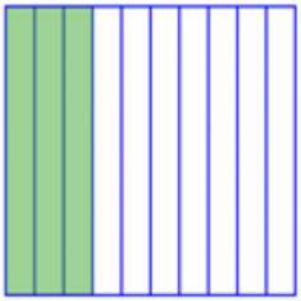
Solution

The grid is divided into 100 smaller squares, with 10 squares in each row. 23 squares out of 100 squares are shaded.

23% of the grid is shaded.

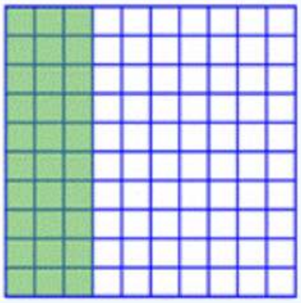
Example

What percent of the large square is shaded?



Solution

The grid is divided into 10 rectangles. For percents, you need to look at 100 equal-sized parts of the whole. You can divide each of the 10 rectangles into 10 pieces, giving 100 parts.

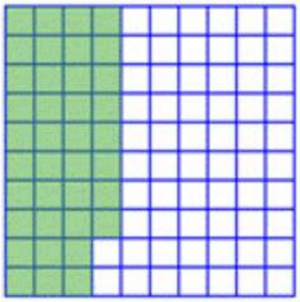


30 small squares out of 100 squares are shaded.

30% of the large square is shaded.

? Exercise

What percent of this grid is shaded?



A. 3%
B. 11%
C. 38%
D. 62%

Answer

A. Incorrect. Three full columns of 10 squares are shaded, plus another 8 squares from the next column. So, there are 30+8, or 38, squares shaded out of the 100 squares in the large square. The correct answer is 38%.
B. Incorrect. Three full columns of 10 squares are shaded, plus another 8 squares from the next column. So, there are 30+8, or 38, squares shaded out of the 100 squares in the large square. The correct answer is 38%.
C. Correct. Three full columns of 10 squares are shaded, plus another 8 squares from the next column. So, there are 30+8, or 38, squares shaded out of the 100 squares in the large square. This means 38% of the large square is shaded.

D. Incorrect. There are 62 small *unshaded* squares out of the 100 in the large square, so the percent of the large square that is *unshaded* is 62%. However, the question asked what percent is shaded. There are 38 shaded squares of the 100 squares in the large square, so the correct answer is 38%.

Rewriting Percents, Decimals, and Fractions

It is often helpful to change the format of a number. For example, you may find it easier to add decimals than to add fractions. If you can write the fractions as decimals, you can add them as decimals. Then you can rewrite your decimal sum as a fraction, if necessary.

Percents can be written as fractions and decimals in very few steps.

✓ Example

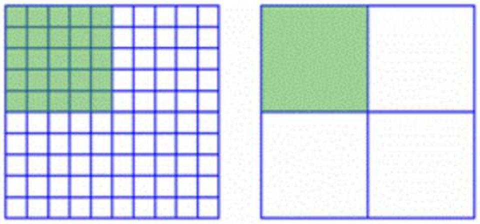
Write 25% as a simplified fraction and as a decimal.

Solution

Write as a fraction.	$25\% = \frac{25}{100}$	Since % means "out of 100," 25% means 25 out of 100. You write this as a fraction, using 100 as the denominator.
	$\frac{25}{100} = \frac{25 \div 25}{100 \div 25} = \frac{1}{4}$	Simplify the fraction by dividing the numerator and denominator by the common factor 25.
Write as a decimal.	$25\% = \frac{25}{100} = 0.25$	You can also just move the decimal point in the whole number 25 two places to the left to get 0.25.

$25\% = \frac{1}{4} = 0.25$

Notice in the diagram below that 25% of a grid is also $\frac{1}{4}$ of the grid, as you found in the example.



Notice that in the previous example, rewriting a percent as a decimal takes just a shift of the decimal point. You can use fractions to understand why this is the case. Any percentage x can be represented as the fraction $\frac{x}{100}$, and any fraction $\frac{x}{100}$ can be written as a decimal by moving the decimal point in x two places to the left. For example, 81% can be written as $\frac{81}{100}$, and dividing 81 by 100 results in 0.81. People often skip over the intermediary fraction step and just convert a percent to a decimal by moving the decimal point two places to the left.

In the same way, rewriting a decimal as a percent (or as a fraction) requires few steps.

✓ Example

Write 0.6 as a percent and as a simplified fraction.

Solution

Write as a percent.	$0.6 = 0.60 = 60\%$	Write 0.6 as 0.60, which is 60 hundredths. 60 hundredths is 60 percent. You can also move the decimal point two places to the *right* to find the percent equivalent.
Write as a fraction.	$0.6 = \frac{6}{10}$	To write 0.6 as a fraction, you read the decimal, 6 tenths, and write 6 tenths in fraction form.
	$\frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}$	Simplify the fraction by dividing the numerator and denominator by 2, a common factor.

$0.6 = 60\% = \frac{3}{5}$

In this example, the percent is not a whole number. You can handle this in the same way, but it’s usually easier to convert the percent to a decimal and then convert the decimal to a fraction.

✓ Example

Write 5.6% as a decimal and as a simplified fraction.

Solution

Write as a decimal.	$5.6\% = 0.056$	Move the decimal point two places to the left. In this case, insert a 0 in front of the 5 (05.6) in order to be able to move the decimal to the left two places.
Write as a fraction.	$0.056 = \frac{56}{1{,}000}$	Write the fraction as you would read the decimal. The last digit is in the *thousandths* place, so the denominator is 1,000.
	$\frac{56}{1{,}000} = \frac{56 \div 8}{1{,}000 \div 8} = \frac{7}{125}$	Simplify the fraction by dividing the numerator and denominator by 8, a common factor.

$5.6\% = \frac{7}{125} = 0.056$

? Exercise

Write 0.645 as a percent and as a simplified fraction.

A. 64.5% and $\frac{129}{200}$
B. 0.645% and $\frac{645}{1{,}000}$
C. 645% and $\frac{645}{1}$
D. 64.5% and $\frac{200}{129}$

Answer

A. Correct. $0.645 = 64.5\% = \frac{129}{200}$
B. Incorrect. $0.645 = 64.5\%$, not 0.645%. Remember that when you convert a decimal to a percent, you have to move the decimal point two places to the right. The correct answer is 64.5% and $\frac{129}{200}$.
C. Incorrect. $0.645 = 64.5\%$, not 645%. Remember that when you convert a decimal to a percent, you have to move the decimal point two places to the right. The correct answer is 64.5% and $\frac{129}{200}$.

D. Incorrect. To write 0.645 as a percent, move the decimal point two places to the right: 64.5%. To write 0.645 as a fraction, use 645 as the numerator. The place value of the last digit (the 5) is thousandths, so the denominator is 1,000. The fraction is $\frac{645}{1,000}$. The greatest common factor of 645 and 1,000 is 5, so you can divide the numerator and denominator by 5 to get $\frac{129}{200}$. The correct answer is 64.5% and $\frac{129}{200}$.

In order to write a fraction as a decimal or a percent, you can write the fraction as an equivalent fraction with a denominator of 10 (or any other power of 10 such as 100 or 1,000), which can be then converted to a decimal and then a percent.

✓ Example

Write $\frac{3}{4}$ as a decimal and as a percent.

Solution

Write as a decimal.	$\frac{3}{4}=\frac{3\cdot 25}{4\cdot 25}=\frac{75}{100}$	Find an equivalent fraction with 10, 100, 1,000, or other power of 10 in the denominator. Since 100 is a multiple of 4, you can multiply 4 by 25 to get 100. Multiply both the numerator and the denominator by 25.
	$\frac{75}{100}=0.75$	Write the fraction as a decimal with the 5 in the hundredths place.
Write as a percent.	$0.75=75\%$	To write the decimal as a percent, move the decimal point two places to the *right*.

$\frac{3}{4}=0.75=75\%$

If it is difficult to find an equivalent fraction with a denominator of 10, 100, 1,000, and so on, you can always divide the numerator by the denominator to find the decimal equivalent.

✓ Example

Write $\frac{3}{8}$ as a decimal and as a percent.

Solution

Write as a decimal.	$\frac{3}{8}=3\div 8$ 0.375 8\longdiv3.000	Divide the numerator by the denominator. $3\div 8=0.375$
Write as a percent.	$0.375=37.5\%$	To write the decimal as a percent, move the decimal point two places to the *right*.

$\frac{3}{8}=0.375=37.5\%$

? Exercise

Write $\frac{4}{5}$ as a decimal and as a percent.

A. 80.0 and 0.8%
B. 0.4 and 4%
C. 0.8 and 80%
D. 0.8 and 8%

Answer

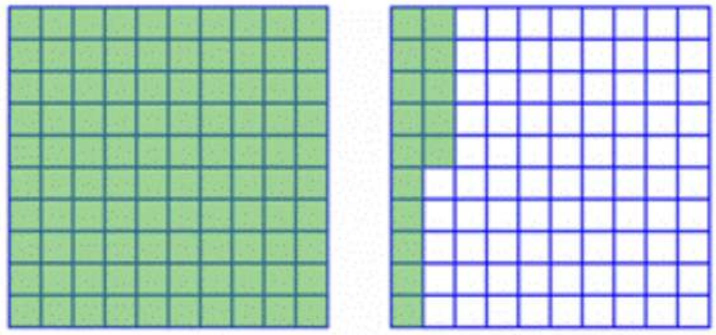
A. Incorrect. Notice that 10 is a multiple of 5, so you can rewrite $\frac{4}{5}$ using 10 as the denominator. Multiply the numerator and denominator by 2 to get $\frac{8}{10}$. The equivalent decimal is 0.8. You can write this as a percent by moving the decimal point two places to the right. Since 0.8 has only one place to the right, include 0 in the hundredths place: $0.8 = 0.80 = 80\%$. The correct answer is 0.8 and 80%.
B. Incorrect. To find a decimal equivalent for $\frac{4}{5}$, first convert the fraction to tenths. Multiply the numerator and denominator by 2 to get $\frac{8}{10}$. The equivalent decimal is 0.8 So, $\frac{4}{5}$ and 0.4 are not equivalent quantities. The correct answer is 0.8 and 80%.
C. Correct. The answer is $\frac{4}{5} = 0.8 = 80\%$.
D. Incorrect. It is true that $\frac{4}{5} = 0.8$, but this does not equal 8%. To write 0.8 as a percent, move the decimal point two places to the right: $0.8 = 0.80 = 80\%$. The correct answer is 0.8 and 80%.

Mixed Numbers

All the previous examples involve fractions and decimals less than 1, so all of the percents you have seen so far have been less than 100%.

Percents greater than 100% are possible as well. Percents more than 100% are used to describe situations where there is more than one whole (fractions and decimals greater than 1 are used for the same reason).

In the diagram below, 115% is shaded. Each grid is considered a whole, and you need two grids for 115%.



Expressed as a decimal, the percent 115% is 1.15; as a fraction, it is $1\frac{15}{100}$, or $1\frac{3}{20}$. Notice that you can still convert among percents, fractions, and decimals when the quantity is greater than one whole.

Numbers greater than one that include a fractional part can be written as the sum of a whole number and the fractional part. For instance, the mixed number $3\frac{1}{4}$ is the sum of the whole number 3 and the fraction $\frac{1}{4} \cdot 3\frac{1}{4} = 3 + \frac{1}{4}$.

✓ Example

Write $2\frac{7}{8}$ as a decimal and as a percent.

Solution

Write as a decimal.	$2\frac{7}{8} = 2 + \frac{7}{8}$	Write the mixed fraction as 2 wholes plus the fractional part.
	$\frac{7}{8} = 7 \div 8$ 0.875 8\longdiv7.000	Write the fractional part as a decimal by dividing the numerator by the denominator $7 \div 8 = 0.875$.
	$2 + 0.875 = 2.875$	Add 2 to the decimal.
Write as a percent.	$2.875 = 287.5\%$	Now you can move the decimal point two places to the *right* to write the decimal as a percent.

$2\frac{7}{8} = 2.875 = 287.5\%$

Note that a whole number can be written as a percent. 100% means one whole; so two wholes would be 200%.

✓ Example

Write 375% as a decimal and as a simplified fraction.

Solution

Write as a decimal.	$375\% = 3.75$	Move the decimal point two places to the left. Note that there is a whole number along with the decimal as the percent is more than 100%.
Write as a fraction.	$3.75 = 3 + 0.75$	Write the decimal as a sum of the whole number and the fractional part.
	$0.75 = \frac{75}{100}$	Write the decimal part as a fraction.
	$\frac{75}{100} = \frac{75 \div 25}{100 \div 25} = \frac{3}{4}$	Simplify the fraction by dividing the numerator and denominator by a common factor of 25.
	$3 + \frac{3}{4} = 3\frac{3}{4}$	Add the whole number part to the fraction.

$375\% = 3.75 = 3\frac{3}{4}$

? Exercise

Write 4.12 as a percent and as a simplified fraction.

A. 0.0412% and $\frac{412}{100}$
B. 412% and $4\frac{3}{25}$
C. 412% and $\frac{103}{250}$
D. 4.12% and $4\frac{3}{25}$

Answer

A. Incorrect. To convert 4.12 to a percent, move the decimal point two places to the right, not the left. The correct answer is 412% and $4\frac{3}{25}$.
B. Correct. 4.12 equals 412%, and the simplified form of $4\frac{12}{100}$ is $4\frac{3}{25}$.
C. Incorrect. 4.12 does equal 412%, but it is also equivalent to $4\frac{12}{100}$, not $\frac{412}{1,000}$. The correct answer is 412% and $4\frac{3}{25}$.
D. Incorrect. To convert 4.12 to a percent, move the decimal point two places to the right. The correct answer is 412% and $4\frac{3}{25}$.

Summary

Percents are a common way to represent fractional amounts, just as decimals and fractions are. Any number that can be written as a decimal, fraction, or percent can also be written using the other two representations.

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SECTION OVERVIEW

5.2: Solving Percent Problems

Topic hierarchy

5.2.1: Solving Percent Problems

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5.2.1: Solving Percent Problems

Learning Objectives

- Identify the amount, the base, and the percent in a percent problem.
- Find the unknown in a percent problem.

Introduction

Percents are a ratio of a number and 100, so they are easier to compare than fractions, as they always have the same denominator, 100. A store may have a 10% off sale. The amount saved is always the same portion or fraction of the price, but a higher price means more money is taken off. Interest rates on a saving account work in the same way. The more money you put in your account, the more money you get in interest. It's helpful to understand how these percents are calculated.

Parts of a Percent Problem

Jeff has a coupon at the Guitar Store for 15% off any purchase of $100 or more. He wants to buy a used guitar that has a price tag of $220 on it. Jeff wonders how much money the coupon will take off the original $220 price.

Problems involving percents have any three quantities to work with: the **percent**, the **amount**, and the **base**.

- The percent has the percent symbol (%) or the word "percent." In the problem above, 15% is the percent off the purchase price.
- The base is the whole amount. In the problem above, the whole price of the guitar is $220, which is the base.
- The amount is the number that relates to the percent. It is always part of the whole. In the problem above, the amount is unknown. Since the percent is the percent *off*, the amount will be the *amount off* of the price.

You will return to this problem a bit later. The following examples show how to identify the three parts: the percent, the base, and the amount.

✓ Example

Identify the percent, amount, and base in this problem.

30 is 20% of what number?

Solution

Percent: The percent is the number with the % symbol: 20%.

Base: The base is the whole amount, which in this case is unknown.

Amount: The amount based on the percent is 30.

Answer:

Percent=20%

Amount=30

Base=unknown

The previous problem states that 30 is a portion of another number. That means 30 is the amount. Note that this problem could be rewritten: 20% of what number is 30?

? Exercise

Identify the percent, base, and amount in this problem:

What percent of 30 is 3?

Answer

> The percent is unknown, because the problem states "*What* percent?" The base is the whole in the situation, so the base is 30. The amount is the portion of the whole, which is 3 in this case.

Solving with Equations

Percent problems can be solved by writing **equations.** An equation uses an equal sign (=) to show that two mathematical expressions have the same value.

Percents are fractions, and just like fractions, when finding a percent (or fraction, or portion) of another amount, you multiply.

The percent of the base is the amount.

Percent **of the** Base **is the** Amount.

$$\text{Percent} \cdot \text{Base} = \text{Amount}$$

In the examples below, the unknown is represented by the letter n. The unknown can be represented by any letter or a box $\square$ or even a question mark.

✓ Example

Write an equation that represents the following problem.

30 is 20% of what number?

Solution

20% of what number is 30?	Rewrite the problem in the form "percent of base is amount."
Percent is: 20% Base is: unknown Amount is: 30	Identify the percent, the base, and the amount.
$\text{Percent} \cdot \text{Base} = \text{Amount}$ $20\% \cdot n = 30$	Write the percent equation. using n for the base, which is the unknown value.

$20\% \cdot n = 30$

Once you have an equation, you can solve it and find the unknown value. To do this, think about the relationship between multiplication and division. Look at the pairs of multiplication and division facts below, and look for a pattern in each row.

Multiplication	**Division**
$2 \cdot 3 = 6$	$6 \div 2 = 3$
$8 \cdot 5 = 40$	$40 \div 8 = 5$
$7 \cdot 4 = 28$	$28 \div 7 = 4$
$6 \cdot 9 = 54$	$54 \div 6 = 9$

Multiplication and division are inverse operations. What one does to a number, the other "undoes."

When you have an equation such as $20\% \cdot n = 30$, you can divide 30 by 20% to find the unknown: $n = 30 \div 20\%$.

You can solve this by writing the percent as a decimal or fraction and then dividing.

$n = 30 \div 20\% = 30 \div 0.20 = 150$

✓ Example

What percent of 72 is 9?

Solution

Percent: unknown Base: 72 Amount: 9	Identify the percent, base, and amount.
$n \cdot 72 = 9$	Write the percent equation: $\text{Percent} \cdot \text{Base} = \text{Amount}$. Use n for the unknown (percent).
$n = 9 \div 72$	Divide to undo the multiplication of n times 72.
0.125 72\longdiv9.000	Divide 9 by 72 to find the value for n, the unknown.
$n = 0.125$ $n = 12.5\%$	Move the decimal point two places to the right to write the decimal as a percent.

12.5% of 72 is 9.

You can estimate to see if the answer is reasonable. Use 10% and 20%, numbers close to 12.5%, to see if they get you close to the answer.

10% of $72 = 0.1 \cdot 72 = 7.2$

20% of $72 = 0.2 \cdot 72 = 14.4$

Notice that 9 is between 7.2 and 14.4, so 12.5% is reasonable since it is between 10% and 20%.

✓ Example

What is 110% of 24?

Solution

Percent: 110% Base: 24 Amount: unknown	Identify the percent, the base, and the amount.
$110\% \cdot 24 = n$	Write the percent equation. $\text{Percent} \cdot \text{Base} = \text{Amount}$. The amount is unknown, so use n.
$1.10 \cdot 24 = n$	Write the percent as a decimal by moving the decimal point two places to the left.
$1.10 \cdot 24 = 26.4 = n$	Multiply 24 by 1.10 or 1.1.

26.4 is 110% of 24.

This problem is a little easier to estimate. 100% of 24 is 24. And 110% is a little bit more than 24. So, 26.4 is a reasonable answer.

? Exercise

18 is what percent of 48?

A. 0.375%
B. 8.64%
C. 37.5%
D. 864%

Answer

A. 0.375%

Incorrect. You may have calculated properly, but you forgot to move the decimal point when you rewrote your answer as a percent. The equation for this problem is $n \cdot 48 = 18$. The corresponding division is $18 \div 48$, so $n = 0.375$. Rewriting this decimal as a percent gives the correct answer, 37.5%.

B. 8.64%

Incorrect. You may have used 18 or 48 as the percent, rather than the amount or base. The equation for this problem is $n \cdot 48 = 18$. The corresponding division is $18 \div 48$, so $n = 0.375$. Rewriting this decimal as a percent gives the correct answer, 37.5%.

C. 37.5%

Correct. The equation for this problem is $n \cdot 48 = 18$. The corresponding division is $18 \div 48$, so $n = 0.375$. Rewriting this decimal as a percent gives 37.5%.

D. 864%

Incorrect. You probably used 18 or 48 as the percent, rather than the amount or base, and also forgot to rewrite the percent as a decimal before multiplying. The equation for this problem is $n \cdot 48 = 18$. The corresponding division is $18 \div 48$, so $n = 0.375$. Rewriting this decimal as a percent gives the correct answer, 37.5%.

Using Proportions to Solve Percent Problems

Percent problems can also be solved by writing a **proportion.** A proportion is an equation that sets two ratios or fractions equal to each other. With percent problems, one of the ratios is the percent, written as $\frac{n}{100}$. The other ratio is the amount to the base.

$$\text{Percent} = \frac{\text{amount}}{\text{base}}$$

Example

Write a proportion to find the answer to the following question.

30 is 20% of what number?

Solution

$\frac{20}{100} = \frac{\text{amount}}{\text{base}}$	The percent in this problem is 20%. Write this percent in fractional form, with 100 as the denominator.
$\frac{20}{100} = \frac{30}{n}$	The percent is written as the ratio $\frac{20}{100}$, the amount is 30, and the base is unknown.
$20 \cdot n = 30 \cdot 100$ $20 \cdot n = 3,000$ $n = 3,000 \div 20$ $n = 150$	Cross multiply and solve for the unknown, n, by dividing 3,000 by 20.

30 is 20% of 150.

Example

What percent of 72 is 9?

Solution

$\text{Percent} = \frac{\text{amount}}{\text{base}}$ $\frac{n}{100} = \frac{9}{72}$	The percent is the ratio of n to 100. The amount is 9, and the base is 72.
$n \cdot 72 = 9 \cdot 100$ $n \cdot 72 = 900$ $n = 900 \div 72$ $n = 12.5$	Cross multiply and solve for n by dividing 900 by 72.
12.5% of 72 is 9	The percent is $\frac{12.5}{100} = 12.5\%$.

✓ Example

What is 110% of 24?

Solution

$\text{Percent} = \frac{\text{amount}}{\text{base}}$ $\frac{110}{100} = \frac{n}{24}$	The percent is the ratio $\frac{110}{100}$. The amount is unknown, and the base is 24.
$24 \cdot 110 = 100 \cdot n$ $2,640 \div 100 = n$ $26.4 = n$	Cross multiply and solve for n by dividing 2,640 by 100.
$26.4 \text{ is } 110\% \text{ of } 24$	

? Exercise

18 is 125% of what number?

A. 0.144
B. 14.4
C. 22.5
D. $694\frac{4}{9}$ (or about 694.4)

Answer

A. 0.144

Incorrect. You probably didn't write a proportion and just divided 18 by 125. Or, you incorrectly set up one fraction as $\frac{18}{125}$ and set this equal to the base, n. The percent in this case is 125%, so one fraction in the proportion should be $\frac{125}{100}$. The base is unknown and the amount is 18, so the other fraction is $\frac{18}{n}$. Solving the proportion $\frac{125}{100} = \frac{18}{n}$ gives $n = 14.4$.

B. 14.4

Correct. The percent in this case is 125%, so one fraction in the proportion should be $\frac{125}{100}$. The base is unknown and the amount is 18, so the other fraction is $\frac{18}{n}$. Solving the proportion $\frac{125}{100} = \frac{18}{n}$ gives $n = 14.4$.

C. 22.5

Incorrect. You probably put the amount (18) over 100 in the proportion, rather than the percent (125). Perhaps you thought 18 was the percent and 125 was the base. The correct percent fraction for the proportion is $\frac{125}{100}$. The base is unknown and the amount is 18, so the other fraction is $\frac{18}{n}$. Solving the proportion $\frac{125}{100} = \frac{18}{n}$ gives $n = 14.4$.

D. $694\frac{4}{9}$ (or about 694.4)

Incorrect. You probably confused the amount (18) with the percent (125) when you set up the proportion. The correct percent fraction for the proportion is $\frac{125}{100}$. The base is unknown and the amount is 18, so the other fraction is $\frac{18}{n}$. Solving the proportion $\frac{125}{100} = \frac{18}{n}$ gives $n = 14.4$.

Let's go back to the problem that was posed at the beginning. You can now solve this problem as shown in the following example.

✓ Example

Jeff has a coupon at the Guitar Store for 15% off any purchase of $100 or more. He wants to buy a used guitar that has a price tag of $220 on it. Jeff wonders how much money the coupon will take off of the $220 original price.

Solution

How much is 15% of $220?	Simplify the problems by eliminating extra words.

Percent: 15% Base: 220 Amount: n	Identify the percent, the base, and the amount.
$15\% \cdot 220 = n$	Write the percent equation. $\text{Percent} \cdot \text{Base} = \text{Amount}$
$0.15 \cdot 220 = 33$	Convert 15% to 0.15, then multiply by 220. 15% of \$220 is \$33.

The coupon will take \$33 off the original price.

You can estimate to see if the answer is reasonable. Since 15% is half way between 10% and 20%, find these numbers.

$$10\% \text{ of } 220 = 0.1 \cdot 220 = 22$$
$$20\% \text{ of } 220 = 0.2 \cdot 220 = 44$$

The answer, 33, is between 22 and 44. So \$33 seems reasonable.

There are many other situations that involve percents. Below are just a few.

✓ Example

Evelyn bought some books at the local bookstore. Her total bill was \$31.50, which included 5% tax. How much did the books cost before tax?

Solution

What number +5% of that number is \$31.50?	In this problem, you know that the tax of 5% is added onto the cost of the books. So if the cost of the books is 100%, the cost plus tax is 105%.
105% of what number = 31.50? Percent: 105% Base: n Amount: 31.50	Identify the percent, the base, and the amount.
$105\% \cdot n = 31.50$	Write the percent equation. $\text{Percent} \cdot \text{Base} = \text{Amount}.$
$1.05 \cdot n = 31.50$	Convert 105% to a decimal.
$n = 31.50 \div 1.05 = 30$	Divide to undo the multiplication of n times 1.05.

The books cost \$30 before tax.

✓ Example

Susana worked 20 hours at her job last week. This week, she worked 35 hours. In terms of a percent, how much more did she work this week than last week?

Solution

35 is what percent of 20?	Simplify the problem by eliminating extra words.
Percent: n Base: 20 Amount: 35	Identify the percent, the base, and the amount.
$n \cdot 20 = 35$	Write the percent equation. $\text{Percent} \cdot \text{Base} = \text{Amount}.$
$n = 35 \div 20$	Divide to undo the multiplication of n times 20.

$n = 1.75 = 175\%$	Convert 1.75 to a percent.

Since 35 is 175% of 20, Susana worked 75% more this week than she did last week. (You can think of this as, “Susana worked 100% of the hours she worked last week, as well as 75% more.”)

Summary

Percent problems have three parts: the percent, the base (or whole), and the amount. Any of those parts may be the unknown value to be found. To solve percent problems, you can use the equation, $\text{Percent} \cdot \text{Base} = \text{Amount}$, and solve for the unknown numbers. Or, you can set up the proportion, $\text{Percent} = \frac{\text{amount}}{\text{base}}$, where the percent is a ratio of a number to 100. You can then use cross multiplication to solve the proportion.

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CHAPTER OVERVIEW

6: Measurement

6.1: U.S. Customary Units of Measurement
  6.1.1: Length
  6.1.2: Weight
  6.1.3: Capacity
6.2: Metric Units of Measurement
  6.2.1: The Metric System
  6.2.2: Converting within the Metric System
  6.2.3: Using Metric Conversions to Solve Problems
6.3: Temperature
  6.3.1: Temperature Scales

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SECTION OVERVIEW

6.1: U.S. Customary Units of Measurement

Topic hierarchy

6.1.1: Length

6.1.2: Weight

6.1.3: Capacity

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6.1.1: Length

Learning Objectives

- Define units of length and convert from one to another.
- Perform arithmetic calculations on units of length.
- Solve application problems involving units of length.

Introduction

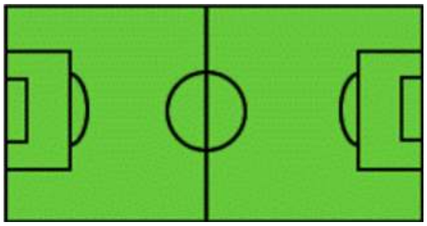
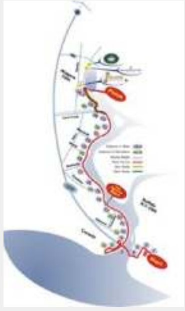
Measurement is a number that describes the size or amount of something. You can measure many things like length, area, capacity, weight, temperature and time. In the United States, two main systems of measurement are used: the **metric system** and the **U.S. customary measurement system**. This topic addresses the measurement of length using the U.S. customary measurement system.

Suppose you want to purchase tubing for a project, and you see two signs in a hardware store: $1.88 for 2 feet of tubing and $5.49 for 3 yards of tubing. If both types of tubing will work equally well for your project, which is the better price? You need to know about two **units of measurement**, yards and feet, in order to determine the answer.

Units of Length

Length is the distance from one end of an object to the other end, or from one object to another. For example, the length of a letter-sized piece of paper is 11 inches. The system for measuring length in the United States is based on the four customary units of length: **inch**, **foot**, **yard**, and **mile**. Below are examples to show measurement in each of these units.

Unit	**Description**	**Image**
Inch/Inches	Some people donate their hair to be made into wigs for cancer patients who have lost hair as a result of treatment. One company requires hair donations to be at least 8 inches long.	
	Frame size of a bike: the distance from the center of the crank to the top of the seat tube. Frame size is usually measured in inches. This frame is 16 inches.	Standover Height Frame Size
Foot/Feet	Rugs are typically sold in standard lengths. One typical size is a rug that is 8 feet wide and 11 feet long. This is often described as an 8 by 11 rug.	

Yard/Yards	Soccer fields vary some in their size. An official field can be any length between 100 and 130 yards.	
Mile/Miles	A marathon is 26.2 miles long. One marathon route is shown in the map to the right.	

You can use any of these four U.S. customary measurement units to describe the length of something, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the length of a rug in feet rather than miles, and to describe a marathon in miles rather than inches.

You may need to convert between units of measurement. For example, you might want to express your height using feet and inches (5 feet 4 inches) or using only inches (64 inches). You need to know the unit equivalents in order to make these conversions between units.

The table below shows equivalents and conversion factors for the four customary units of measurement of length.

Unit Equivalents	**Conversion Factors (longer to shorter units of measurement)**	**Conversion Factors (shorter to longer units of measurement)**
$1 \text{ foot } = 12 \text{ inches}$	$\frac{12 \text{ inches}}{1 \text{ foot}}$	$\frac{1 \text{ foot}}{12 \text{ inches}}$
$1 \text{ yard } = 3 \text{ feet}$	$\frac{3 \text{ feet}}{1 \text{ yard}}$	$\frac{1 \text{ yard}}{3 \text{ feet}}$
$1 \text{ mile } = 5,280 \text{ feet}$	$\frac{5{,}280 \text{ feet}}{1 \text{ mile}}$	$\frac{1 \text{ mile}}{5{,}280 \text{ feet}}$

Note that each of these conversion factors is a ratio of equal values, so each conversion factor equals 1. Multiplying a measurement by a conversion factor does not change the size of the measurement at all since it is the same as multiplying by 1; it just changes the units that you are using to measure.

Converting Between Units of Length

You can use the conversion factors to convert a measurement, such as feet, to another type of measurement, such as inches.

Note that there are many more inches for a measurement than there are feet for the same measurement, as feet is a longer unit of measurement. You could use the conversion factor $\frac{12 \text{ inches}}{1 \text{ foot}}$.

If a length is measured in feet, and you'd like to convert the length to yards, you can think, "I am converting from a shorter unit to a longer one, so the length in yards will be less than the length in feet." You could use the conversion factor $\frac{1 \text{ yard}}{3 \text{ feet}}$.

If a distance is measured in miles, and you want to know how many feet it is, you can think, "I am converting from a longer unit of measurement to a shorter one, so the number of feet would be greater than the number of miles." You could use the conversion factor $\frac{5{,}280 \text{ feet}}{1 \text{ mile}}$.

You can use the **factor label method** to convert a length from one unit of measure to another using the conversion factors. In the factor label method, you multiply by unit fractions to convert a measurement from one unit to another. Study the example below to see how the factor label method can be used to convert $3\frac{1}{2}$ feet into an equivalent number of inches.

✓ Example

How many inches are in $3\frac{1}{2}$ feet?

Solution

$3\frac{1}{2}\text{ feet } =?\text{ inches}$	Begin by reasoning about your answer. Since a foot is longer than an inch, this means the answer would be greater than $3\frac{1}{2}$.
$3\frac{1}{2}\text{ feet }\cdot\frac{12\text{ inches}}{1\text{ foot}} =?\text{ inches}$	Find the conversion factor that compares inches and feet, with “inches” in the numerator, and multiply.
$\frac{7\text{ feet}}{2}\cdot\frac{12\text{ inches}}{1\text{ foot}} =?\text{ inches}$	Rewrite the mixed number as an improper fraction before multiplying.
$\frac{7\cancel{\text{ feet}}}{2}\cdot\frac{12\text{ inches}}{1\cancel{\text{ foot}}} =?\text{ inches}$ $\frac{7}{2}\cdot\frac{12\text{ inches}}{1} =?\text{ inches}$	You can cancel similar units when they appear in the numerator *and* the denominator. So here, cancel the similar units “feet” and “foot.” This eliminates this unit from the problem.
$\frac{7\cdot 12\text{ inches}}{2\cdot 1} =?\text{ inches}$	Rewrite as multiplication of numerators and denominators.
$\frac{84\text{ inches}}{2} =?\text{ inches}$	Multiply.
$\frac{84\text{ inches}}{2} = 42\text{ inches}$	Divide.

There are 42 inches in $3\frac{1}{2}$ feet.

Notice that by using the factor label method you can cancel the units out of the problem, just as if they were numbers. You can only cancel if the unit being cancelled is in both the numerator and denominator of the fractions you are multiplying.

In the problem above, you cancelled *feet* and *foot* leaving you with *inches*, which is what you were trying to find.

$$\frac{7\cancel{\text{ feet}}}{2}\cdot\frac{12\text{ inches}}{1\cancel{\text{ foot}}} =?\text{ inches}$$

What if you had used the wrong conversion factor?

$$\frac{7\text{ feet}}{2}\cdot\frac{1\text{ foot}}{12\text{ inches}} =$$

You could not cancel the feet because the unit is not the same in *both* the numerator and the denominator. So if you complete the computation, you would still have both feet and inches in the answer and no conversion would take place.

Here is another example of a length conversion using the factor label method.

✓ Example

How many yards is 7 feet?

Solution

$7\text{ feet } =?\text{ yards}$	Start by reasoning about the size of your answer. Since a yard is longer than a foot, there will be fewer yards. So your answer will be less than 7.
$7\text{ feet }\cdot\frac{1\text{ yard}}{3\text{ feet}} =?\text{ yards}$	Find the conversion factor that compares feet and yards, with yards in the numerator.
$\frac{7\text{ feet}}{1}\cdot\frac{1\text{ yard}}{3\text{ feet}} =?\text{ yards}$	Rewrite the whole number as a fraction in order to multiply.
$\frac{7\cancel{\text{ feet}}}{1}\cdot\frac{1\text{ yard}}{3\cancel{\text{ feet}}} =?\text{ yards}$	Cancel the similar units “feet” and “feet” leaving only yards.
$\frac{7}{1}\cdot\frac{1\text{ yard}}{3} =?\text{ yards}$ $\frac{7\cdot 1\text{ yard}}{1\cdot 3} =?\text{ yards}$	Multiply.
$\frac{7\text{ yards}}{3} = 2\frac{1}{3}\text{ yards}$	Divide, and write as a mixed number.

7 feet equals $2\frac{1}{3}$ yards.

Note that if the units do not cancel to give you the answer you are trying to find, you may not have used the correct conversion factor.

? Exercise

How many feet are in $2\frac{1}{2}$ miles?

A. 10,560 feet
B. 30 feet
C. 2,112 feet
D. 13,200 feet

Answer

A. Incorrect. There are 5,280 feet in a mile, so multiply $2\frac{1}{2}$, not 2, by 5,280. The correct answer is 13,200 feet.
B. Incorrect. A mile is much longer than 30 feet. There are 5,280 feet in a mile, so multiply 5,280 by $2\frac{1}{2}$, not $2\frac{1}{2}$ by 12, to find the number of feet in $2\frac{1}{2}$ miles. The correct answer is 13,200 feet
C. Incorrect. Multiply, don't divide, 5,280 by $2\frac{1}{2}$. The correct answer is 13,200 feet.
D. Correct. There are 5,280 feet in a mile, so multiply $2\frac{1}{2}$ by 5,280 to get 13,200 feet.

Applying Unit Conversions

There are times when you will need to perform computations on measurements that are given in different units. For example, consider the tubing problem given earlier. You must decide which of the two options is a better price, and you have to compare prices given in different unit measurements.

In order to compare, you need to convert the measurements into one single, common unit of measurement. To be sure you have made the computation accurately, think about whether the unit you are converting to is smaller or larger than the number you have. Its relative size will tell you whether the number you are trying to find is greater or lesser than the given number.

✓ Example

An interior decorator needs border trim for a home she is wallpapering. She needs 15 feet of border trim for the living room, 30 feet of border trim for the bedroom, and 26 feet of border trim for the dining room. How many yards of border trim does she need?

Solution

$15 \text{ feet } + 30 \text{ feet } + 26 \text{ feet } = 71 \text{ feet}$	You need to find the total length of border trim that is needed for all three rooms in the house. Since the measurements for each room are given in feet, you can add the numbers.
$71 \text{ feet } = \ ? \text{ yards}$	How many yards is 71 feet?eason about the size of your answer. Since a yard is longer than a foot, there will be fewer yards. Expect your answer to be less than 71.
$\frac{71 \text{ feet}}{1} \cdot \frac{1 \text{ yard}}{3 \text{ feet}} =? \text{ yards}$	Use the conversion factor $\frac{1 \text{ yard}}{3 \text{ feet}}$.
$\frac{71 \cancel{\text{feet}}}{1} \cdot \frac{1 \text{ yard}}{3 \cancel{\text{feet}}} =? \text{ yards}$	Since "feet" is in the numerator and denominator, you can cancel this unit.
$\frac{71}{1} \cdot \frac{1 \text{ yard}}{3} =? \text{ yards}$ $\frac{71 \cdot 1 \text{ yard}}{1 \cdot 3} =? \text{ yards}$ $\frac{71 \text{ yards}}{3} =? \text{ yards}$	Multiply.
$\frac{71 \text{ yards}}{3} = 23\frac{2}{3} \text{ yards}$	Divide, and write as a mixed number.

The interior decorator needs $23\frac{2}{3}$ yards of border trim.

The next example uses the factor label method to solve a problem that requires converting from miles to feet.

✓ Example

Two runners were comparing how much they had trained earlier that day. Jo said, “According to my pedometer, I ran 8.3 miles.” Alex said, “That’s a little more than what I ran. I ran 8.1 miles.” How many more feet did Jo run than Alex?

Solution

$8.3 \text{ miles } - 8.1 \text{ miles } = 0.2$ $0.2 \text{ mile } = \frac{2}{10} \text{ mile}$	You need to find the difference between the distance Jo ran and the distance Alex ran. Since both distances are given in the same unit, you can subtract and keep the unit the same.
$\frac{2}{10} \text{ mile } =? \text{ feet}$	Since the problem asks for the difference in *feet*, you must convert from miles to feet. How many feet is 0.2 miles? Reason about the size of your answer. Since a mile is longer than a foot, the distance when expressed as feet will be a number greater than 0.2.
$\frac{2 \text{ miles}}{10} \cdot \frac{5{,}280 \text{ feet}}{1 \text{ mile}} =? \text{ feet}$	Use the conversion factor $\frac{5{,}280 \text{ feet}}{1 \text{ mile}}$.
$\frac{2 \cancel{\text{ miles}}}{10} \cdot \frac{5{,}280 \text{ feet}}{1 \cancel{\text{ mile}}} =? \text{ feet}$	Since “mile” is in the numerator and denominator, you can cancel this unit.
$\frac{2}{10} \cdot \frac{5{,}280 \text{ feet}}{1} =? \text{ feet}$ $\frac{2 \cdot 5{,}280 \text{ feet}}{10 \cdot 1} =? \text{ feet}$ $\frac{10{,}560 \text{ feet}}{10} =? \text{ feet}$	Multiply. Divide.
$\frac{10{,}560 \text{ feet}}{10} = 1,056 \text{ feet}$	

Jo ran 1,056 feet farther than Alex.

Now let’s revisit the question from earlier.

✓ Example

You are walking through a hardware store and notice two sales on tubing.

3 yards of Tubing A costs \$5.49.

Tubing B sells for \$1.88 for 2 feet.

Either tubing is acceptable for your project. Which tubing is less expensive?

Solution

Tubing A	
$3 \text{ yards } = \$5.49$	Find the unit price for each tubing. This will make it easier to compare.
$\frac{\$5.49 \div 3}{3 \text{ yards } \div 3} = \frac{\$1.83}{1 \text{ yard}}$	Find the cost per yard of Tubing A by dividing the cost of 3 yards of the tubing by 3.
Tubing B	
$2 \text{ feet } = \$1.88$	Tubing B is sold by the foot. Find the cost per foot by dividing \$1.88 by 2 feet.
$\frac{\$1.88 \div 2}{2 \text{ feet } \div 2} = \frac{\$0.94}{1 \text{ foot}}$ $\frac{\$0.94}{1 \text{ foot}} \cdot \frac{3 \text{ feet}}{1 \text{ yard}} = \frac{\$?}{? \text{ yard}}$	To compare the prices, you need to have the same unit of measure.

$\frac{\$0.94}{1\ \cancel{\text{foot}}} \cdot \frac{3\ \cancel{\text{feet}}}{1\ \text{yard}} = \frac{\$2.82}{1\ \text{yard}}$ \$2.82 per yard	Use the conversion factor $\frac{3\text{ feet}}{1\text{ yard}}$, cancel and multiply.
Tubing A: \$1.83 per yard Tubing B: \$2.82 per yard	Compare prices for 1 yard of each tubing.

Tubing A is less expensive than Tubing B.

In the problem above, you could also have found the price per foot for each kind of tubing and compared the unit prices of each per foot.

? Exercise

A fence company is measuring a rectangular area in order to install a fence around its perimeter. If the length of the rectangular area is 130 yards and the width is 75 feet, what is the total length of the distance to be fenced?

A. 410 yards
B. 930 feet
C. 710 feet
D. 465 feet

Answer

A. Incorrect. The distance around the rectangle is two times the length plus two times the width, but you cannot perform this computation unless the units are the same. Convert yards to feet and then compute. The correct answer is 930 feet.
B. Correct. 130 yards is equivalent to 390 feet. To find the perimeter, add length + length + width + width: $390\text{ feet} + 390\text{ feet} + 75\text{ feet} + 75\text{ feet} = 930\text{ feet}$
C. Incorrect. Convert 130 yards to feet by multiplying by 3. Then, double to get the distance of the two long sides of the rectangle in feet. Distance for the width is already given in feet, so multiply by 2 to get the length of both short sides of the rectangle. The correct answer is 930 feet.
D. Incorrect. Fencing is needed for 4 sides, not just two sides. The correct answer is 930 feet.

Summary

The four basic units of measurement that are used in the U.S. customary measurement system are: inch, foot, yard, and mile. Typically, people use yards, miles, and sometimes feet to describe long distances. Measurement in inches is common for shorter objects or lengths.

You need to convert from one unit of measure to another if you are solving problems that include measurements involving more than one type of measurement. Each of the units can be converted to one of the other units using the table of equivalents, the conversion factors, and/or the factor label method shown in this topic.

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6.1.2: Weight

Learning Objectives

- Define units of weight and convert from one to another.
- Perform arithmetic calculations on units of weight.
- Solve application problems involving units of weight.

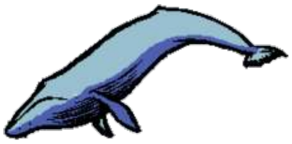
Introduction

When you mention how heavy or light an object is, you are referring to its weight. In the U.S. customary system of measurement, weight is measured in ounces, pounds, and tons. Like other units of measurement, you can convert between these units and you sometimes need to do this to solve problems.

In 2010, the post office charged \$0.44 to mail something that weighed an ounce or less. The post office charged \$0.17 for each additional ounce, or fraction of an ounce, of weight. How much would it have cost to mail a package weighing two pounds three ounces? To answer this question, you need to understand the relationship between ounces and pounds.

Units of Weight

You often use the word **weight** to describe how heavy or light an object or person is. Weight is measured in the U.S. customary system using three units: ounces, pounds, and tons. An **ounce** is the smallest unit for measuring weight, a **pound** is a larger unit, and a **ton** is the largest unit.

Whales are some of the largest animals in the world. Some species can reach weights of up to 200 tons: that's equal to 400,000 pounds.	
Meat is a product that is typically sold by the pound. One pound of ground beef makes about four hamburger patties.	
Ounces are used to measure lighter objects. A stack of 11 pennies is equal to about one ounce.	

You can use any of the customary measurement units to describe the weight of something, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the weight of a human being in pounds rather than tons. It makes more sense to describe the weight of a car in tons rather than ounces.

1 pound = 16 ounces

1 ton = 2,000 pounds

Converting Between Units of Weight

Four ounces is a typical serving size of meat. Since meat is sold by the pound, you might want to convert the weight of a package of meat from pounds to ounces in order to determine how many servings are contained in a package of meat.

The weight capacity of a truck is often provided in tons. You might need to convert pounds into tons if you are trying to determine whether a truck can safely transport a big shipment of heavy materials.

The table below shows the unit conversions and conversion factors that are used to make conversions between customary units of weight.

Unit Equivalents	**Conversion Factors (heavier to lighter units of measurement)**	**Conversion Factors (lighter to heavier units of measurement)**
$1 \text{ pound } = 16 \text{ ounces}$	$\frac{16 \text{ ounces}}{1 \text{ pound}}$	$\frac{1 \quad \text{pound}}{16 \quad \text{ounces}}$
$1 \text{ ton } = 2000 \text{ pounds}$	$\frac{2000 \text{ pounds}}{1 \text{ ton}}$	$\frac{1 \text{ ton}}{2000 \text{ pounds}}$

You can use the **factor label method** to convert one customary unit of weight to another customary unit of weight. This method uses conversion factors, which allow you to “cancel” units to end up with your desired unit of measurement.

Each of these conversion factors is a ratio of equal values, so each conversion factor equals 1. Multiplying a measurement by a conversion factor does not change the size of the measurement at all, since it is the same as multiplying by 1. It just changes the units that you are using to measure it in.

Two examples illustrating the factor label method are shown below.

✓ Example

How many ounces are in $2\frac{1}{4}$ pounds?

Solution

$2\frac{1}{4} \text{ pounds } =? \text{ ounces}$	Begin by reasoning about your answer. Since a pound is heavier than an ounce, expect your answer to be a number greater than $2\frac{1}{4}$.
$2\frac{1}{4} \text{ pounds } \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} =? \text{ ounces}$	Multiply by the conversion factor that relates ounces and pounds: $\frac{16 \text{ ounces}}{1 \text{ pound}}$
$\frac{9 \text{ pounds}}{4} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} =? \text{ ounces}$	Write the mixed number as an improper fraction.
$\frac{9 \cancel{\text{pounds}}}{4} \cdot \frac{16 \text{ ounces}}{1 \cancel{\text{pound}}} =? \text{ ounces}$	The common unit “pound” can be cancelled because it appears in both the numerator and denominator.
$\frac{9}{4} \cdot \frac{16 \text{ ounces}}{1} =? \text{ ounces}$ $\frac{9 \cdot 16 \text{ ounces}}{4 \cdot 1} =? \text{ ounces}$ $\frac{144 \text{ ounces}}{4} =? \text{ ounces}$ $\frac{144 \text{ ounces}}{4} = 36 \text{ ounces}$	Multiply and simplify.

Multiply and simplify 36 ounces in $2\frac{1}{4}$ pounds.

✓ Example

How many tons is 6,500 pounds?

Solution

$6,500 \text{ pounds } =? \text{ tons}$	Begin by reasoning about your answer. Since a ton is heavier than a pound, expect your answer to be a number less than 6,500.

$6,500 \text{ pounds } \cdot \frac{1 \text{ ton}}{2,000 \text{ pounds}} =? \text{ tons}$	Multiply by the conversion factor that relates tons to pounds: $\frac{1 \text{ ton}}{2,000 \text{ pounds}}$
$\frac{6,500 \text{ pounds}}{1} \cdot \frac{1 \text{ ton}}{2,000 \text{ pounds}} =? \text{ tons}$ $\frac{6,500 \cancel{\text{ pounds}}}{1} \cdot \frac{1 \text{ ton}}{2,000 \cancel{\text{ pounds}}} =? \text{ tons}$	Apply the factor label method.
$\frac{6,500}{1} \cdot \frac{1 \text{ ton}}{2,000} =? \text{ tons}$ $\frac{6,500 \text{ tons}}{2,000} =? \text{ tons}$ $\frac{6,500 \text{ tons}}{2,000} = 3\frac{1}{4} \text{ tons}$	Multiply and simplify.

6,500 pounds is equal to $3\frac{1}{4}$ tons.

Exercise

How many pounds is 72 ounces?

A. $4\frac{1}{2}$ pounds
B. 6 pounds
C. 24 pounds
D. 1, 152 pounds

Answer

A. Correct. There are ounces in one pound, so $72 \text{ ounces } \cdot \frac{1 \text{ pound}}{16 \text{ ounces}} = 4\frac{1}{2} \text{ pounds}$.
B. Incorrect. There are 16 ounces in a pound, not 12. The correct answer is $4\frac{1}{2}$ pounds.
C. Incorrect. There are 16 ounces in a pound, not 3. The correct answer is $4\frac{1}{2}$ pounds.
D. Incorrect. Pounds are heavier than ounces, so the answer must be less than 72. Multiply by $\frac{1 \text{ pound}}{16 \text{ ounces}}$, not $\frac{16 \text{ ounces}}{1 \text{ pound}}$. The correct answer is $4\frac{1}{2}$ pounds.

Applying Unit Conversions

There are times when you need to perform calculations on measurements that are given in different units. To solve these problems, you need to convert one of the measurements to the same unit of measurement as the other measurement.

Think about whether the unit you are converting to is smaller or larger than the unit you are converting from. This will help you be sure that you are making the right computation. You can use the factor label method to make the conversion from one unit to another.

Here is an example of a problem that requires converting between units.

Example

A municipal trash facility allows a person to throw away a maximum of 30 pounds of trash per week. Last week, 140 people threw away the maximum allowable trash. How many tons of trash did this equal?

Solution

$140 \cdot 30 \text{ pounds } = 4,200 \text{ pounds}$	Determine the total trash for the week expressed in pounds. If 140 people each throw away 30 pounds, you can find the total by multiplying.
$4,200 \text{ pounds } =? \text{ tons}$	Then convert 4,200 pounds to tons. Reason about your answer. Since a ton is heavier than a pound, expect your answer to be a number less than 4,200.
$\frac{4,200 \text{ pounds}}{1} \cdot \frac{1 \text{ ton}}{2,000 \text{ pounds}} =? \text{ tons}$ $\frac{4,200 \cancel{\text{ pounds}}}{1} \cdot \frac{1 \text{ ton}}{2,000 \cancel{\text{ pounds}}} =? \text{ tons}$	Find the conversion factor appropriate for the situation: $\frac{1 \text{ ton}}{2,000 \text{ pounds}}$

$\frac{4{,}200}{1} \cdot \frac{1 \text{ ton}}{2{,}000} =? \text{ tons}$ $\frac{4{,}200 \cdot 1 \text{ ton}}{1 \cdot 2{,}000} =? \text{ tons}$ $\frac{4{,}200 \text{ tons}}{2{,}000} =? \text{ tons}$ $\frac{4{,}200 \text{ tons}}{2{,}000} = 2\frac{1}{10} \text{ tons}$	Multiply and simplify.

The total amount of trash generated is $2\frac{1}{10}$ tons.

Let's revisit the post office problem that was posed earlier. We can use unit conversion to solve this problem.

✓ Example

The post office charges $0.44 to mail something that weighs an ounce or less. The charge for each additional ounce, or fraction of an ounce, of weight is $0.17. At this rate, how much will it cost to mail a package that weighs 2 pounds 3 ounces?

Solution

$2 \text{ pounds } 3 \text{ ounces } =? \text{ ounces}$	Since the pricing is for ounces, convert the weight of the package from pounds and ounces into just ounces.
$\frac{2 \text{ pounds}}{1} \cdot \frac{16 \text{ ounces}}{\text{pound}} =? \text{ ounces}$ $\frac{2 \cancel{\text{ pounds}}}{1} \cdot \frac{16 \text{ ounces}}{\cancel{\text{pound}}} =? \text{ ounces}$	First use the factor label method to convert 2 pounds to ounces.
$\frac{2}{1} \cdot \frac{16 \text{ ounces}}{1} = 32 \text{ ounces}$	$2 \text{ pounds } = 32 \text{ ounces}$
$32 \text{ ounces } + 3 \text{ ounces } = 35 \text{ ounces}$	Add the additional 3 ounces to find the weight of the package. The package weighs 35 ounces. There are 34 additional ounces, since 35-1=34.
$\$0.44 + \$0.17(34)$ $\$0.44 + \$5.78$ $\$0.44 + \$5.78 = \$6.22$	Apply the pricing formula. $0.44 for the first ounce and $0.17 for each additional ounce.

It will cost $6.22 to mail a package that weighs 2 pounds 3 ounces.

? Exercise

The average weight of a northern bluefin tuna is 1,800 pounds. The average weight of a great white shark is $2\frac{1}{2}$ tons. On average, how much more does a great white shark weigh, in pounds, than a northern bluefin tuna?

A. $5,000$ pounds
B. $3,200$ pounds
C. $182\frac{1}{2}$ pounds
D. You cannot answer this because the units of weight are different.

Answer

A. Incorrect. A great white shark has an average weight of 5,000 pounds, and you have to subtract 1,800 to find the difference in the weights of the shark and the tuna. The correct answer is 3,200 pounds.
B. Correct. $2\frac{1}{2} \text{ tons } = 5,000 \text{ pounds }$.
C. Incorrect. You cannot subtract these weights because they are given in different units. To find the difference, first convert the weight of the shark to pounds and then subtract the weight of the tuna in pounds. The correct answer is 3,200 pounds.
D. Incorrect. The units are different, but you can express the weights in the same unit and then compute. First convert the weight of the shark to pounds and then subtract the weight of the tuna in pounds. The correct answer is 3,200 pounds.

Summary

In the U.S. customary system of measurement, weight is measured in three units: ounces, pounds, and tons. A pound is equivalent to 16 ounces, and a ton is equivalent to 2,000 pounds. While an object's weight can be described using any of these units, it is typical to describe very heavy objects using tons and very light objects using an ounce. Pounds are used to describe the weight of many objects and people. Often, in order to compare the weights of two objects or people or to solve problems involving weight, you must convert from one unit of measurement to another unit of measurement. Using conversion factors with the factor label method is an effective strategy for converting units and solving problems.

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6.1.3: Capacity

Learning Objectives

- Define units of capacity and convert from one to another.
- Perform arithmetic calculations on units of capacity.
- Solve application problems involving units of capacity.

Introduction

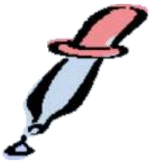
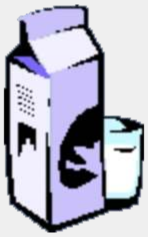
Capacity is the amount of liquid (or other pourable substance) that an object can hold when it's full. When a liquid, such as milk, is being described in gallons or quarts, this is a measure of capacity.

Understanding units of capacity can help you solve problems like this: Sven and Johanna were hosting a potluck dinner. They did not ask their guests to tell them what they would be bringing, and three people ended up bringing soup. Erin brought 1 quart, Richard brought 3 pints, and LeVar brought 9 cups. How many cups of soup did they have all together?

Units of Capacity

There are five main units for measuring capacity in the U.S. customary measurement system. The smallest unit of measurement is a **fluid ounce**. "Ounce" is also used as a measure of weight, so it is important to use the word "fluid" with ounce when you are talking about capacity. Sometimes the prefix "fluid" is not used when it is clear from the context that the measurement is capacity, not weight.

The other units of capacity in the customary system are the **cup**, **pint**, **quart**, and **gallon**. The table below describes each unit of capacity and provides an example to illustrate the size of the unit of measurement.

Fluid Ounce A unit of capacity equal to $\frac{1}{8}$ of a cup. One fluid ounce of water at 62° Fahrenheit weighs about one ounce. The amount of liquid medicine is often measured in fluid ounces.	
Cup A unit equal to 8 fluid ounces. The capacity of a standard measuring cup is one cup.	
Pint A unit equal to 16 fluid ounces, or 2 cups. The capacity of a carton of ice cream is often measured in pints.	
Quart A unit equal to 32 fluid ounces, or 4 cups. You often see quarts of milk being sold in the supermarket.	

Gallon A unit equal to 4 quarts, or 128 fluid ounces. When you fill up your car with gasoline, the price of gas is often listed in dollars per gallon.	

You can use any of these five measurement units to describe the capacity of an object, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the capacity of a swimming pool in gallons and the capacity of an expensive perfume in fluid ounces.

Sometimes you will need to convert between units of measurement. For example, you might want to express 5 gallons of lemonade in cups if you are trying to determine how many 8-fluid ounce servings the amount of lemonade would yield.

The table below shows some of the most common equivalents and conversion factors for the five customary units of measurement of capacity.

Unit Equivalents	**Conversion Factors (heavier to lighter units of measurement)**	**Conversion Factors (lighter to heavier units of measurement)**
$1\text{ cup } = 8\text{ fluid ounces}$	$\frac{1\text{ cup}}{8\text{ fluid ounces}}$	$\frac{8\text{ fluid ounces}}{1\text{ cup}}$
$1\text{ pint } = 2\text{ cups}$	$\frac{1\text{ pint}}{2\text{ cups}}$	$\frac{2\text{ cups}}{1\text{ pint}}$
$1\text{ quart } = 2\text{ pints}$	$\frac{1\text{ quart}}{2\text{ pints}}$	$\frac{2\text{ pints}}{1\text{ quart}}$
$1\text{ quart } = 4\text{ cups}$	$\frac{1\text{ quart}}{4\text{ cups}}$	$\frac{4\text{ cups}}{1\text{ quart}}$
$1\text{ gallon } = 4\text{ quarts}$	$\frac{1\quad\text{gallon}}{4\quad\text{quarts}}$	$\frac{4\quad\text{quarts}}{1\quad\text{gallon}}$
$1\text{ gallon } = 16\text{ cups}$	$\frac{1\text{ gallon}}{16\text{ cups}}$	$\frac{16\text{ cups}}{1\text{ gallon}}$

Converting Between Units of Capacity

As with converting units of length and weight, you can use the factor label method to convert from one unit of capacity to another. An example of this method is shown below.

✓ Example

How many pints is $2\frac{3}{4}$ gallons?

Solution

$2\frac{3}{4}\text{ gallons } = \ ?\text{ pints}$	Begin by reasoning about your answer. Since a gallon is larger than a pint, expect the answer in pints to be a number greater than $2\frac{3}{4}$.
$\frac{11\text{ gallons}}{4}\cdot\frac{4\text{ quarts}}{1\text{ gallon}}\cdot\frac{2\text{ pints}}{1\text{ quart}} = ?\text{ pints}$	The table above does not contain a conversion factor for gallons and pints, so you cannot convert it in one step. However, you can use quarts as an intermediate unit, as shown here.
$\frac{11\ \cancel{\text{gallons}}}{4}\cdot\frac{4\ \cancel{\text{quarts}}}{1\ \cancel{\text{gallon}}}\cdot\frac{2\text{ pints}}{1\ \cancel{\text{quart}}} = ?\text{ pints}$	Set up the equation so that two sets of labels cancel: gallons and quarts.
$\frac{11}{4}\cdot\frac{4}{1}\cdot\frac{2\text{ pints}}{1} = ?\text{ pints}$ $\frac{11\cdot 4\cdot 2\text{ pints}}{4\cdot 1\cdot 1} = ?\text{ pints}$ $\frac{88\text{ pints}}{4} = 22\text{ pints}$	Multiply and simplify.

$2\frac{3}{4}$ gallons is 22 pints.

Example

How many gallons is 32 fluid ounces?

Solution

$\text{32 fluid ounces } =? \text{ gallons}$	Begin by reasoning about your answer. Since gallons is a larger unit than fluid ounces, expect the answer to be less than 32.
$\frac{\text{32 fl oz}}{1} \cdot \frac{\text{1 cup}}{\text{8 fl oz}} \cdot \frac{\text{1 pint}}{\text{2 cups}} \cdot \frac{\text{1 quart}}{\text{2 pints}} \cdot \frac{\text{1 gallon}}{\text{4 quarts}} =? \text{ gallons}$	The table above does not contain a conversion factor for gallons and fluid ounces, so you cannot convert it in one step. Use a series of intermediate units, as shown here.
$\frac{32 \cancel{\text{ fl oz}}}{1} \cdot \frac{1 \cancel{\text{ cup}}}{8 \cancel{\text{ fl oz}}} \cdot \frac{1 \cancel{\text{ pt}}}{2 \cancel{\text{ cups}}} \cdot \frac{1 \cancel{\text{ qt}}}{2 \cancel{\text{ pt}}} \cdot \frac{\text{1 gal}}{4 \cancel{\text{ qt}}} =? \text{ gallons}$	Cancel units that appear in both the numerator and denominator.
$\frac{32}{1} \cdot \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\text{1 gallon}}{4} =? \text{ gallons}$ $\frac{32 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \text{ gallon}}{1 \cdot 8 \cdot 2 \cdot 2 \cdot 4} =? \text{ gallons}$ $\frac{\text{32 gallons}}{128} = \frac{1}{4} \text{ gallon}$	Multiply and simplify.

32 fluid ounces is the same as $\frac{1}{4}$ gallon.

Exercise

Find the sum of 4 gallons and 2 pints. Express your answer in cups.

A. 68 cups
B. 6 cups
C. 2 gallons and 4 pints
D. 34 pints

Answer

A. Correct. Each gallon has 16 cups, so $4 \cdot 16 = 64$ will give you the number of cups in 4 gallons. Each pint has 2 cups, so $2 \cdot 2 = 4$ will give you the number of cups in 2 pints. $64 + 4 = 68$ cups.
B. Incorrect. You cannot add measurements expressed in different units. First convert each amount to cups and then add. The correct answer is 68 cups.
C. Incorrect. This is not the correct amount. First convert each amount to cups and then add them together. Also, notice that this answer is not expressed in cups, as the problem requires. The correct answer is 68 cups.
D. Incorrect. This is the correct amount, but it is expressed in the incorrect unit: pints. Multiply 34 pints by 2 (since there are 2 cups in a pint) to give the amount in cups as required. The correct answer is 68 cups.

Applying Unit Conversions

There are times when you will need to combine measurements that are given in different units. In order to do this, you need to convert first so that the units are the same.

Consider the situation posed earlier in this topic.

Example

Sven and Johanna were hosting a potluck dinner. They did not ask their guests to tell them what they would be bringing, and three people ended up bringing soup. Erin brought 1 quart, Richard brought 3 pints, and LeVar brought 9 cups. How much soup did they have total?

Solution

$\text{1 quart } + \text{3 pints } + \text{9 cups}$	Since the problem asks for the total amount of soup, you must add the three quantities. Before adding, you must convert the quantities to the same unit.

$1 \text{ quart } = 4 \text{ cups}$	The problem does not require a particular unit, so you can choose. Cups might be the easiest computation. This is given in the table of equivalents.
$\frac{3 \text{ pints}}{1} \cdot \frac{2 \text{ cups}}{1 \text{ pint}} =? \text{ cups}$ $\frac{3 \cancel{\text{ pints}}}{1} \cdot \frac{2 \text{ cups}}{1 \cancel{\text{ pint}}} = 6 \text{ cups}$	Use the factor label method to convert pints to cups.
$4 \text{ cups } + 6 \text{ cups } + 9 \text{ cups } = 19 \text{ cups}$	Add the 3 quantities.

There are 19 cups of soup for the dinner.

✓ Example

Natasha is making lemonade to bring to the beach. She has two containers. One holds one gallon and the other holds 2 quarts. If she fills both containers, how many cups of lemonade will she have?

Solution

$1 \text{ gallon } + 2 \text{ quarts } =? \text{ cups}$	This problem requires you to find the sum of the capacity of each container and then convert that sum to cups.
$4 \text{ quarts } + 2 \text{ quarts } = 6 \text{ quarts}$	First, find the sum in quarts. 1 gallon is equal to 4 quarts.
$\frac{6 \text{ quarts}}{1} \cdot \frac{2 \text{ pints}}{1 \text{ quart}} \cdot \frac{2 \text{ cups}}{1 \text{ pint}} =? \text{ cups}$	Since the problem asks for the capacity in cups, convert 6 quarts to cups.
$\frac{6 \cancel{\text{ quarts}}}{1} \cdot \frac{2 \cancel{\text{ pints}}}{1 \cancel{\text{ quart}}} \cdot \frac{2 \text{ cups}}{1 \cancel{\text{ pint}}} =? \text{ cups}$	Cancel units that appear in both the numerator and denominator.
$6 \cdot 2 \cdot 2 = 24 \text{ cups}$	Multiply.

Natasha will have 24 cups of lemonade.

Another way to work the problem above would be to first change 1 gallon to 16 cups and change 2 quarts to 8 cups. Then add: $16 + 8 = 24$ cups.

? Exercise

Alan is making chili. He is using a recipe that makes 24 cups of chili. He has a 5-quart pot and a 2-gallon pot and is trying to determine whether the chili will all fit in one of these pots. Which of the pots will fit the chili?

A. The chili will not fit into either of the pots.
B. The chili can fit into either pot.
C. The chili will fit into the 5-quart pot only.
D. The chili will fit into the 2-gallon pot only.

Answer

A. Incorrect. The 5-quart pot will hold only 20 cups, leaving 4 cups that won’t fit into the pot. However, the 2-gallon pot is large enough to fit the chili. The correct answer is the 2-gallon pot only.
B. Incorrect. The 5-quart pot will hold only 20 cups, so the 24 cups of chili cannot fit in there. The correct answer is the 2-gallon pot only.
C. Incorrect. The 5-quart pot will hold only 20 cups. The 2-gallon pot will hold 32 cups. The correct answer is the 2-gallon pot only.
D. Correct. $5 \text{ quarts } = 5 \cdot 4 \text{ cups } = 20 \text{ cups}$, so 24 cups of chili will not fit into the 5-quart pot. $2 \text{ gallons } = 32 \text{ cups}$, so 24 cups of chili will fit in this pot.

Summary

There are five basic units for measuring capacity in the U.S. customary measurement system. These are the fluid ounce, cup, pint, quart, and gallon. These measurement units are related to one another, and capacity can be described using any of the units. Typically, people use gallons to describe larger quantities and fluid ounces, cups, pints, or quarts to describe smaller quantities. Often, in order to compare or to solve problems involving the amount of liquid in a container, you need to convert from one unit of measurement to another.

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SECTION OVERVIEW

6.2: Metric Units of Measurement

Topic hierarchy

6.2.1: The Metric System

6.2.2: Converting within the Metric System

6.2.3: Using Metric Conversions to Solve Problems

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6.2.1: The Metric System

Learning Objectives

- Describe the general relationship between the U.S. customary units and metric units of length, weight/mass, and volume.
- Define the metric prefixes and use them to perform basic conversions among metric units.

Introduction

In the United States, both the **U.S. customary measurement system** and the **metric system** are used, especially in medical, scientific, and technical fields. In most other countries, the metric system is the primary system of measurement. If you travel to other countries, you will see that road signs list distances in kilometers and milk is sold in liters. People in many countries use words like “kilometer,” “liter,” and “milligram” to measure the length, volume, and weight of different objects. These measurement units are part of the metric system.

Unlike the U.S. customary system of measurement, the metric system is based on 10s. For example, a liter is 10 times larger than a deciliter, and a centigram is 10 times larger than a milligram. This idea of “10” is not present in the U.S. customary system: there are 12 inches in a foot, 3 feet in a yard, and 5,280 feet in a mile!

So, what if you have to find out how many milligrams are in a decigram? Or, what if you want to convert meters to kilometers? Understanding how the metric system works is a good start.

What is Metric?

The metric system uses units such as **meter**, **liter**, and **gram** to measure length, liquid volume, and mass, just as the U.S. customary system uses feet, quarts, and ounces to measure these.

In addition to the difference in the basic units, the metric system is based on 10s, and different measures for length include kilometer, meter, decimeter, centimeter, and millimeter. Notice that the word “meter” is part of all of these units.

The metric system also applies the idea that units within the system get larger or smaller by a power of 10. This means that a meter is 100 times larger than a centimeter, and a kilogram is 1,000 times heavier than a gram. You will explore this idea a bit later. For now, notice how this idea of “getting bigger or smaller by 10” is very different than the relationship between units in the U.S. customary system, where 3 feet equals 1 yard and 16 ounces equals 1 pound.

Length, Mass, and Volume

The table below shows the basic units of the metric system. Note that the names of all metric units follow from these three basic units.

Length	Mass	Volume
basic units		
meter	gram	liter
other units you may see		
kilometer	kilogram	dekaliter
centimeter	centigram	centiliter
millimeter	milligram	milliliter

In the metric system, the basic unit of length is the meter. A meter is slightly larger than a yardstick, or just over three feet.

The basic metric unit of mass is the gram. A regular-sized paperclip has a mass of about 1 gram.

Among scientists, one gram is defined as the mass of water that would fill a 1-centimeter cube. You may notice that the word “mass” is used here instead of “weight.” In the sciences and technical fields, a distinction is made between weight and mass. Weight is a measure of the pull of gravity on an object. For this reason, an object’s weight would be different if it was weighed on Earth or on the moon because of the difference in the gravitational forces. However, the object’s mass would remain the same in both places because mass measures the amount of substance in an object. As long as you are planning on only measuring objects on Earth, you can use mass/weight fairly interchangeably, but it is worth noting that there is a difference!

Finally, the basic metric unit of volume is the liter. A liter is slightly larger than a quart.

The handle of a shovel is about 1 meter.	A paperclip weighs about 1 gram.	A medium-sized container of milk is about 1 liter.

Though it is rarely necessary to convert between the customary and metric systems, sometimes it helps to have a mental image of how large or small some units are. The table below shows the relationship between some common units in both systems.

	Common Measurements in Customary and Metric Systems
Length	1 centimeter is a little less than half an inch.
	1.6 kilometers is about 1 mile.
	1 meter is about 3 inches longer than 1 yard.
Mass	1 kilogram is a little more than 2 pounds.
	28 grams is about the same as 1 ounce.
Volume	1 liter is a little more than 1 quart.
	4 liters is a little more than 1 gallon.

Prefixes in the Metric System

The metric system is a base 10 system. This means that each successive unit is 10 times larger than the previous one.

The names of metric units are formed by adding a prefix to the basic unit of measurement. To tell how large or small a unit is, you look at the **prefix**. To tell whether the unit is measuring length, mass, or volume, you look at the base.

Prefixes in the Metric System

kilo-	1,000 times **larger** than base unit
hecto-	100 times **larger** than base unit
deka-	10 times **larger** than base unit
meter, gram, liter	base units
deci-	10 times **smaller** than base unit
centi-	100 times **smaller** than base unit
milli-	1,000 times **smaller** than base unit

Using this table as a reference, you can see the following:

- A kilogram is 1,000 times larger than one gram (so $1 \text{ kilogram } = 1,000 \text{ grams}$).
- A centimeter is 100 times smaller than one meter (so $1 \text{ meter } = 100 \text{ centimeters}$).
- A dekaliter is 10 times larger than one liter (so $1 \text{ dekaliter } = 10 \text{ liters}$).

Here is a similar table that just shows the metric units of measurement for mass, along with their size relative to 1 gram (the base unit). The common abbreviations for these metric units have been included as well.

Measuring Mass in the Metric System

kilogram (kg)	$1,000\text{ grams}$
hectogram (hg)	100 grams
dekagram (dag)	10 grams
gram (g)	gram
decigram (dg)	0.1 gram
centigram (cg)	0.01 gram
milligram (mg)	0.001 gram

Since the prefixes remain constant through the metric system, you could create similar charts for length and volume. The prefixes have the same meanings whether they are attached to the units of length (meter), mass (gram), or volume (liter).

? Exercise

Which of the following sets of three units are all metric measurements of length?

A. inch, foot, yard
B. kilometer, centimeter, millimeter
C. kilogram, gram, centigram
D. kilometer, foot, decimeter

Answer

A. Incorrect. Although these units do measure length, they are all units of measurement from the U.S. customary system. The correct answer is kilometer, centimeter, millimeter.
B. Correct. All of these measurements are from the metric system. You can tell they are measurements of length because they all contain the word “meter.”
C. Incorrect. These measurements are from the metric system, but they are measurements of mass, not length. The correct answer is kilometer, centimeter, millimeter.
D. Incorrect. Kilometer and decimeter are metric units of length, but foot is not. The correct answer is kilometer, centimeter, millimeter.

Converting Units Up and Down the Metric Scale

Converting between metric units of measure requires knowledge of the metric prefixes and an understanding of the decimal system. That’s about it.

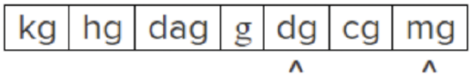
For instance, you can figure out how many centigrams are in one dekagram by using the table above. One dekagram is larger than one centigram, so you expect that one dekagram will equal many centigrams.

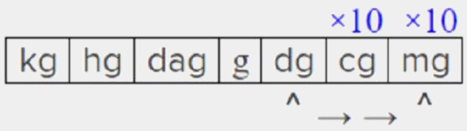
In the table, each unit is 10 times larger than the one to its immediate right. This means that $1\text{ dekagram }=10\text{ grams}$; $10\text{ grams }=100\text{ decigrams}$; and $100\text{ decigrams }=1,000\text{ centigrams}$. So, $1\text{ dekagram }=1,000\text{ centigrams}$.

✓ Example

How many milligrams are in one decigram?

Solution

 kg | hg | dag | g | dg | cg | mg ^ ^	Identify locations of milligrams and decigrams. Decigrams (dg) are larger than milligrams (mg), so you expect there to be many mg in one dg.

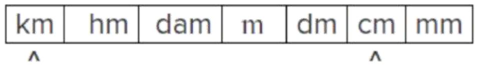
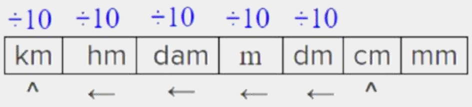
 ×10 ×10 kg hg dag g dg cg mg ^ → → ^	Dg is 10 times larger than a cg, and a cg is 10 times larger than a mg.
$1\text{dg}\cdot 10\cdot 10=100\text{mg}$	Since you are going from a larger unit to a smaller unit, multiply. Multiply: $1\cdot 10\cdot 10$, to find the number of milligrams in one decigram.

There are 100 milligrams (mg) in 1 decigram (dg).

Example

Convert 1 centimeter to kilometers.

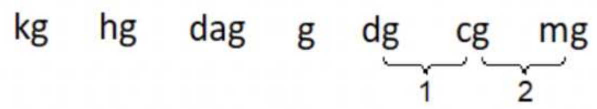
Solution

	Identify locations of kilometers and centimeters. Kilometers (km) are larger than centimeters (cm), so you expect there to be less than one km in a cm.
 ÷10 ÷10 ÷10 ÷10 ÷10 km hm dam m dm cm mm ^ ← ← ← ← ^	Cm is 10 times smaller than a dm; a dm is 10 times smaller than a m, etc. Since you are going from a smaller unit to a larger unit, divide.
$1\text{ centimeter }\div 10\div 10\div 10\div 10\div 10=0.00001\text{ kilometers}$	Divide: $1\div 10\div 10\div 10\div 10\div 10$ to find the number of kilometers in one centimeter.

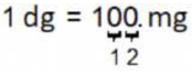
$1\text{ centimeter }(\text{cm})=0.00001\text{ kilometers }(\text{km})$.

Once you begin to understand the metric system, you can use a shortcut to convert among different metric units. The size of metric units increases tenfold as you go up the metric scale. The decimal system works the same way: a tenth is 10 times larger than a hundredth; a hundredth is 10 times larger than a thousandth, etc. By applying what you know about decimals to the metric system, converting among units is as simple as moving decimal points.

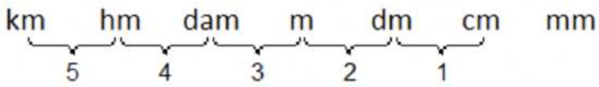
Here is the first problem from above: How many milligrams are in one decigram? You can recreate the order of the metric units as shown below:



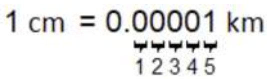
This question asks you to start with 1 decigram and convert that to milligrams. As shown above, milligrams is two places to the right of decigrams. You can just move the decimal point two places to the right to convert decigrams to milligrams:



The same method works when you are converting from a smaller to a larger unit, as in the problem: Convert 1 centimeter to kilometers.



Note that instead of moving to the right, you are now moving to the left, so the decimal point must do the same:



Exercise

How many milliliters are in 1 liter?

A. 0.001
B. 0.1
C. 100
D. 1,000

Answer

A. Incorrect. One liter is larger than one milliliter, so you would expect there to be more than one milliliter in 1 liter. The correct answer is 1,000.
B. Incorrect. One liter is larger than one milliliter, so you would expect there to be more than one milliliter in 1 liter. The correct answer is 1,000.
C. Incorrect. There are 100 milliliters in 1 deciliter. The correct answer is 1,000.
D. Correct. There are 10 milliliters in a centiliter, 10 centiliters in a deciliter, and 10 deciliters in a liter. Multiply: $10 \cdot 10 \cdot 10$, to find the number of milliliters in a liter, 1,000.

Summary

The metric system is the system of measurement used in most countries, and is an alternative system in the United States as well. The metric system is based on joining one of a series of prefixes, including kilo-, hecto-, deka-, deci-, centi-, and milli-, with a base unit of measurement, such as meter, liter, or gram. Units in the metric system are all related by a power of 10, which means that each successive unit is 10 times larger than the previous one. This makes converting one metric measurement to another a straightforward process, and is often as simple as moving a decimal point. It is always important, though, to consider the direction of the conversion. If you are converting a smaller unit to a larger unit, then the decimal point has to move to the left (making your number smaller); if you are converting a larger unit to a smaller unit, then the decimal point has to move to the right (making your number larger).

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6.2.2: Converting within the Metric System

Learning Objectives

- Perform arithmetic calculations on metric units of length, mass, and volume.

Introduction

While knowing the different units used in the metric system is important, the real purpose behind learning the metric system is for you to be able to use these measurement units to calculate the size, mass, or volume of different objects. In practice, it is often necessary to convert one metric measurement to another unit. This happens frequently in the medical, scientific, and technical fields, where the metric system is commonly used.

If you have a prescription for 5,000 milligrams of medicine, and upon getting it filled, the dosage reads 5 grams of medicine, did the pharmacist make a mistake?

For a moment, imagine that you are a pharmacist. You receive three prescriptions for liquid amoxicillin: one calls for 2.5 centiliters, one calls for 0.3 deciliters, and one calls for 450 milliliters. Amoxicillin is stored in the refrigerator in 1 liter, 1 deciliter, and 1 centiliter containers. Which container should you use to ensure you are not wasting any of the unused drug?

To solve this problem, you need to know how to convert from one measurement to another as well as how to add different quantities together. Let's take a look at how to do this.

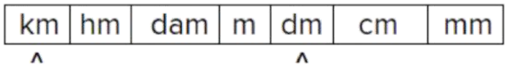
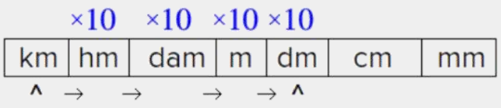
Converting from Larger to Smaller Units

Converting between measurements in the metric system is simply a matter of identifying the unit that you have, the unit that you want to convert to, and then counting the number of units between them. A basic example of this is shown below.

Example

Convert 1 kilometer to decimeters.

Solution

	Identify locations of kilometers and decimeters. Kilometers (km) are larger than decimeters (dm), so you expect there to be more than one dm in a km.
 ×10 ×10 ×10 ×10 km hm dam m dm cm mm ^ → → → → ^	Count the intermediate units, multiplying by 10 as you go.
$1 \text{ kilometer } \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10,000 \text{ decimeters}$	(Since you are going from a larger unit to a smaller unit, you multiply.) Multiply to find the number of decimeters in one kilometer.

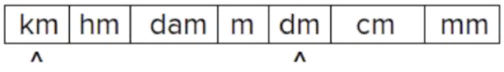
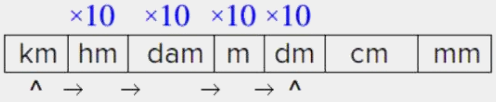
$1 \text{ kilometer } = 10,000 \text{ decimeters}$

This problem is straightforward because you are converting 1 kilometer to another unit. The example below shows how you would solve this problem if you were asked to convert 8.2 *kilometers* to *decimeters*. Notice that most steps are the same; the critical difference is that you multiply by 8.2 in the final step.

Example

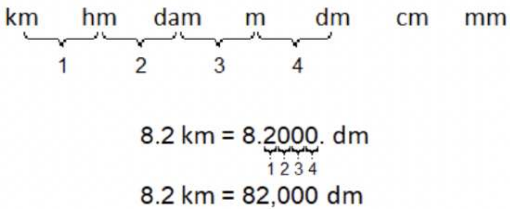
Convert 8.2 kilometers to decimeters.

Solution

	Identify locations of kilometers and decimeters. Kilometers (km) are larger than decimeters (dm), so you expect there to be more than one dm in a km.
 ×10 ×10 ×10 ×10 km hm dam m dm cm mm ^ → → → → ^	Count the intermediate units, multiplying by 10 as you go.
$8.2\text{ kilometers }\cdot 10\cdot 10\cdot 10\cdot 10 = 82,000\text{ decimeters}$	Since you are going from a larger unit to a smaller unit, multiply. Multiply to find the number of decimeters in 8.2 kilometers.

$8.2\text{ kilometers } = 82,000\text{ decimeters}$

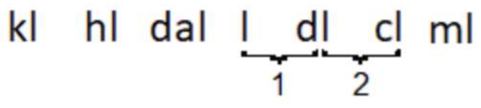
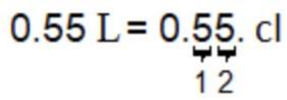
You can also apply the rules of base 10 to use the “move the decimal” shortcut method in this example. Notice how decimeters (dm) is four places to the right of kilometers (km); similarly, you move the decimal point four places to the right when converting 8.2 kilometers to decimeters.



Example

Convert 0.55 liters to centiliters.

Solution

	Count two places from liters to centiliters.
	In 0.55 liters, move the decimal point two places to the right.

$0.55\text{ liters } = 55\text{ centiliters}$

Exercise

How many dekaliters are in 0.5 deciliters?

A. 500
B. 5
C. 0.5
D. 0.005

Answer

A. Incorrect. A dekaliter is larger than a deciliter, so you would expect the number of dekaliters in 0.5 deciliters to be smaller than 0.5. The correct answer is 0.005.
B. Incorrect. A dekaliter is larger than a deciliter, so you would expect the number of dekaliters in 0.5 deciliters to be smaller than 0.5. The correct answer is 0.005.

C. Incorrect. Deciliters and dekaliters are different units of measurement, so you would not expect 0.5 deciliters to equal 0.5 dekaliters. The correct answer is 0.005.
D. Correct. One deciliter is 100 times smaller than a dekaliter, so you move the decimal point two places to the left to convert 0.5 deciliters to 0.005 dekaliters.

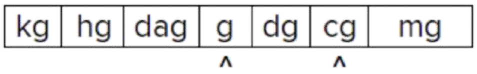
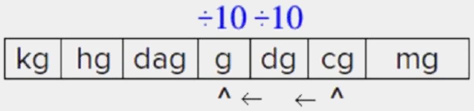
Converting from Smaller to Larger Units

You can use similar processes when converting from smaller to larger units. When converting a larger unit to a smaller one, you multiply; when you convert a smaller unit to a larger one, you divide. Here is an example.

✓ Example

Convert 739 centigrams to grams.

Solution

	Identify locations of centigrams and grams. Centigrams (cg) are smaller than grams (g), so you expect there to be less than 739 grams in 739 centigrams.
 ÷10 ÷10 kg hg dag g dg cg mg ^ ← ← ^	Count the intermediate units, dividing by 10 as you go.
$739 \div 10 \div 10 = 7.39 \text{ grams}$	Since you are going from a smaller unit to a larger unit, divide. Divide to find the number of grams in 739 centigrams.

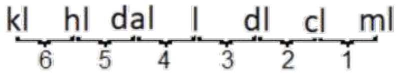
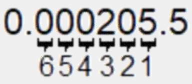
$739 \text{ centigrams } = 7.39 \text{ grams}$

Notice that the shortcut method of counting prefixes and moving the decimal the same number of places also works here. Just make sure you are moving the decimal point in the correct direction for the conversion.

✓ Example

Convert 205.5 milliliters to kiloliters.

Solution

 kl hl dal l dl cl ml 6 5 4 3 2 1	Count six places from milliliters to kiloliters.
	Milliliters is smaller than kiloliters, so you expect the number 205.5 to get smaller as you move up the metric chart.
$205.5 \text{ milliliters } = 0.0002055 \text{ kiloliters}$	In 205.5 milliliters, move the decimal point six places to the left.

$205.5 \text{ milliliters } = 0.0002055 \text{ kiloliters}$

? Exercise

Convert 3,085 milligrams to grams.

A. $3,085,000 \text{ grams}$
B. 308.5 grams
C. 3.085 grams
D. 0.3085 grams

Answer

A. Incorrect. Grams are larger than milligrams, so you would expect the number of grams in 3,085 milligrams to be less than 3,085. The correct answer is 3,085 grams.
B. Incorrect. One gram is more than 10 times larger than a milligram, so you would expect the number of grams to be less than 308.5. The correct answer is 3,085 grams.
C. Correct. One gram is 1,000 times larger than a milligram, so you can move the decimal point in 3,085 three places to the left.
D. Incorrect. This is too small; one gram is 1,000, not 10,000, times larger than a milligram. The correct answer is 3,085 grams.

Factor Label Method

There is yet another method that you can use to convert metric measurements: the **factor label method**. You used this method when you were converting measurement units within the U.S. customary system.

The factor label method works the same in the metric system; it relies on the use of unit fractions and the cancelling of intermediate units. The table below shows some of the **unit equivalents** and **unit fractions** for length in the metric system. (You should notice that all of the unit fractions contain a factor of 10. Remember that the metric system is based on the notion that each unit is 10 times larger than the one that came before it.)

Also, notice that two new prefixes have been added here: mega- (which is very big) and micro- (which is very small).

Unit Equivalents	Conversion Factors	
$1 \text{ meter } = 1,000,000 \text{ micrometers}$	$\frac{1 \text{ m}}{1,000,000 \mu\text{m}}$	$\frac{1,000,000 \mu\text{m}}{1 \text{ m}}$
$1 \text{ meter } = 1,000 \text{ millimeters}$	$\frac{1 \text{ m}}{1,000 \text{ mm}}$	$\frac{1,000 \text{ mm}}{1 \text{ m}}$
$1 \text{ meter } = 100 \text{ centimeters}$	$\frac{1 \text{ m}}{100 \text{ cm}}$	$\frac{100 \text{ cm}}{1 \text{ m}}$
$1 \text{ meter } = 10 \text{ decimeters}$	$\frac{1 \quad \text{m}}{10 \quad \text{dm}}$	$\frac{10 \text{ dm}}{1 \text{ m}}$
$1 \text{ dekameter } = 10 \text{ meters}$	$\frac{1 \text{ dam}}{10 \text{ m}}$	$\frac{10 \text{ m}}{1 \text{ dam}}$
$1 \text{ hectometer } = 100 \text{ meters}$	$\frac{1 \text{ hm}}{100 \text{ m}}$	$\frac{100 \text{ m}}{1 \text{ hm}}$
$1 \text{ kilometer } = 1,000 \text{ meters}$	$\frac{1 \text{ km}}{1,000 \text{ m}}$	$\frac{1,000 \text{ m}}{1 \text{ km}}$
$1 \text{ megameter } = 1,000,000 \text{ meters}$	$\frac{1 \text{ Mm}}{1,000,000 \text{ m}}$	$\frac{1,000,000 \text{ m}}{1\text{Mm}}$

When applying the factor label method in the metric system, be sure to check that you are not skipping over any intermediate units of measurement!

Example

Convert 7,225 centimeters to meters.

Solution

$7,225 \text{ centimeters } =? \text{ meters}$	Meters is larger than centimeters, so you expect your answer to be less than 7,225.
$\frac{7,225 \text{ centimeters}}{1} \cdot \frac{1 \text{ meter}}{100 \text{ centimeters}} =? \text{ meters}$	Using the factor label method, write 7,225 centimeters as a fraction and use unit fractions to convert it to meters.
$\frac{7,225. \cancel{\text{cm}}}{1} \cdot \frac{1 \text{ m}}{100 \cancel{\text{cm}}} =? \text{ m}$ $\frac{7,225}{1} \cdot \frac{1 \text{ meter}}{100} = \frac{7,225 \text{ meters}}{100}$ $\frac{7,225 \text{ meters}}{100} = 72.25 \text{ meters}$	Cancel similar units, multiply, and simplify.

$7,225 \text{ centimeters } = 72.25 \text{ meters}$

Exercise

Using whichever method you prefer, convert 32.5 kilometers to meters.

A. $32,500$ meters
B. 325 meters
C. 0.325 meters
D. 0.00325 meters

Answer

A. Correct. To find the number of meters in 32.5 kilometers, you can set up the following equation: $\frac{32.5 \text{ kilometers}}{1} \cdot \frac{1{,}000 \text{ meters}}{1 \text{ kilometer}} = \frac{32{,}500 \text{ meters}}{1}$. The kilometers units cancel, leaving the answer in meters.
B. Incorrect. A kilometer is more than 10 times the size of a meter; look at the unit fractions and try your calculations again. The correct answer is 32,500 meters.
C. Incorrect. A kilometer is larger than a meter, so you would expect the number of meters in 32.5 kilometers to be more than 32.5. Look at the unit fractions and try your calculations again. The correct answer is 32,500 meters.
D. Incorrect. A kilometer is larger than a meter, so you would expect the number of meters in 32.5 kilometers to be more than 32.5. Look at the unit fractions and try your calculations again. The correct answer is 32,500 meters.

Now that you have seen how to convert among metric measurements in multiple ways, let's revisit the problem posed earlier.

Example

If you have a prescription for 5,000 milligrams of medicine, and upon getting it filled, the dosage reads 5 grams of medicine, did the pharmacist make a mistake?

Solution

$5,000 \text{ milligrams } =? \text{ grams}$	Need to convert milligrams to grams.
$\frac{5{,}000 \text{ milligrams}}{1} \cdot \frac{1 \text{ gram}}{1{,}000 \text{ milligrams}} =? \text{ grams}$	
$\frac{5{,}000 \ \cancel{\text{mg}}}{1} \cdot \frac{1 \text{ g}}{1{,}000 \ \cancel{\text{mg}}} =? \text{ g}$ $\frac{5{,}000 \cdot 1 \text{ gram}}{1 \cdot 1{,}000} = \frac{5{,}000 \text{ grams}}{1{,}000}$ $\frac{5{,}000 \text{ grams}}{1{,}000} = 5 \text{ grams}$	

$5 \text{ grams } = 5,000 \text{ milligrams}$, so the pharmacist did not make a mistake.

Summary

To convert among units in the metric system, identify the unit that you have, the unit that you want to convert to, and then count the number of units between them. If you are going from a larger unit to a smaller unit, you multiply by 10 successively. If you are going from a smaller unit to a larger unit, you divide by 10 successively. The factor label method can also be applied to conversions within the metric system. To use the factor label method, you multiply the original measurement by unit fractions; this allows you to represent the original measurement in a different measurement unit.

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6.2.3: Using Metric Conversions to Solve Problems

Learning Objectives

- Solve application problems involving metric units of length, mass, and volume.

Introduction

Learning how to solve real-world problems using metric conversions is as important as learning how to do the conversions themselves. Mathematicians, scientists, nurses, and even athletes are often confronted with situations where they are presented with information using metric measurements, and must then make informed decisions based on that data.

To solve these problems effectively, you need to understand the context of a problem, perform conversions, and then check the reasonableness of your answer. Do all three of these steps and you will succeed in whatever measurement system you find yourself using.

Understanding Context and Performing Conversions

The first step in solving any real-world problem is to understand its context. This will help you figure out what kinds of solutions are reasonable (and the problem itself may give you clues about what types of conversions are necessary). Here is an example.

Example

In the Summer Olympic Games, athletes compete in races of the following lengths: 100 meters, 200 meters, 400 meters, 800 meters, 1500 meters, 5000 meters and 10,000 meters. If a runner were to run in all these races, how many kilometers would he run?

Solution

$\begin{array}{r} 10,000 \\ 5,000 \\ 1,500 \\ 800 \\ 400 \\ 200 \\ + \quad 100 \\ \hline 18,000 \end{array}$	To figure out how many kilometers he would run, you need to first add all of the lengths of the races together and *then* convert that measurement to kilometers.
$\frac{18{,}000 \text{ meters}}{1} \cdot \frac{1 \text{ kilometer}}{1{,}000 \text{ meters}} =? \text{ kilometers}$	Use the factor label method and unit fractions to convert from meters to kilometers.
$\frac{18{,}000 \cancel{\text{m}}}{1} \cdot \frac{1 \text{ km}}{1{,}000 \cancel{\text{m}}} =? \text{ km}$ $\frac{18{,}000}{1} \cdot \frac{1 \text{ kilometer}}{1{,}000} = \frac{18{,}000 \text{ kilometers}}{1{,}000}$ $\frac{18{,}000 \text{ kilometers}}{1{,}000} = 18 \text{ kilometers}$	Cancel, multiply, and solve.

The runner would run 18 kilometers.

This may not be likely to happen (a runner would have to be quite an athlete to compete in all of these races) but it is an interesting question to consider. The problem required you to find the total distance that the runner would run (in kilometers). The example showed how to add the distances, in meters, and then convert that number to kilometers.

An example with a different context, but still requiring conversions, is shown below.

Example

One bottle holds 295 deciliters while another one holds 28,000 milliliters. What is the difference in capacity between the two bottles?

Solution

$\text{295 deciliters } = \text{ ? liters}$ $\text{28,000 milliliters } =\text{? liters}$	The two measurements are in different units. You can convert both units to liters and then compare them.
$\frac{\text{295 deciliters}}{1}\cdot\frac{\text{1 liter}}{\text{10 deciliters}} =\text{? liters}$	Convert deciliters to liters.
$\frac{\text{295 dl}}{1}\cdot\frac{\text{1 L}}{\text{10 dl}} =\text{? L}$	Cancel similar units and multiply.
$\frac{295}{1}\cdot\frac{\text{1 liter}}{10} = \frac{\text{295 liters}}{10}$ $\frac{\text{295 liters}}{10} = \text{29.5 liters}$	$\text{295 deciliters } = \text{29.5 liters}$
$\frac{\text{28,000 milliliters}}{1}\cdot\frac{\text{1 liter}}{\text{1,000 milliliters}} =\text{? liters}$	Convert milliliters to liters.
$\frac{\text{28,000 }\cancel{\text{ml}}}{1}\cdot\frac{\text{1 L}}{\text{1,000 }\cancel{\text{ml}}} =\text{? L}$ $\frac{\text{28,000}}{1}\cdot\frac{\text{1 liter}}{\text{1,000}} = \frac{\text{28,000 liters}}{\text{1,000}}$ $\frac{\text{28,000 liters}}{\text{1,000}} = \text{28 liters}$	$\text{28,000 milliliters } = \text{28 liters}$
$\text{29.5 liters } - \text{28 liters } = \text{1.5 liters}$	The question asks for "difference in capacity" between the bottles.

There is a difference in capacity of 1.5 liters between the two bottles.

This problem asked for the difference between two quantities. The easiest way to find this is to convert one quantity so that both quantities are measured in the same unit, and then subtract one from the other.

? Exercise

One boxer weighs in at 85 kilograms. He is 80 dekagrams heavier than his opponent. How much does his opponent weigh?

A. 5 kilograms
B. 84.2 kilograms
C. 84.92 kilograms
D. 85.8 kilograms

Answer

A. Incorrect. Look at the unit labels. The boxer is 80 dekagrams heavier, not 80 kilograms heavier. The correct answer is 84.2 kilograms.
B. Correct. $\text{80 dekagrams } = \text{0.8 kilograms}$, and $85 - 0.8 = 84.2$.
C. Incorrect. This would have been true if the difference in weight was 8 dekagrams, not 80 dekagrams. The correct answer is 84.2 kilograms.
D. Incorrect. The first boxer is 80 dekagrams *heavier*, not *lighter* than his opponent. This question asks for the opponent's weight. The correct answer is 84.2 kilograms.

Checking your Conversions

Sometimes it is a good idea to check your conversions using a second method. This usually helps you catch any errors that you may make, such as using the wrong unit fractions or moving the decimal point the wrong way.

✓ Example

A two-liter bottle contains 87 centiliters of oil and 4.1 deciliters of water. How much more liquid is needed to fill the bottle?

Solution

$\text{87 centiliters } + \text{4.1 deciliters } +? = \text{2 liters}$	You are looking for the amount of liquid needed to fill the bottle. Convert both measurements to liters and then solve the problem.

$87 \text{ centiliters } = \text{ ? liters}$ $\frac{87 \text{ centiliters}}{1} \cdot \frac{1 \text{ liters}}{100 \text{ centiliters}} =? \text{ liters}$ $\frac{87 \cancel{\text{cl}}}{1} \cdot \frac{1 \text{ L}}{100 \cancel{\text{cl}}} =? \text{ L}$ $\frac{87}{1} \cdot \frac{1 \text{ liter}}{100} = \frac{87 \text{ liters}}{100}$ $\frac{87 \text{ liters}}{100} = 0.87 \text{ liters}$	Convert 87 centiliters to liters.
$4.1 \text{ deciliters } = \text{ ? liters}$ $\frac{4.1 \text{ deciliters}}{1} \cdot \frac{1 \text{ liter}}{10 \text{ deciliters}} =? \text{ liters}$ $\frac{4.1 \cancel{\text{dl}}}{1} \cdot \frac{1 \text{ L}}{10 \cancel{\text{dl}}} =? \text{ L}$ $\frac{4.1}{1} \cdot \frac{1 \text{ liter}}{10} = \frac{4.1 \text{ liters}}{10}$ $\frac{4.1 \text{ liters}}{10} = 0.41 \text{ liters}$	Convert 4.1 deciliters to liters.
$87 \text{ centiliters } + 4.1 \text{ deciliters } +? = 2 \text{ liters}$ $0.87 \text{ liter } + 0.41 \text{ liter } +? = 2 \text{ liters}$ $2 \text{ liters } - 0.87 \text{ liter } - 0.41 \text{ liter } = 0.72 \text{ liter}$	Subtract to find how much more liquid is needed to fill the bottle.

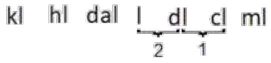
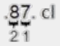
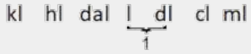
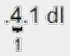
The amount of liquid needed to fill the bottle is 0.72 liter.

Having come up with the answer, you could also check your conversions using the quicker “move the decimal” method, shown below.

✓ Example

A two-liter bottle contains 87 centiliters of oil and 4.1 deciliters of water. How much more liquid is needed to fill the bottle?

Solution

$87 \text{ centiliters } + 4.1 \text{ deciliters } +? = 2 \text{ liters}$	You are looking for the amount of liquid needed to fill the bottle.
$87 \text{ centiliters } = \text{ ? liters}$	Convert 87 centiliters to liters.
	On the chart, liters (l) are two places to the left of centiliters (cl).
 $87 \text{ centiliters } = 0.87 \text{ liters}$	Move the decimal point two places to the left in 87 centiliters.
$4.1 \text{ deciliters } = \text{ ? liters}$	Convert 4.1 deciliters to liters.
 	On the chart, liter (l) is one place to the left of deciliter (dl).
$4.1 \text{ deciliters } = 0.41 \text{ liters}$	Move the decimal point one place to the left in 4.1 deciliters.
$87 \text{ centiliters } + 4.1 \text{ deciliters } +? = 2 \text{ liters}$ $0.87 \text{ liter } + 0.41 \text{ liter } +? = 2 \text{ liters}$ $2 \text{ liters } - 0.87 \text{ liter } - 0.41 \text{ liter } = 0.72 \text{ liter}$	Subtract to find how much more liquid is needed to fill the bottle.

The amount of liquid needed to fill the bottle is 0.72 liters.

The initial answer checks out. 0.72 liter of liquid is needed to fill the bottle. Checking one conversion with another method is a good practice for catching any errors in scale.

Summary

Understanding the context of real-life application problems is important. Look for words within the problem that help you identify what operations are needed, and then apply the correct unit conversions. Checking your final answer by using another conversion method (such as the “move the decimal” method, if you have used the factor label method to solve the problem) can cut down on errors in your calculations.

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SECTION OVERVIEW

6.3: Temperature

Topic hierarchy

6.3.1: Temperature Scales

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6.3.1: Temperature Scales

Learning Objectives

- State the freezing and boiling points of water on the Celsius and Fahrenheit temperature scales.
- Convert from one temperature scale to the other, using conversion formulas.

Introduction

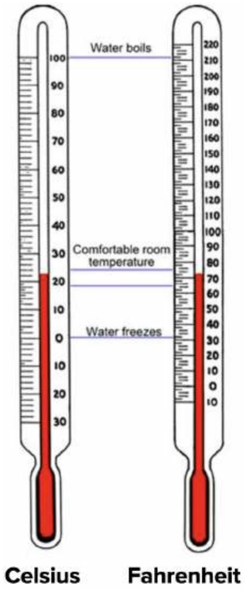
Turn on the television any morning and you will see meteorologists talking about the day’s weather forecast. In addition to telling you what the weather conditions will be like (sunny, cloudy, rainy, muggy), they also tell you the day’s forecast for high and low temperatures. A hot summer day may reach 100° in Philadelphia, while a cool spring day may have a low of 40° in Seattle.

If you have been to other countries, though, you may notice that meteorologists measure heat and cold differently outside of the United States. For example, a TV weatherman in San Diego may forecast a high of 89°, but a similar forecaster in Tijuana, Mexico, which is only 20 miles south, may look at the same weather pattern and say that the day’s high temperature is going to be 32°. What’s going on here?

The difference is that the two countries use different temperature scales. In the United States, temperatures are usually measured using the **Fahrenheit** scale, while most countries that use the metric system use the **Celsius** scale to record temperatures. Learning about the different scales, including how to convert between them, will help you figure out what the weather is going to be like, no matter which country you find yourself in.

Measuring Temperature on Two Scales

Fahrenheit and Celsius are two different scales for measuring temperature.

A thermometer measuring a temperature of 22° Celsius is shown here.	 Water boils Comfortable room temperature Water freezes Celsius Fahrenheit	A thermometer measuring a temperature of 72° Fahrenheit shown here.
On the Celsius scale, water freezes at 0° and boils at 100°.		On the Fahrenheit scale, water freezes at 32° and boils at 212°.
If the United States were to adopt the Celsius scale, forecast temperatures would rarely go below -30° or above 45°. (A temperature of -18° may be forecast for a cold winter day in Michigan, while a temperature of 43° may be predicted for a hot summer day in Arizona.)		In the United States, forecast temperatures measured in Fahrenheit rarely go below -20° or above 120°. (A temperature of 0° may be forecast for a cold winter day in Michigan, while a temperature of 110° may be predicted for a hot summer day in Arizona.)
Most office buildings maintain an indoor temperature between 18° Celsius and 24° Celsius to keep employees comfortable.		Most office buildings maintain an indoor temperature between 65° Fahrenheit and 75° Fahrenheit to keep employees comfortable.

Exercise

A cook puts a thermometer into a pot of water to see how hot it is. The thermometer reads 132°, but the water is not boiling yet. Which temperature scale is the thermometer measuring?

A. Celsius
B. Fahrenheit

Answer

A. Incorrect. On the Celsius scale, water boils at 100^{o}, so if the water is not boiling and the measurement is over 100^{o}, then it cannot be Celsius. The correct answer is Fahrenheit.
B. Correct. Water boils at 212^{o} on the Fahrenheit scale, so a measurement of 132^{o} on a Fahrenheit scale is legitimate for hot (but non-boiling) water.

Converting Between the Scales

By looking at the two thermometers shown, you can make some general comparisons between the scales. For example, many people tend to be comfortable in outdoor temperatures between 50^{o} Fahrenheit and 80^{o} Fahrenheit (or between 10^{o} Celsius and 25^{o} Celsius). If a meteorologist predicts an average temperature of 0^{o} Celsius (or 32^{o} Fahrenheit), then it is a safe bet that you will need a winter jacket.

Sometimes, it is necessary to convert a Celsius measurement to its exact Fahrenheit measurement or vice versa. For example, what if you want to know the temperature of your child in Fahrenheit, and the only thermometer you have measures temperature in Celsius measurement? Converting temperature between the systems is a straightforward process as long as you use the formulas provided below.

Temperature Conversion Formulas

To convert a Fahrenheit measurement to a Celsius measurement, use this formula.

$C=\frac{5}{9}(F-32)$

To convert a Celsius measurement to a Fahrenheit measurement, use this formula.

$F=\frac{9}{5}C+32$

How were these formulas developed? They came from comparing the two scales. Since the freezing point is 0^{o} on the Celsius scale and 32^{o} on the Fahrenheit scale, we subtract 32 when converting from Fahrenheit to Celsius, and add 32 when converting from Celsius to Fahrenheit.

There is also a reason for the fractions $\frac{5}{9}$ and $\frac{9}{5}$. There are 100 degrees between the freezing (0^{o}) and boiling points (100^{o}) of water on the Celsius scale and 180 degrees between the similar points (32^{o} and 212^{o}) on the Fahrenheit scale. Writing these two scales as a ratio, $\frac{F^{\circ}}{C^{\circ}}$, gives $\frac{180^{\circ}}{100^{\circ}}=\frac{180^{\circ}\div 20}{100^{\circ}\div 20}=\frac{9}{5}$. If you flip the ratio to be $\frac{C^{\circ}}{F^{\circ}}$, you get $\frac{100^{\circ}}{180^{\circ}}=\frac{100^{\circ}\div 20}{180^{\circ}\div 20}=\frac{5}{9}$. Notice how these fractions are used in the conversion formulas.

The example below illustrates the conversion of Celsius temperature to Fahrenheit temperature, using the boiling point of water, which is 100^{o} Celsius.

Example

The boiling point of water is 100^{o} Celsius. What temperature does water boil at in the Fahrenheit scale?

Solution

$F=\frac{9}{5}C+32$	A Celsius temperature is given. To convert it to the Fahrenheit scale, use the formula at the left.
$F=\frac{9}{5}(100)+32$ $F=\frac{900}{5}+32$	Substitute 100 for C and multiply.
$F=\frac{900\div 5}{5\div 5}+32$ $F=\frac{180}{1}+32$	Simplify $\frac{900}{5}$ by dividing numerator and denominator by 5.
$F=212$	Add 180+32

The boiling point of water is 212^{o} Fahrenheit.

Example

Water freezes at 32^o Fahrenheit. On the Celsius scale, what temperature is this?

Solution

$C=\frac{5}{9}(F-32)$	A Fahrenheit temperature is given. To convert it to the Celsius scale, use the formula at the left.
$C=\frac{5}{9}(32-32)$	Substitute 32 for F and subtract.
$C=\frac{5}{9}(0)$	Any number multiplied by 0 is 0.
$C=0$	

The freezing point of water is 0^o Celsius.

The two previous problems used the conversion formulas to verify some temperature conversions that were discussed earlier: the boiling and freezing points of water. The next example shows how these formulas can be used to solve a real-world problem using different temperature scales.

Example

Two scientists are doing an experiment designed to identify the boiling point of an unknown liquid. One scientist gets a result of 120^o Celsius; the other gets a result of 250^o Fahrenheit. Which temperature is higher and by how much?

Solution

What is the difference between 120^o Celsius and 250^o Fahrenheit?	One temperature is given in degrees Celsius, and the other is given in degrees Fahrenheit. To find the difference between them, we need to measure them on the same scale.
$F=\frac{9}{5}C+32$	Use the conversion formula to convert 120^o Celsius to degrees Fahrenheit. (You could convert 250^o Fahrenheit to degrees Celsius instead; this is explained in the text after this example.)
$F=\frac{9}{5}(120)+32$	Substitute 120 for C.
$F=\frac{1080}{5}+32$	Multiply.
$F=\frac{1080\div 5}{5\div 5}+32$	Simplify $\frac{1080}{5}$ by dividing numerator and denominator by 5.
$F=\frac{216}{1}+32$	Add $216+32$.
$F=248$	You have found that $120^{\circ}\text{C}=248^{\circ}\text{F}$.
$250^{\circ}-248^{\circ}=2^{\circ}\text{ Fahrenheit}$	To find the difference between 248^o Fahrenheit and 250^o Fahrenheit, subtract.

250^o Fahrenheit is the higher temperature by 2^o Fahrenheit.

You could have converted 250^o Fahrenheit to degrees Celsius instead, and then found the difference in the two measurements. (Had you done it this way, you would have found that 250^oF=121.1^oC, and that 121.1^o Celsius is 1.1^o Celsius higher than 120^o Celsius.) Whichever way you choose, it is important to compare the temperature measurements within the same scale, and to apply the conversion formulas accurately.

Exercise

Tatiana is researching vacation destinations, and she sees that the average summer temperature in Barcelona, Spain is around 26^o Celsius. What is the average temperature in degrees Fahrenheit?

A. 79° Fahrenheit
B. -3° Fahrenheit
C. 45° Fahrenheit
D. 58° Fahrenheit

Answer

A. Correct. Tatiana can find the Fahrenheit equivalent by solving the equation $F=\frac{9}{5}(26)+32$. The result is 78.8^{o} Fahrenheit, which rounds to 79^{o} Fahrenheit.
B. Incorrect. You used the wrong formula. To find the Fahrenheit equivalent, use the formula $F=\frac{9}{5}C+32$. The correct answer is 79^{o} Fahrenheit.
C. Incorrect. You misapplied the formula; try substituting 26 for C in the formula $F=\frac{9}{5}C+32$. The correct answer is 79^{o} Fahrenheit.
D. Incorrect. You misapplied the formula; try substituting 26 for C in the formula $F=\frac{9}{5}C+32$. The correct answer is 79^{o} Fahrenheit.

Summary

Temperature is often measured in one of two scales: the Celsius scale and the Fahrenheit scale. A Celsius thermometer will measure the boiling point of water at 100^{o} and its freezing point at 0^{o}; a Fahrenheit thermometer will measure the same events at 212^{o} for the boiling point of water and 32^{o} as its freezing point. You can use conversion formulas to convert a measurement made in one scale to the other scale.

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CHAPTER OVERVIEW

7: Geometry

7.1: Basic Geometric Concepts and Figures
  7.1.1: Figures in 1 and 2 Dimensions
  7.1.2: Properties of Angles
  7.1.3: Triangles
  7.1.4: The Pythagorean Theorem
7.2: Perimeter, Circumference, and Area
  7.2.1: Quadrilaterals
  7.2.2: Perimeter and Area
  7.2.3: Circles
7.3: Volume of Geometric Solids
  7.3.1: Solids

Thumbnail: A two-dimensional perspective projection of a sphere (CC BY-3.0; Geek3 via Wikipedia).

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SECTION OVERVIEW

7.1: Basic Geometric Concepts and Figures

Topic hierarchy

7.1.1: Figures in 1 and 2 Dimensions

7.1.2: Properties of Angles

7.1.3: Triangles

7.1.4: The Pythagorean Theorem

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7.1.1: Figures in 1 and 2 Dimensions

Learning Objectives

- Identify and define points, lines, line segments, rays, and planes.
- Classify angles as acute, right, obtuse, or straight.

Introduction

You use geometric terms in everyday language, often without thinking about it. For example, any time you say “walk along this line” or “watch out, this road quickly angles to the left” you are using geometric terms to make sense of the environment around you. You use these terms flexibly, and people generally know what you are talking about.

In the world of mathematics, each of these geometric terms has a specific definition. It is important to know these definitions, as well as how different figures are constructed, to become familiar with the language of geometry. Let’s start with a basic geometric figure: the plane.

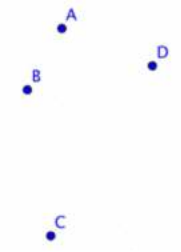
Figures on a Plane

A **plane** is a flat surface that continues forever (or, in mathematical terms, infinitely) in every direction. It has two dimensions: length and width.

You can visualize a plane by placing a piece of paper on a table. Now imagine that the piece of paper stays perfectly flat and extends as far as you can see in two directions, left-to-right and front-to-back. This gigantic piece of paper gives you a sense of what a geometric plane is like: it continues infinitely in two directions. (Unlike the piece of paper example, though, a geometric plane has no height.)

A plane can contain a number of geometric figures. The most basic geometric idea is a **point**, which has no dimensions. A point is simply a location on the plane. It is represented by a dot. Three points that don’t lie in a straight line will determine a plane.

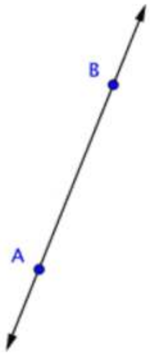
The image below shows four points, labeled A, B, C, and D.



Two points on a plane determine a line. A **line** is a one-dimensional figure that is made up of an infinite number of individual points placed side by side. In geometry, all lines are assumed to be straight; if they bend, they are called a curve. A line continues infinitely in two directions.

Below is line AB or, in geometric notation, $\overleftrightarrow{AB}$. The arrows indicate that the line keeps going forever in the two directions. This line could also be called line BA. While the order of the points does not matter for a line, it is customary to name the two points in alphabetical order.

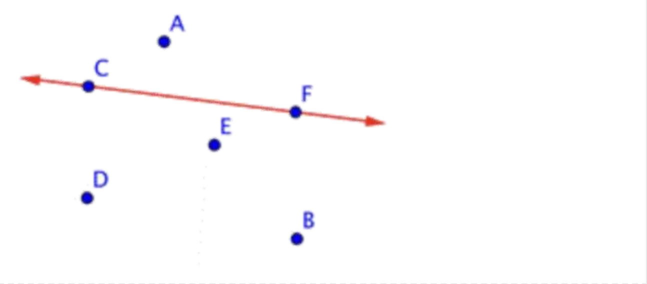
The image below shows the points A and B and the line $\overleftrightarrow{AB}$.



✓ Example

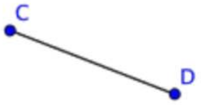
Name the line shown in red.

Solution

	The red line goes through the points C and F, so the line is $\overleftrightarrow{CF}$.

$\overleftrightarrow{CF}$

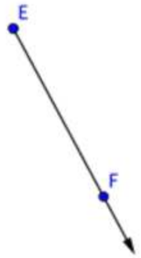
There are two more figures to consider. The section between any two points on a line is called a **line segment**. A line segment can be very long, very short, or somewhere in between. The difference between a line and a line segment is that the line segment has two endpoints and a line goes on forever. A line segment is denoted by its two endpoints, as in $\overline{CD}$.



A **ray** has one endpoint and goes on forever in one direction. Mathematicians name a ray with notation like $\overrightarrow{EF}$, where point E is the endpoint and F is a point on the ray. When naming a ray, we always say the endpoint first. Note that $\overrightarrow{FE}$ would have the endpoint at F, and continue through E, which is a different ray than $\overrightarrow{EF}$, which would have an endpoint at E, and continue through F.

The term “ray” may be familiar because it is a common word in English. “Ray” is often used when talking about light. While a ray of light resembles the geometric term “ray,” it does not go on forever, and it has some width. A geometric ray has no width; only length.

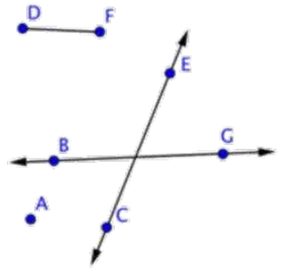
Below is an image of ray EF or $\overrightarrow{EF}$. Notice that the end point is E.



Example

Identify each line and line segment in the picture below.

Solution

	Two points define a line, and a line is denoted with arrows. There are two lines in this picture: $\overleftrightarrow{CE}$ and $\overleftrightarrow{BG}$. A line segment is a section between two points. $\overline{DF}$ is a line segment. But there are also two more line segments on the lines themselves: $\overline{CE}$ and $\overline{BG}$.

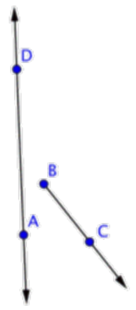
Lines: $\overleftrightarrow{CE}, \overleftrightarrow{BG}$

Line segments: $\overline{DF}, \overline{CE}, \overline{BG}$

Example

Identify each point and ray in the picture below.

Solution

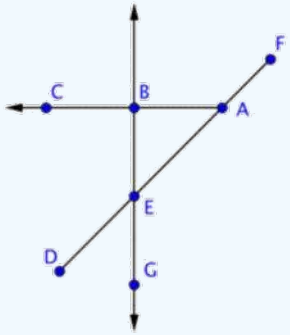
	There are four points: A, B, C and D. There are also three rays, though only one may be obvious. Ray $\overrightarrow{BC}$ begins at point B and goes through C. Two more rays exist on line $\overleftrightarrow{AD}$: they are $\overrightarrow{DA}$ and $\overrightarrow{AD}$.

Points: A, B, C, D

Rays: $\overrightarrow{BC}, \overrightarrow{AD}, \overrightarrow{DA}$

Exercise

Which of the following is *not* represented in the image below?



A. $\overleftrightarrow{BG}$
B. $\overrightarrow{BA}$
C. $\overline{DF}$
D. $\overrightarrow{AC}$

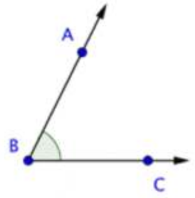
Answer

A. Incorrect. A line goes through points B and G, so $\overleftrightarrow{BG}$ is shown. $\overleftrightarrow{BA}$, is not shown in this image.
B. Correct. This image does not show any ray that begins at point B and goes through point A.
C. Incorrect. There is a line segment connecting points D and F, so $\overleftrightarrow{DF}$ is shown. $\overleftrightarrow{BA}$, is not shown in this image.
D. Incorrect. There is a ray beginning at point A and going through point C, so $\overrightarrow{AC}$ is shown. $\overrightarrow{BA}$, is not shown in this image.

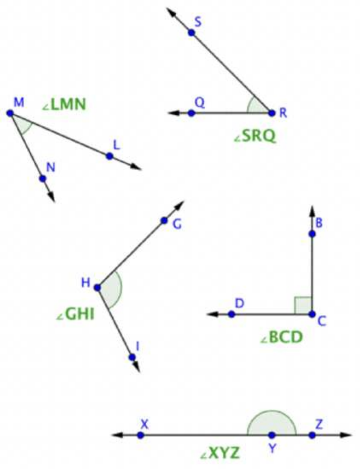
Angles

Lines, line segments, points, and rays are the building blocks of other figures. For example, two rays with a common endpoint make up an **angle**. The common endpoint of the angle is called the **vertex**.

The angle ABC is shown below. This angle can also be called $\angle ABC$, $\angle CBA$, or simply $\angle B$. When you are naming angles, be careful to include the vertex (here, point B as the middle letter.



The image below shows a few angles on a plane. Notice that the label of each angle is written "point-vertex-point," and the geometric notation is in the form $\angle ABC$.



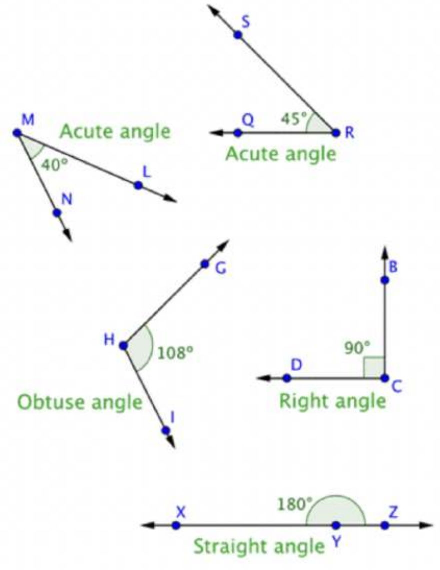
Sometimes angles are very narrow; sometimes they are very wide. When people talk about the “size” of an angle, they are referring to the arc between the two rays. The length of the rays has nothing to do with the size of the angle itself. Drawings of angles will often include an arc (as shown above) to help the reader identify the correct ‘side’ of the angle.

Think about an analog clock face. The minute and hour hands are both fixed at a point in the middle of the clock. As time passes, the hands rotate around the fixed point, making larger and smaller angles as they go. The length of the hands does not impact the angle that is made by the hands.

An angle is measured in degrees, represented by the degree symbol, which is a small circle at the upper right of a number. For example, a circle is defined as having 360^o. (In skateboarding and basketball, “doing a 360" refers to jumping and doing one complete body rotation.)

A **right angle** is any degree that measures exactly 90^o. This represents exactly one-quarter of the way around a circle. Rectangles contain exactly four right angles. A corner mark is often used to denote a right angle, as shown in right angle DCB below.

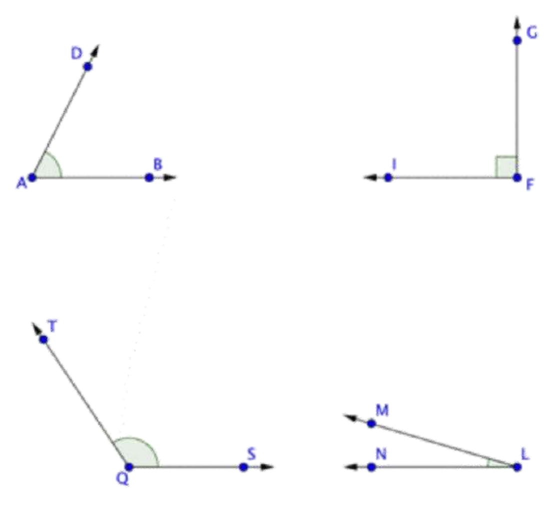
Angles that are between 0^o and 90^o (smaller than right angles) are called **acute angles**. Angles that are between 90^o and 180^o (larger than right angles and less than 180^o) are called **obtuse angles**. And an angle that measures exactly 180^o is called a **straight angle** because it forms a straight line!



✓ Example

Label each angle below as acute, right, or obtuse.

Solution

	You can start by identifying any right angles. $\angle GFI$ is a right angle, as indicated by the corner mark at vertex F. Acute angles will be smaller than $\angle GFI$ (or less than 90^{o}). This means that $\angle DAB$ and $\angle MLN$ are acute. $\angle TQS$ is larger than $\angle GFI$, so it is an obtuse angle.

$\angle DAB$ and $\angle MLN$ are acute angles.

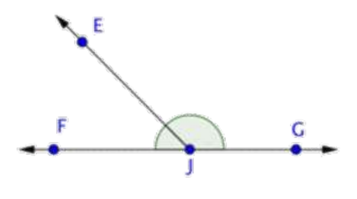
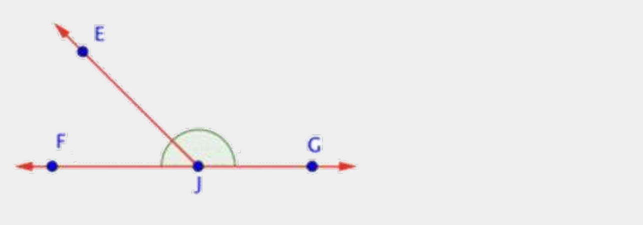
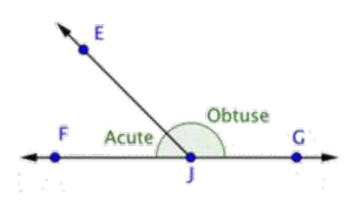
$\angle GFI$ is a right angle.

$\angle TQS$ is an obtuse angle.

✓ Example

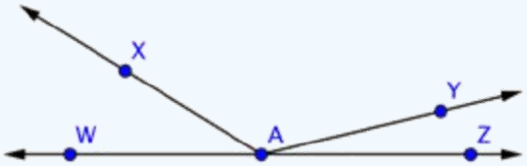
Identify each point, ray, and angle in the figure below.

Solution

	Begin by identifying each point in the figure. There are 4: $E, F, G, \text{ and } J$.
	Now find rays. A ray begins at one point, and then continues through another point towards infinity (indicated by an arrow). Three rays start at point J: $\overrightarrow{JE}$, $\overrightarrow{JF}$, and $\overrightarrow{JG}$. But also notice that a ray could start at point F and go through J and G, and another could start at point G and go through J and F. These rays can be represented by $\overrightarrow{GF}$ and $\overrightarrow{FG}$
	Finally, look for angles. $\angle EJG$ is obtuse, $\angle EJF$ is acute, and $\angle FJG$ is straight. (Don't forget those straight angles!)
Points: E, F, G, J Rays: $\overrightarrow{JE}, \overrightarrow{JG}, \overrightarrow{JF}, \overrightarrow{GF}, \overrightarrow{FG}$ Angles: $\angle EJG, \angle EJF, \angle FJG$	

Exercise

Identify the acute angles in the image below.



A. $\angle WAX, \angle XAY,$ and $\angle YAZ$
B. $\angle WAY$ and $\angle YAZ$
C. $\angle WAX$ and $\angle YAZ$
D. $\angle WAZ$ and $\angle XAY$

Answer

A. Incorrect. $\angle WAX$ and $\angle YAZ$ are both acute angles, but $\angle XAY$ is an obtuse angle. So only $\angle WAX$ and $\angle YAZ$ are acute angles.
B. Incorrect. $\angle YAZ$ is an acute angle, but $\angle WAY$ is an obtuse angle. Both $\angle WAX$ and $\angle YAZ$ are acute angles.
C. Correct. Both $\angle WAX$ and $\angle YAZ$ are acute angles.
D. Incorrect. $\angle WAZ$ is a straight angle, and $\angle XAY$ is an obtuse angle. Both $\angle WAX$ and $\angle YAZ$ are acute angles.

Measuring Angles with a Protractor

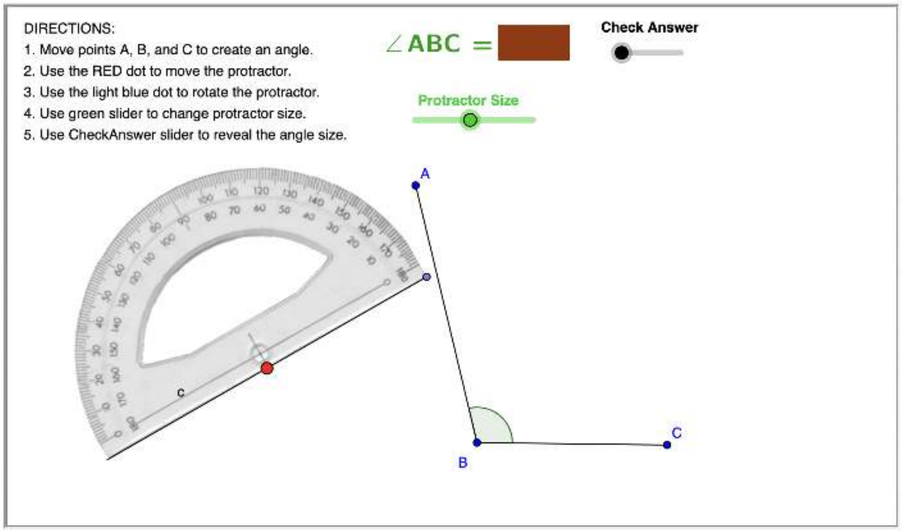
Learning how to measure angles can help you become more comfortable identifying the difference between angle measurements. For instance, how is a 135^o angle different from a 45^o angle?

Measuring angles requires a **protractor**, which is a semi-circular tool containing 180 individual hash marks. Each hash mark represents 1^o. (Think of it like this: a circle is 360^o, so a semi-circle is 180^o.) To use the protractor, do the following three steps:

1. line up the vertex of the angle with the dot in the middle of the flat side (bottom) of the protractor,
2. align one side of the angle with the line on the protractor that is at the zero degree mark, and
3. look at the curved section of the protractor to read the measurement.

Supplemental Interactive Activity

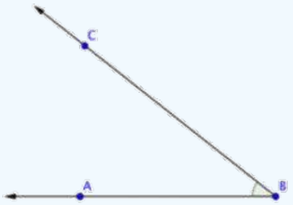
For practice using a protractor, try out the activity below:


DIRECTIONS:
1. Move points A, B, and C to create an angle.
2. Use the RED dot to move the protractor.
3. Use the light blue dot to rotate the protractor.
4. Use green slider to change protractor size.
5. Use CheckAnswer slider to reveal the angle size.

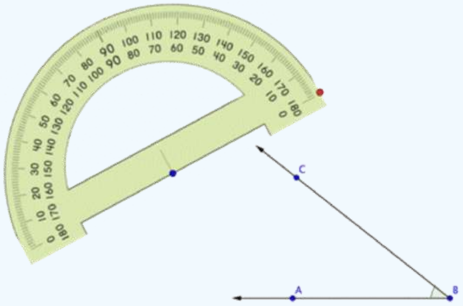
The example below shows you how to use a protractor to measure the size of an angle.

✓ Example

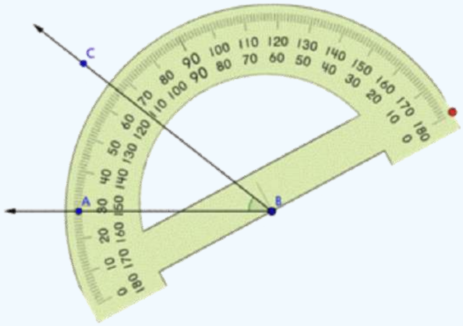
Use a protractor to measure the angle shown below.



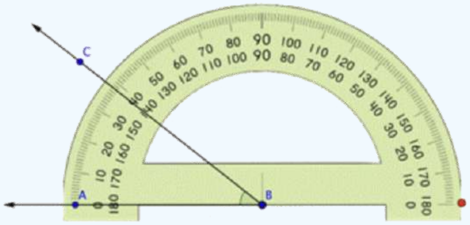
Solution



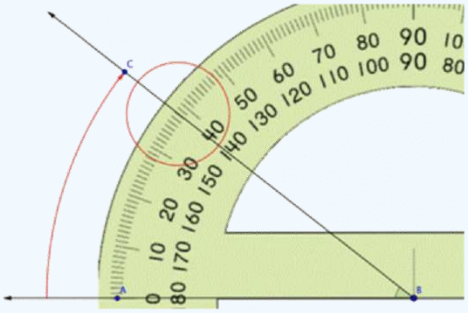
Use a protractor to measure the angle.



Align the blue dot on the protractor with the vertex of the angle you want to measure.



Rotate the protractor around the vertex of the angle until the side of the angle is aligned with the 0 degree mark of the protractor.

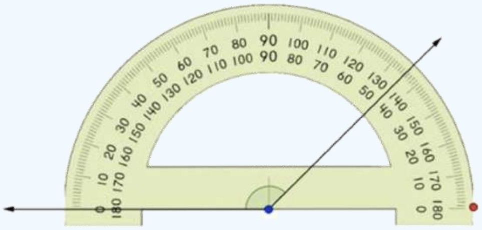


Read the measurement, in degrees, of the angle. Begin with the side of the angle that is aligned with the 0^o mark of the protractor and count up from 0^o. This angle measures 38^o.

The angle measures 38^o.

? Exercise

What is the measurement of the angle shown below?



A. 45^o
B. 135^o
C. 145^o
D. 180^o

Answer

A.
B.
C.
D.

Summary

Geometric shapes and figures are all around us. A point is a zero-dimensional object that defines a specific location on a plane. A line is made up of an infinite number of points, all arranged next to each other in a straight pattern, and going on forever. A ray begins at one point and goes on towards infinity in one direction only. A plane can be described as a two-dimensional canvas that goes on forever.

When two rays share an endpoint, an angle is formed. Angles can be described as acute, right, obtuse, or straight, and are measured in degrees. You can use a protractor (a special math tool) to closely measure the size of any angle.

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7.1.2: Properties of Angles

Learning Objectives

- Identify parallel and perpendicular lines.
- Find measures of angles.
- Identify complementary and supplementary angles.

Introduction

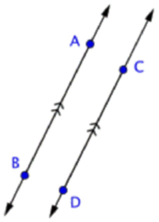
Imagine two separate and distinct lines on a plane. There are two possibilities for these lines: they will either intersect at one point, or they will never intersect. When two lines intersect, four angles are formed. Understanding how these angles relate to each other can help you figure out how to measure them, even if you only have information about the size of one angle.

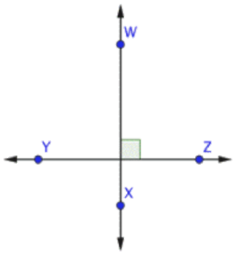
Parallel and Perpendicular

Parallel lines are two or more lines that never intersect. Likewise, parallel line segments are two line segments that never intersect even if the line segments were turned into lines that continued forever. Examples of parallel line segments are all around you, in the two sides of this page and in the shelves of a bookcase. When you see lines or structures that seem to run in the same direction, never cross one another, and are always the same distance apart, there's a good chance that they are parallel.

Perpendicular lines are two lines that intersect at a 90^o (right) angle. And perpendicular line segments also intersect at a 90^o (right) angle. You can see examples of perpendicular lines everywhere as well: on graph paper, in the crossing pattern of roads at an intersection, to the colored lines of a plaid shirt. In our daily lives, you may be happy to call two lines perpendicular if they merely seem to be at right angles to one another. When studying geometry, however, you need to make sure that two lines intersect at a 90^o angle before declaring them to be perpendicular.

The image below shows some parallel and perpendicular lines. The geometric symbol for parallel is ||, so you can show that $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$. Parallel lines are also often indicated by the marking >> on each line (or just a single > on each line). Perpendicular lines are indicated by the symbol $\perp$, so you can write $\overleftrightarrow{WX} \perp \overleftrightarrow{YZ}$.

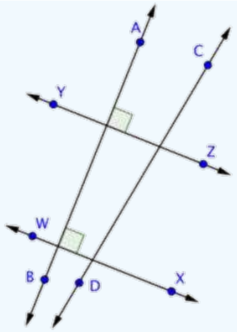




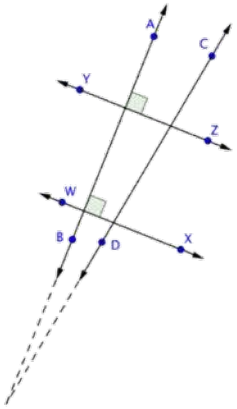
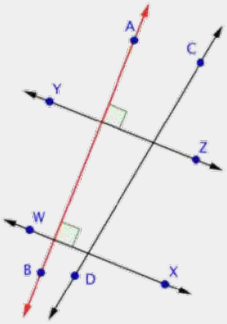
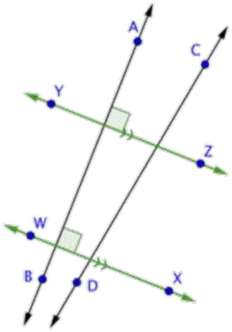
If two lines are parallel, then any line that is perpendicular to one line will also be perpendicular to the other line. Similarly, if two lines are both perpendicular to the same line, then those two lines are parallel to each other. Let's take a look at one example and identify some of these types of lines.

Example

Identify a set of parallel lines and a set of perpendicular lines in the image below.



Solution

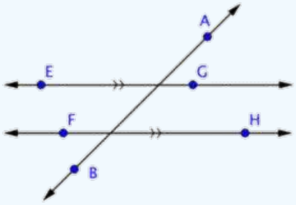
	Parallel lines never meet, and perpendicular lines intersect at a right angle. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ do not intersect in this image, but if you imagine extending both lines, they will intersect soon. So, they are neither parallel nor perpendicular.
	$\overleftrightarrow{AB}$ is perpendicular to both $\overleftrightarrow{WX}$ and $\overleftrightarrow{YZ}$, as indicated by the right-angle marks at the intersection of those lines.
	Since $\overleftrightarrow{AB}$ is perpendicular to both lines, then $\overleftrightarrow{WX}$ and $\overleftrightarrow{YZ}$ are parallel.

$\overleftrightarrow{WX} \| \overleftrightarrow{YZ}$

$\overleftrightarrow{AB} \perp \overleftrightarrow{WX}, \overleftrightarrow{AB} \perp \overleftrightarrow{YZ}$

Exercise

Which statement most accurately represents the image below?



A. $\overleftrightarrow{EF} \| \overleftrightarrow{GH}$
B. $\overleftrightarrow{AB} \perp \overleftrightarrow{EG}$
C. $\overleftrightarrow{FH} \| \overleftrightarrow{EG}$
D. $\overleftrightarrow{AB} \| \overleftrightarrow{FH}$

Answer

A. Incorrect. This image shows the lines $\overleftrightarrow{EG}$ and $\overleftrightarrow{FH}$, not $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$. Both $\overleftrightarrow{EG}$ and $\overleftrightarrow{FH}$ are marked with >> on each line, and those markings mean they are parallel. The correct answer is $\overleftrightarrow{FH} \| \overleftrightarrow{EG}$.
B. Incorrect. $\overleftrightarrow{AB}$ does intersect $\overleftrightarrow{EG}$, but the intersection does not form a right angle. This means that they cannot be perpendicular. The correct answer is $\overleftrightarrow{FH} \| \overleftrightarrow{EG}$.
C. Correct. Both $\overleftrightarrow{EG}$ and $\overleftrightarrow{FH}$ are marked with >> on each line, and those markings mean they are parallel.
D. Incorrect. $\overleftrightarrow{AB}$ and $\overleftrightarrow{FH}$ intersect, so they cannot be parallel. Both $\overleftrightarrow{EG}$ and $\overleftrightarrow{FH}$ are marked with >> on each line, and those markings mean they are parallel. The correct answer is $\overleftrightarrow{FH} \| \overleftrightarrow{EG}$.

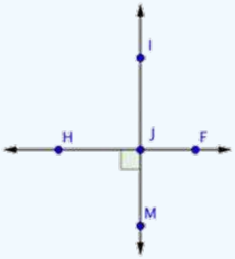
Finding Angle Measurements

Understanding how parallel and perpendicular lines relate can help you figure out the measurements of some unknown angles. To start, all you need to remember is that perpendicular lines intersect at a 90° angle, and that a straight angle measures 180°.

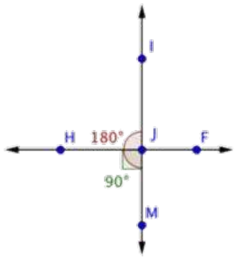
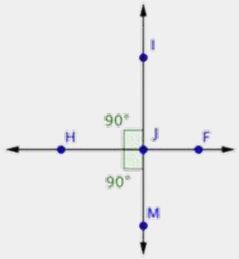
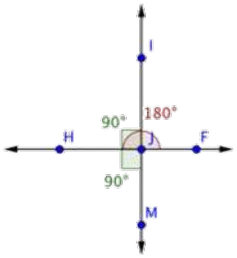
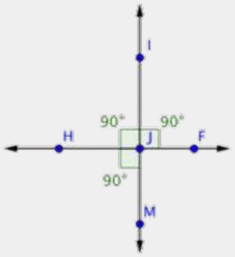
The measure of an angle such as $\angle A$ is written as $m\angle A$. Look at the example below. How can you find the measurements of the unmarked angles?

Example

Find the measurement of $\angle IJF$.



Solution

	Only one angle, $\angle HJM$, is marked in the image. Notice that it is a right angle, so it measures 90^{o}. $\angle HJM$ is formed by the intersection of lines $\overleftrightarrow{IM}$ and $\overleftrightarrow{HF}$. Since $\overleftrightarrow{IM}$ is a line, $\angle IJM$ is a straight angle measuring 180^{o}.
	You can use this information to find the measurement of $\angle HJI$: $m\angle HJM + m\angle HJI = m\angle IJM$ $90^\circ + m\angle HJI = 180^\circ$ $m\angle HJI = 90^\circ$
	Now use the same logic to find the measurement of $\angle IJF$. $\angle IJF$ is formed by the intersection of lines $\overleftrightarrow{IM}$ and $\overleftrightarrow{HF}$. Since $\overleftrightarrow{HF}$ is a line, $\angle HJF$ will be a straight angle measuring 180^{o}.
	You know that $\angle HJI$ measures 90^{o}. Use this information to find the measurement of $\angle IJF$: $m\angle HJI + m\angle IJF = m\angle HJF$ $90^\circ + m\angle IJF = 180^\circ$ $m\angle IJF = 90^\circ$

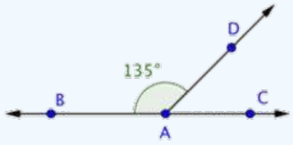
$m\angle IJF = 90^\circ$

In this example, you may have noticed that angles $\angle HJI, \angle IJF,$ and $\angle HJM$ are all right angles. (If you were asked to find the measurement of $\angle FJM$, you would find that angle to be 90^{o}, too.) This is what happens when two lines are perpendicular: the four angles created by the intersection are all right angles.

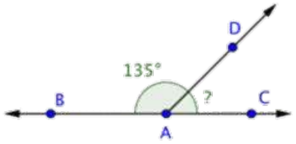
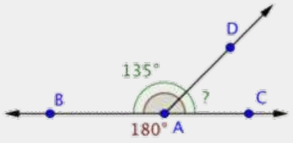
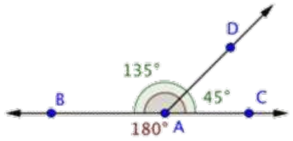
Not all intersections happen at right angles, though. In the example below, notice how you can use the same technique as shown above (using straight angles) to find the measurement of a missing angle.

✓ Example

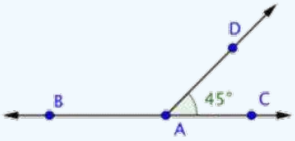
Find the measurement of $\angle DAC$.



Solution

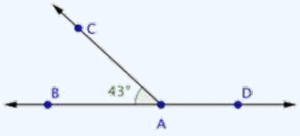
	This image shows the line $\overleftrightarrow{BC}$ and the ray $\overrightarrow{AD}$ intersecting at point A. The measurement of $\angle BAD$ is 135^o. You can use straight angles to find the measurement of $\angle DAC$.
	$\angle BAC$ is a straight angle, so it measures 180^o.
	Use this information to find the measurement of $\angle DAC$. $m\angle BAD + m\angle DAC = m\angle BAC$ $135^\circ + m\angle DAC = 180^\circ$ $m\angle DAC = 45^\circ$

$m\angle DAC = 45^\circ$



? Exercise

Find the measurement of $\angle CAD$.



A. 43^o
B. 137^o
C. 147^o
D. 317^o

Answer

A.
B.
C.
D.

Supplementary and Complementary

In the example above, $m\angle BAC$ and $m\angle DAC$ add up to 180^o. Two angles whose measures add up to 180^o are called **supplementary angles**. There's also a term for two angles whose measurements add up to 90^o; they are called **complementary angles**.

One way to remember the difference between the two terms is that "corner" and "complementary" each begin with c (a 90^o angle looks like a corner), while straight and "supplementary" each begin with s (a straight angle measures 180^o).

If you can identify supplementary or complementary angles within a problem, finding missing angle measurements is often simply a matter of adding or subtracting.

Example

Two angles are supplementary. If one of the angles measures 48^o, what is the measurement of the other angle?

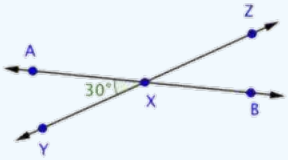
Solution

$m\angle A + m\angle B = 180^\circ$	Two supplementary angles make up a straight angle, so the measurements of the two angles will be 180^o.
$48^\circ + m\angle B = 180^\circ$ $m\angle B = 180^\circ - 48^\circ$ $m\angle B = 132^\circ$	You know the measurement of one angle. To find the measurement of the second angle, subtract 48^o from 180^o.

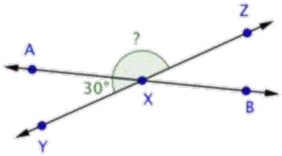
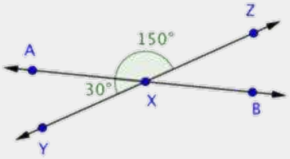
The measurement of the other angle is 132^o.

Example

Find the measurement of $\angle AXZ$.



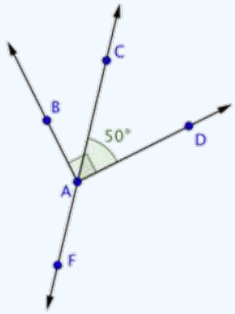
Solution

	This image shows two intersecting lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{YZ}$. They intersect at point X, forming four angles. Angles $\angle AXY$ and $\angle AXZ$ are supplementary because together they make up the straight angle $\angle YXZ$.
	Use this information to find the measurement of $\angle AXZ$. $m\angle AXY + m\angle AXZ = m\angle YXZ$ $30^\circ + m\angle AXZ = 180^\circ$ $m\angle AXZ = 150^\circ$

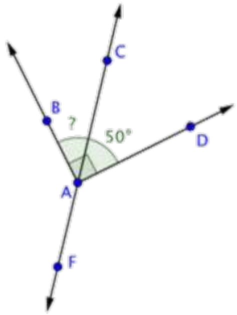
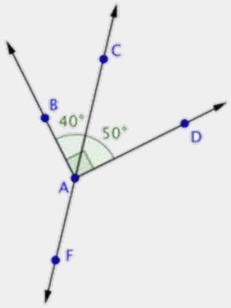
$m\angle AXZ = 150^\circ$

Example

Find the measurement of $\angle BAC$.



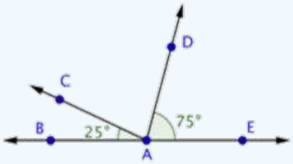
Solution

	This image shows the line $\overleftrightarrow{CF}$ and the rays $\overrightarrow{AB}$ and $\overrightarrow{AD}$, all intersecting at point A. Angle $\angle BAD$ is a right angle. Angles $\angle BAC$ and $\angle CAD$ are complementary, because together they create $\angle BAD$.
	Use this information to find the measurement of $\angle BAC$. $m\angle BAC + m\angle CAD = m\angle BAD$ $m\angle BAC + 50^\circ = 90^\circ$ $m\angle BAC = 40^\circ$

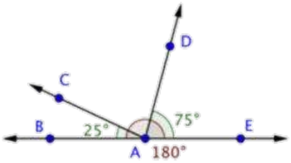
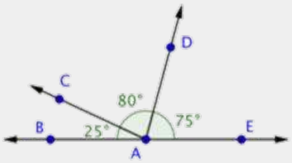
$m\angle BAC = 40^\circ$

✓ Example

Find the measurement of $\angle CAD$.



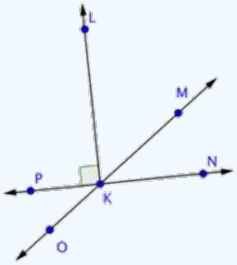
Solution

	You know the measurements of two angles here: $\angle CAB$ and $\angle DAE$. You also know that $m\angle BAE = 180^\circ$.
	Use this information to find the measurement of $\angle CAD$. $m\angle BAC + m\angle CAD + m\angle DAE = m\angle BAE$ $25^\circ + m\angle CAD + 75^\circ = 180^\circ$ $m\angle CAD + 100^\circ = 180^\circ$ $m\angle CAD = 80^\circ$

$m\angle CAD = 80^\circ$

? Exercise 7.1.2.1

Which pair of angles is complementary?



A. $\angle PKO$ and $\angle MKN$
B. $\angle PKO$ and $\angle PKM$
C. $\angle LKP$ and $\angle LKN$
D. $\angle LKM$ and $\angle MKN$

Answer

A. Incorrect. The measures of complementary angles add up to 90^{o}. It looks like the measures of these angles may add up to 90^{o}, but there is no way to be sure, so you cannot say that they are complementary. The correct answer is $\angle LKM$ and $\angle MKN$.
B. Incorrect. $\angle PKO$ and $\angle PKM$ are supplementary angles (not complementary angles) because together they comprise the straight angle $\angle OKM$. The correct answer is $\angle LKM$ and $\angle MKN$.
C. Incorrect. $\angle LKP$ and $\angle LKN$ are supplementary angles (not complementary angles) because together they comprise the straight angle $\angle PKN$. The correct answer is $\angle LKM$ and $\angle MKN$.
D. Correct. The measurements of two complementary angles will add up to 90^{o}. $\angle LKP$ is a right angle, so $\angle LKN$ must be a right angle as well. $\angle LKM + \angle MKN = \angle LKN$, so $\angle LKM$ and $\angle MKN$ are complementary.

Summary

Parallel lines do not intersect, while perpendicular lines cross at a 90^{o}. angle. Two angles whose measurements add up to 180^{o} are said to be supplementary, and two angles whose measurements add up to 90^{o} are said to be complementary. For most pairs of intersecting lines, all you need is the measurement of one angle to find the measurements of all other angles formed by the intersection.

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7.1.3: Triangles

Learning Objectives

- Identify equilateral, isosceles, scalene, acute, right, and obtuse triangles.
- Identify whether triangles are similar, congruent, or neither.
- Identify corresponding sides of congruent and similar triangles.
- Find the missing measurements in a pair of similar triangles.
- Solve application problems involving similar triangles.

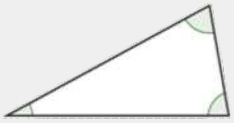
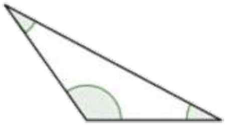
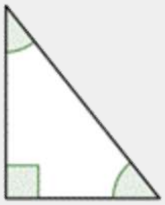
Introduction

Geometric shapes, also called figures, are an important part of the study of geometry. The **triangle** is one of the basic shapes in geometry. It is the simplest shape within a classification of shapes called **polygons**. All triangles have three sides and three angles, but they come in many different shapes and sizes. Within the group of all triangles, the characteristics of a triangle's sides and angles are used to classify it even further. Triangles have some important characteristics, and understanding these characteristics allows you to apply the ideas in real-world problems.

Classifying and Naming Triangles

A polygon is a closed plane figure with three or more straight sides. Polygons each have a special name based on the number of sides they have. For example, the polygon with three sides is called a triangle because "tri" is a prefix that means "three." Its name also indicates that this polygon has three angles. The prefix "poly" means many.

The table below shows and describes three classifications of triangles. Notice how the types of angles in the triangle are used to classify the triangle.

Name of Triangle	Picture of Triangle	Description
Acute Triangle		A triangle with 3 acute angles (3 angles measuring between 0^{o} and 90^{o}).
Obtuse Triangle		A triangle with 1 obtuse angle (1 angle measuring between 90^{o} and 180^{o})
Right Triangle		A triangle containing one right angle (1 angle that measures 90^{o}). Note that the right angle is shown with a corner mark and does not need to be labeled 90^{o}.

The sum of the measures of the three interior angles of a triangle is always 180^{o}. This fact can be applied to find the measure of the third angle of a triangle, if you are given the other two. Consider the examples below.

Example

A triangle has two angles that measure 35^{o} and 75^{o}. Find the measure of the third angle.

Solution

$35^{\circ}+75^{\circ}+x=180^{\circ}$	The sum of the three interior angles of a triangle is 180^{o}.

$110^\circ + x = 180^\circ$ $x = 180^\circ - 110^\circ$ $x = 70^\circ$	Find the value of x.

The third angle of the triangle measures 70^o.

✓ Example

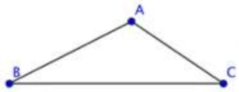
One of the angles in a right triangle measures 57^o. Find the measurement of the third angle.

Solution

$57^\circ + 90^\circ + x = 180^\circ$	The sum of the three angles of a triangle is always 180^o. One of the angles has a measure of 90^o as it is a right triangle.
$147^\circ + x = 180^\circ$ $x = 180^\circ - 147^\circ$ $x = 33^\circ$	Simplify. Find the value of x.

The third angle of the right triangle measures 33^o.

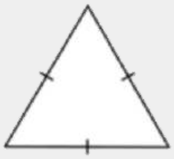
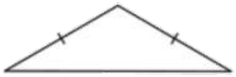
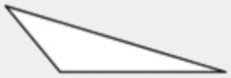
There is an established convention for naming triangles. The labels of the vertices of the triangle, which are generally capital letters, are used to name a triangle.



You can call this triangle ABC or $\triangle ABC$ since A, B, and C are vertices of the triangle. When naming the triangle, you can begin with any vertex. Then keep the letters in order as you go around the polygon. The triangle above could be named in a variety of ways: $\triangle ABC$, or $\triangle CBA$. The sides of the triangle are line segments AB, AC, and CB.

Just as triangles can be classified as acute, obtuse, or right based on their angles, they can also be classified by the length of their sides. Sides of equal length are called **congruent** sides. While we designate a segment joining points A and B by the notation $\overline{AB}$, we designate the length of a segment joining points A and B by the notation AB without a segment bar over it. The length AB is a number, and the segment $\overline{AB}$ is the collection of points that make up the segment.

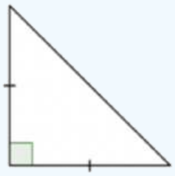
Mathematicians show congruency by putting a hash mark symbol through the middle of sides of equal length. If the hash mark is the same on one or more sides, then those sides are congruent. If the sides have different hash marks, they are *not* congruent. The table below shows the classification of triangles by their side lengths.

Name of Triangle	Picture of Triangle	Description
Equilateral Triangle		A triangle whose three sides have the same length. These sides of equal length are called congruent sides.
Isosceles Triangle		A triangle with exactly two congruent sides.
Scalene Triangle		A triangle in which all three sides are a different length.

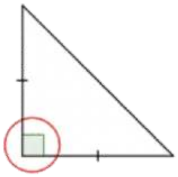
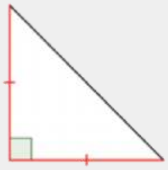
To describe a triangle even more specifically, you can use information about both its sides and its angles. Consider this example.

Example

Classify the triangle below.



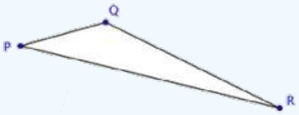
Solution

	Notice what kind of angles the triangle has. Since one angle is a right angle, this is a right triangle.
	Notice the lengths of the sides. Are there congruence marks or other labels? The congruence marks tell us there are two sides of equal length. So, this is an isosceles triangle.

This is an isosceles right triangle.

Exercise

Classify the triangle shown below.



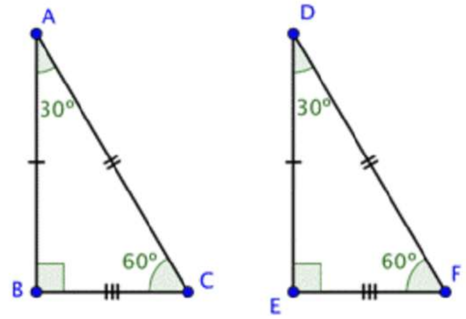
A. acute scalene
B. right isosceles
C. obtuse scalene
D. obtuse isosceles

Answer

A. Incorrect. This triangle has one angle (angle Q) that is between 90° and 180°, so it is an obtuse triangle. It is also scalene because all the sides have different lengths. The correct answer is obtuse scalene.
B. Incorrect. This triangle does not contain a right angle. It has one angle (angle Q) that is somewhere between 90° and 180°, so it is an obtuse triangle. It is also scalene because all the sides have different lengths. The correct answer is obtuse scalene.
C. Correct. This triangle has vertices P, Q, and R, one angle (angle Q) that is between 90° and 180°, and sides of three different lengths.
D. Incorrect. Although this triangle is obtuse, it does not have two sides of equal length. Its three sides are all different lengths, so it is scalene. The correct answer is obtuse scalene.

Identifying Congruent and Similar Triangles

Two triangles are congruent if they are exactly the same size and shape. In congruent triangles, the measures of **corresponding angles** and the lengths of **corresponding sides** are equal. Consider the two triangles shown below:



Since both $\angle B$ and $\angle E$ are right angles, these triangles are right triangles. Let's call these two triangles $\triangle ABC$ and $\triangle DEF$. These triangles are congruent if every pair of corresponding sides has equal lengths and every pair of corresponding angles has the same measure.

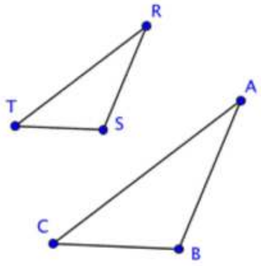
The corresponding sides are opposite the corresponding angles.

A double headed arrow means "corresponds to"

$\angle B \leftrightarrow \angle E$
$\angle A \leftrightarrow \angle D$
$\angle C \leftrightarrow \angle F$
$\overline{AB} \leftrightarrow \overline{DE}$
$\overline{AC} \leftrightarrow \overline{DF}$
$\overline{BC} \leftrightarrow \overline{EF}$

$\triangle ABC$ and $\triangle DEF$ are congruent triangles as the corresponding sides and corresponding angles are equal.

Let's take a look at another pair of triangles. Below are the triangles $\triangle ABC$ and $\triangle RST$.



These two triangles are surely not congruent because $\triangle RST$ is clearly smaller in size than $\triangle ABC$. But, even though they are not the same size, they do resemble one another. They are the same shape. The corresponding angles of these triangles look like they might have the same exact measurement, and if they did they would be congruent angles and we would call the triangles similar triangles.

Congruent angles are marked with hash marks, just as congruent sides are.

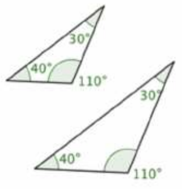


Image showing angle measurements of both triangles.

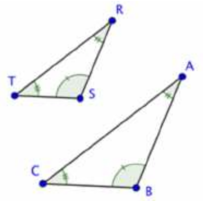


Image showing triangles ABC and RST using hash marks to show angle congruency.

We can also show congruent angles by using multiple bands within the angle, rather than multiple hash marks on one band. Below is an image using multiple bands within the angle.

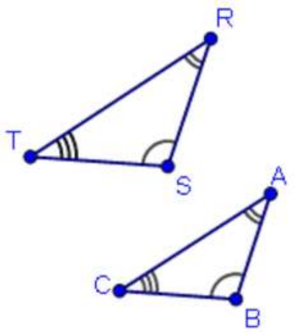


Image showing triangles ABC and RST using bands to show angle congruency.

If the corresponding angles of two triangles have the same measurements, they are called **similar** triangles. This name makes sense because they have the same shape, but not necessarily the same size. When a pair of triangles is similar, the corresponding sides are proportional to one another. That means that there is a consistent scale factor that can be used to compare the corresponding sides. In the previous example, the side lengths of the larger triangle are all 1.4 times the length of the smaller. So, similar triangles are proportional to one another.

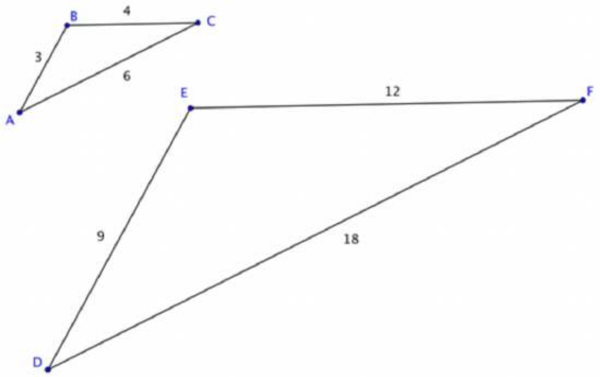
Just because two triangles *look* similar does not mean they *are* similar triangles in the mathematical sense of the word. Checking that the corresponding angles have equal measure is one way of being sure the triangles are similar.

Corresponding Sides of Similar Triangles

There is another method for determining similarity of triangles that involves comparing the ratios of the lengths of the corresponding sides.

If the ratios of the pairs of corresponding sides are equal, the triangles are similar.

Consider the two triangles below.



$\triangle ABC$ is *not* congruent to $\triangle DEF$ because the side lengths of $\triangle DEF$ are longer than those of $\triangle ABC$. So, are these triangles similar? If they are, the corresponding sides should be proportional.

Since these triangles are oriented in the same way, you can pair the left, right, and bottom sides: $\overline{AB}$ and $\overline{DE}$, $\overline{BC}$ and $\overline{EF}$, $\overline{AC}$ and $\overline{DE}$. (You might have called these the two shortest sides, the two longest sides, and the two leftover sides and arrived at the same ratios). Now we will look at the ratios of their lengths.

$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$

Substituting the side length values into the proportion, you see that it is true:

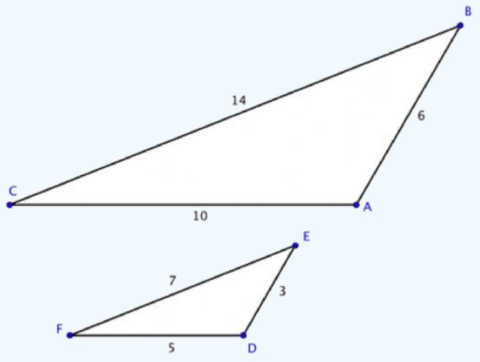
$\frac{3}{9}=\frac{4}{12}=\frac{6}{18}$

If the corresponding sides are proportional, then the triangles are similar. Triangles ABC and DEF are similar, but not congruent.

Let’s use this idea of proportional corresponding sides to determine whether two more triangles are similar.

✓ Example

Determine if the triangles below are similar by seeing if their corresponding sides are proportional.



Solution

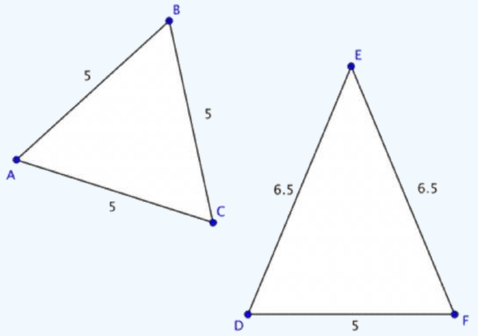
$\overline{CA} \leftrightarrow \overline{FD}$ $\overline{AB} \leftrightarrow \overline{DE}$ $\overline{BC} \leftrightarrow \overline{EF}$	First determine the corresponding sides, which are opposite corresponding angles.
$\frac{CA}{FD} = \frac{AB}{DE} = \frac{BC}{EF}$	Write the corresponding side lengths as ratios.
$\frac{10}{5} = \frac{6}{3} = \frac{14}{7}$ $2 = 2 = 2$	Substitute the side lengths into the ratios, and determine if the ratios of the corresponding sides are equivalent. They are, so the triangles are similar.

$\triangle ABC$ and $\triangle DEF$ are similar.

The mathematical symbol ~ means “is similar to.” So, you can write $\triangle ABC$ is similar to $\triangle DEF$ as $\triangle ABC \sim \triangle DEF$.

? Exercise

Determine whether the two triangles are similar, congruent, or neither.



A. $\triangle ABC$ and $\triangle DEF$ are congruent.
B. $\triangle ABC$ and $\triangle DEF$ are similar.
C. $\triangle ABC$ and $\triangle DEF$ are similar and congruent.
D. $\triangle ABC$ and $\triangle DEF$ are neither similar nor congruent.

Answer

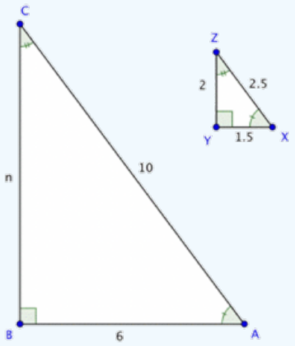
A. Incorrect. Congruent triangles have corresponding sides of equal length and corresponding angles of equal measure. They are the same exact size and shape. $\triangle ABC$ is equilateral and $\triangle DEF$ is isosceles, so they are not the same exact shape. The correct answer is $\triangle ABC$ and $\triangle DEF$ are neither similar nor congruent.
B. Incorrect. The ratios of the corresponding sides are not equal, so the triangles cannot be similar: $\frac{6.5}{5} = \frac{6.5}{5} \neq \frac{5}{5}$. The correct answer is $\triangle ABC$ and $\triangle DEF$ are neither similar nor congruent.
C. Incorrect. All congruent triangles are similar, but these triangles are not congruent. Congruent triangles have corresponding sides of equal length and corresponding angles of equal measure. $\triangle ABC$ is equilateral and $\triangle DEF$ is isosceles, so they are not the same exact shape. The correct answer is $\triangle ABC$ and $\triangle DEF$ are neither similar nor congruent.
D. Correct. The corresponding angle measures are not known to be equal as shown by the absence of congruence marks on the angles. Also, the ratios of the corresponding sides are not equal: $\frac{6.5}{5} = \frac{6.5}{5} \neq \frac{5}{5}$.

Finding Missing Measurements in Similar Triangles

You can find the missing measurements in a triangle if you know some measurements of a similar triangle. Let's look at an example.

✓ Example

$\triangle ABC$ and $\triangle XYZ$ are similar triangles. What is the length of side BC?



Solution

$\frac{BC}{YZ} = \frac{AB}{XY}$	In similar triangles, the ratios of corresponding sides are proportional. Set up a proportion of two ratios, one that includes the missing side.
$\frac{n}{2} = \frac{6}{1.5}$	Substitute in the known side lengths for the side names in the ratio. Let the unknown side length be n.
$2 \cdot 6 = 1.5 \cdot n$ $12 = 1.5n$ $8 = n$	Solve for n using cross multiplication.

The missing length of side BC is 8 units.

This process is fairly straightforward, but be careful that your ratios represent corresponding sides, recalling that corresponding sides are opposite corresponding angles.

Solving Application Problems Involving Similar Triangles

Applying knowledge of triangles, similarity, and congruence can be very useful for solving problems in real life. Just as you can solve for missing lengths of a triangle drawn on a page, you can use triangles to find unknown distances between locations or objects.

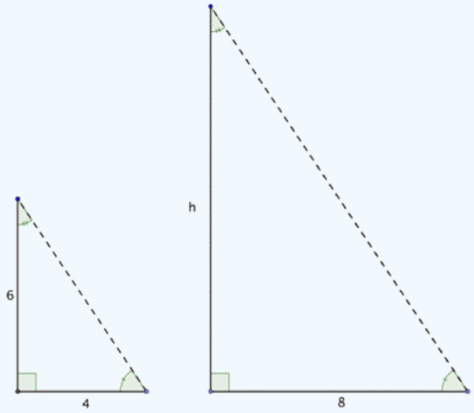
Let’s consider the example of two trees and their shadows. Suppose the sun is shining down on two trees, one that is 6 feet tall and the other whose height is unknown. By measuring the length of each shadow on the ground, you can use triangle similarity to find the unknown height of the second tree.

First, let’s figure out where the triangles are in this situation! The trees themselves create one pair of corresponding sides. The shadows cast on the ground are another pair of corresponding sides. The third side of these imaginary similar triangles runs from the top of each tree to the tip of its shadow on the ground. This is the hypotenuse of the triangle.

If you know that the trees and their shadows form similar triangles, you can set up a proportion to find the height of the tree.

✓ Example

When the sun is at a certain angle in the sky, a 6--foot tree will cast a 4-foot shadow. How tall is a tree that casts an 8-foot shadow?



Solution

$\frac{\text{Tree 1}}{\text{Tree 2}} = \frac{\text{Shadow 1}}{\text{Shadow 2}}$	The angle measurements are the same, so the triangles are similar triangles. Since they are similar triangles, you can use proportions to find the size of the missing side.
	Set up a proportion comparing the heights of the trees and the lengths of their shadows.
$\frac{6}{h} = \frac{4}{8}$	Substitute in the known lengths. Call the missing tree height h.
$6 \cdot 8 = 4h$ $48 = 4h$ $12 = h$	Solve for h using cross-multiplication.

The tree is 12 feet tall.

Summary

Triangles are one of the basic shapes in the real world. Triangles can be classified by the characteristics of their angles and sides, and triangles can be compared based on these characteristics. The sum of the measures of the interior angles of any triangle is 180º. Congruent triangles are triangles of the same size and shape. They have corresponding sides of equal length and corresponding angles of the same measurement. Similar triangles have the same shape, but not necessarily the same size. The lengths of their sides are proportional. Knowledge of triangles can be a helpful in solving real-world problems.

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7.1.4: The Pythagorean Theorem

Learning Objectives

- Use the Pythagorean Theorem to find the unknown side of a right triangle.
- Solve application problems involving the Pythagorean Theorem.

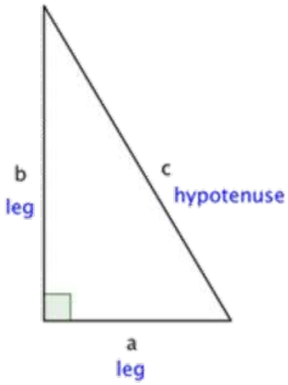
Introduction

A long time ago, a Greek mathematician named **Pythagoras** discovered an interesting property about **right triangles**: the sum of the squares of the lengths of each of the triangle's **legs** is the same as the square of the length of the triangle's **hypotenuse**. This property, which has many applications in science, art, engineering, and architecture, is now called the **Pythagorean Theorem**.

Let's take a look at how this theorem can help you learn more about the construction of triangles. And the best part is you don't even have to speak Greek to apply Pythagoras' discovery.

The Pythagorean Theorem

Pythagoras studied right triangles, and the relationships between the legs and the hypotenuse of a right triangle, before deriving his theory.



The Pythagorean Theorem

If a and b are the lengths of the legs of a right triangle and is the length of the hypotenuse, then the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

This relationship is represented by the formula:

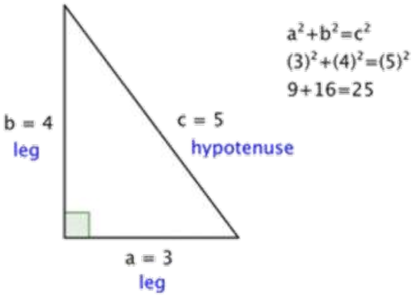
$$a^2 + b^2 = c^2$$

In the box above, you may have noticed the word "square," as well as the small 2s to the top right of the letters in $a^2 + b^2 = c^2$. To **square** a number means to multiply it by itself. So, for example, to square the number 5 you multiply $5 \cdot 5$, and to square the number 12, you multiply $12 \cdot 12$. Some common squares are shown in the table below.

Number	**Number Times Itself**	**Square**
1	$1^2 = 1 \cdot 1$	1
2	$2^2 = 2 \cdot 2$	4
3	$3^2 = 3 \cdot 3$	9
4	$4^2 = 4 \cdot 4$	16
5	$5^2 = 5 \cdot 5$	25
10	$10^2 = 10 \cdot 10$	100

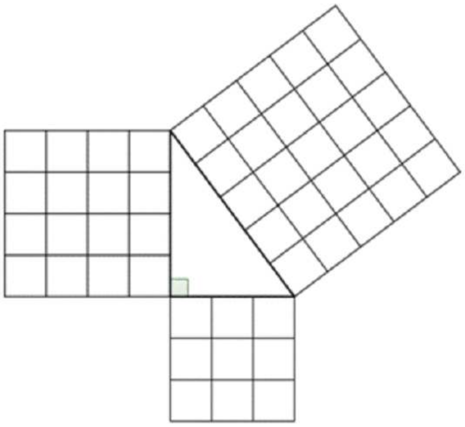
When you see the equation $a^2+b^2=c^2$, you can think of this as “the length of side a times itself, plus the length of side b times itself is the same as the length of side c times itself.”

Let’s try out all of the Pythagorean Theorem with an actual right triangle.



This theorem holds true for this right triangle: the sum of the squares of the lengths of both legs is the same as the square of the length of the hypotenuse. And, in fact, it holds true for all right triangles.

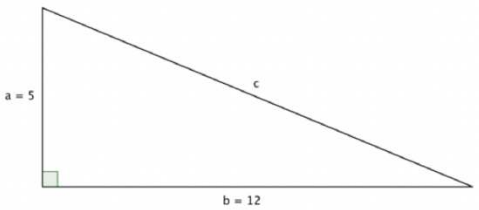
The Pythagorean Theorem can also be represented in terms of area. In any right triangle, the area of the square drawn from the hypotenuse is equal to the sum of the areas of the squares that are drawn from the two legs. You can see this illustrated below in the same 3-4-5 right triangle.



Note that the Pythagorean Theorem only works with *right* triangles.

Finding the Length of the Hypotenuse

You can use the Pythagorean Theorem to find the length of the hypotenuse of a right triangle if you know the length of the triangle’s other two sides, called the legs. Put another way, if you know the lengths of a and b, you can find c.



In the triangle above, you are given measures for legs a and b: 5 and 12, respectively. You can use the Pythagorean Theorem to find a value for the length of c, the hypotenuse.

$a^2+b^2=c^2$	The Pythagorean Theorem
$(5)^2+(12)^2=c^2$	Substitute known values for a and b.
$25+144=c^2$	Evaluate.

$169 = c^2$	Simplify. To find the value of c, think about a number that, when multiplied by itself, equals 169. Does 10 work? How about 11? Does 12? 13? (You can use a calculator to multiply if the numbers are unfamiliar.)
$13 = c$	The square root of 169 is 13.

Using the formula, you find that the length of c, the hypotenuse, is 13.

In this case, you did not know the value of c, you were given the square of the length of the hypotenuse, and had to figure it out from there. When you are given an equation like $169 = c^2$ and are asked to find the value of c, this is called finding the **square root** of a number. (Notice you found a number, c, whose square was 169.)

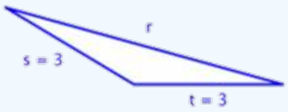
Finding a square root takes some practice, but it also takes knowledge of multiplication, division, and a little bit of trial and error. Look at the table below.

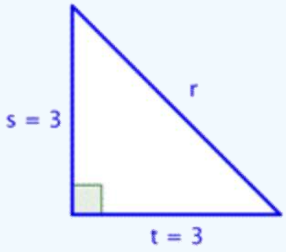
Number x	**Number y which, when multiplied by itself, equals number x**	**Square root y**
1	$1 \cdot 1$	1
4	$2 \cdot 2$	2
9	$3 \cdot 3$	3
16	$4 \cdot 4$	4
25	$5 \cdot 5$	5
100	$10 \cdot 10$	10

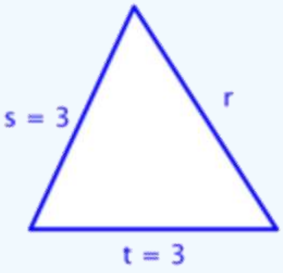
It is a good habit to become familiar with the squares of the numbers from 0 through 10, as these arise frequently in mathematics. If you can remember those square numbers, or if you can use a calculator to find them, then finding many common square roots will be just a matter of recall.

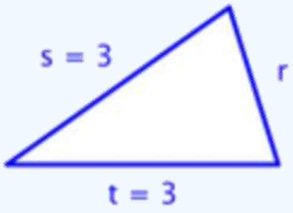
? Exercise

For which of these triangles is $(3)^2 + (3)^2 = r^2$?

A.


B.


C. 

D. 

Answer

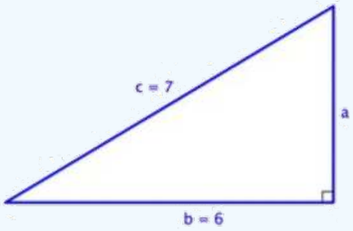
A.
B.
C.
D.

Finding the Length of a Leg

You can use the same formula to find the length of a right triangle’s leg if you are given measurements for the lengths of the hypotenuse and the other leg. Consider the example below.

Example

Find the length of side a in the triangle below. Use a calculator to estimate the square root to one decimal place.



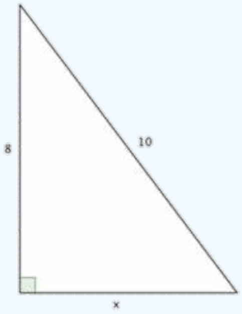
Solution

$a = ?$ $b = 6$ $c = 7$	In this right triangle, you are given the measurements for the hypotenuse, c, and one leg, b. The hypotenuse is always opposite the right angle and it is always the longest side of the triangle.
$a^2 + b^2 = c^2$ $a^2 + 6^2 = 7^2$	To find the length of leg a, substitute the known values into the Pythagorean Theorem.
$a^2 + 36 = 49$ $a^2 = 13$	Solve for a^2. Think: what number, when added to 36, gives you 49?
$a \approx 3.6$	Use a calculator to find the square root of 13. The calculator gives an answer of 3.6055..., which you can round to 3.6. (Since you are approximating, you use the symbol $\approx$.)

$a \approx 3.6$

Exercise

Which of the following correctly uses the Pythagorean Theorem to find the missing side, x?



A. $8^2 + 10^2 = x^2$
B. $x + 8 = 10$
C. $x^2 + 8^2 = 10^2$
D. $x^2 + 10^2 = 8^2$

Answer

A. Incorrect. In this triangle, you know the hypotenuse (the side opposite the right angle) has a length of 10. The lengths of the legs are 8 and x. The correct answer is $x^2 + 8^2 = 10^2$.
B. Incorrect. The Pythagorean Theorem is a relationship between the lengths of the sides squared. The correct answer is $x^2 + 8^2 = 10^2$.
C. Correct. In this triangle, the hypotenuse has length 10, and the legs have length 8 and x. Substituting into the Pythagorean Theorem you have: $x^2 + 8^2 = 10^2$; this equation is the same as $x^2 + 64 = 100$, or $x^2 = 36$. What number, times itself, equals 36? That would make $x = 6$.
D. Incorrect. In this triangle, the hypotenuse has length 10 (always the longest side of the triangle and the side opposite the right angle) not 8. The correct answer is $x^2 + 8^2 = 10^2$.

Using the Theorem to Solve Real World Problems

The Pythagorean Theorem is perhaps one of the most useful formulas you will learn in mathematics because there are so many applications of it in real-world settings. Architects and engineers use this formula extensively when building ramps, bridges, and buildings. Look at the following examples.

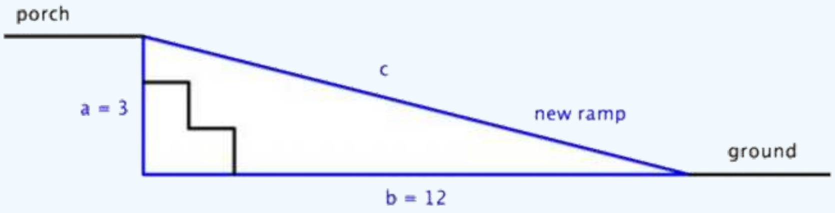
Example

The owners of a house want to convert a stairway leading from the ground to their back porch into a ramp. The porch is 3 feet off the ground and, due to building regulations, the ramp must start 12 feet away from the base of the porch. How long will the ramp be?

Use a calculator to find the square root, and round the answer to the nearest tenth.

Solution

To solve a problem like this one, it often makes sense to draw a simple diagram showing where the legs and hypotenuse of the triangle lie.



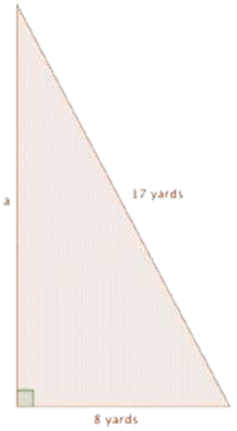
$a=3$ $b=12$ $c=?$	Identify the legs and the hypotenuse of the triangle. You know that the triangle is a *right* triangle since the ground and the raised portion of the porch are perpendicular. This means you can use the Pythagorean Theorem to solve this problem. Identify a, b, and c.
$a^2+b^2=c^2$ $3^2+12^2=c^2$ $9+144=c^2$ $153=c^2$	Use the Pythagorean Theorem to find the length of c.
$12.4=c$	Use your calculator to find c. The square root of 153 is 12.369..., so you can round that to 12.4.

The ramp will be 12.4 feet long.

Example

A sailboat has a large sail in the shape of a right triangle. The longest edge of the sail measures 17 yards and the bottom edge of the sail is 8 yards. How tall is the sail?

Solution

	Draw an image to help you visualize the problem. In a right triangle, the hypotenuse will always be the longest side, so here it must be 17 yards. The problem also tells you that the bottom edge of the triangle is 8 yards.
$a^2+b^2=c^2$ $a^2+8^2=17^2$ $a^2+64=289$ $a^2=225$	Set up the Pythagorean Theorem.
$a=15$	$15\cdot 15=225,\ \text{so } a=15$

The height of the sail is 15 yards.

Summary

The Pythagorean Theorem states that in any right triangle, the sum of the squares of the lengths of the triangle's legs is the same as the square of the length of the triangle's hypotenuse. This theorem is represented by the formula $a^2+b^2=c^2$. Put simply, if you know the lengths of two sides of a right triangle, you can apply the Pythagorean Theorem to find the length of the third side. Remember, this theorem only works for right triangles.

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SECTION OVERVIEW

7.2: Perimeter, Circumference, and Area

Topic hierarchy

7.2.1: Quadrilaterals

7.2.2: Perimeter and Area

7.2.3: Circles

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7.2.1: Quadrilaterals

Learning Objectives

- Identify properties, including angle measurements, of quadrilaterals.

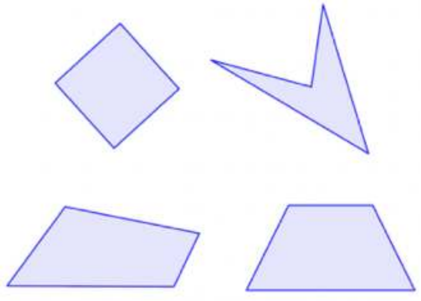
Introduction

Quadrilaterals are a special type of polygon. As with triangles and other polygons, quadrilaterals have special properties and can be classified by characteristics of their angles and sides. Understanding the properties of different quadrilaterals can help you in solving problems that involve this type of polygon.

Defining a Quadrilateral

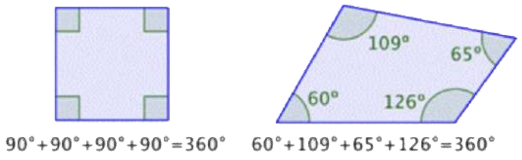
Picking apart the name “quadrilateral” helps you understand what it refers to. The prefix “quad-” means “four,” and “lateral” is derived from the Latin word for “side.” So a quadrilateral is a four-sided polygon.

Since it is a **polygon**, you know that it is a two-dimensional figure made up of straight sides. A quadrilateral also has four angles formed by its four sides. Below are some examples of quadrilaterals. Notice that each figure has four straight sides and four angles.



Interior Angles of a Quadrilateral

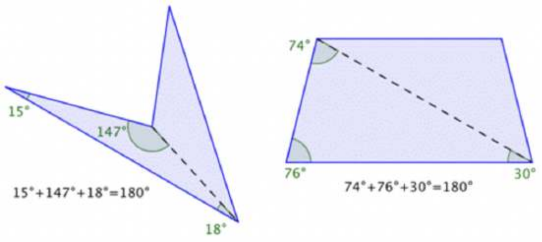
The sum of the interior angles of any quadrilateral is 360^{o}. Consider the two examples below.



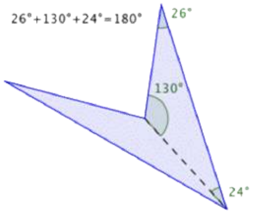
You could draw many quadrilaterals such as these and carefully measure the four angles. You would find that for every quadrilateral, the sum of the interior angles will always be 360^{o}.

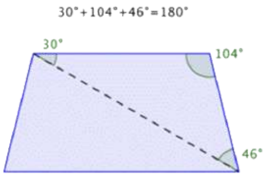
You can also use your knowledge of triangles as a way to understand why the sum of the interior angles of any quadrilateral is 360^{o}. Any quadrilateral can be divided into two triangles as shown in the images below.

In the first image, the quadrilaterals have each been divided into two triangles. The angle measurements of one triangle are shown for each.



These measurements add up to 180^{o}. Now look at the measurements for the other triangles. They also add up to 180^{o}!

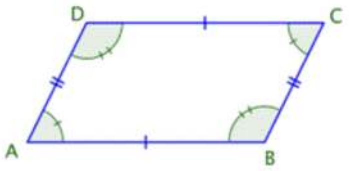




Since the sum of the interior angles of any triangle is 180^o and there are two triangles in a quadrilateral, the sum of the angles for each quadrilateral is 360^o.

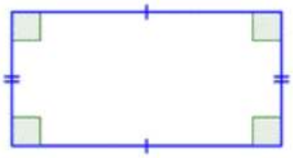
Specific Types of Quadrilaterals

Let’s start by examining the group of quadrilaterals that have two pairs of parallel sides. These quadrilaterals are called **parallelograms** They take a variety of shapes, but one classic example is shown below.

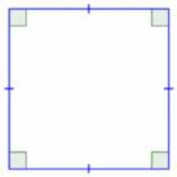


Imagine extending the pairs of opposite sides. They would never intersect because they are parallel. Notice, also, that the opposite angles of a parallelogram are congruent, as are the opposite sides. (Remember that “congruent” means “the same size.”) The geometric symbol for congruent is $\cong$, so you can write $\angle A \cong \angle C$ and $\angle B \cong \angle D$. The parallel sides are also the same length: $\overline{AB} \cong \overline{DC}$ and $\overline{BC} \cong \overline{AD}$. These relationships are true for all parallelograms.

There are two special cases of parallelograms that will be familiar to you from your earliest experiences with geometric shapes. The first special case is called a **rectangle**. By definition, a rectangle is a parallelogram because its pairs of opposite sides are parallel. A rectangle also has the special characteristic that all of its angles are right angles; all four of its angles are congruent.

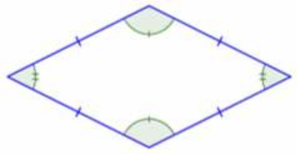


The other special case of a parallelogram is a special type of rectangle, a **square**. A square is one of the most basic geometric shapes. It is a special case of a parallelogram that has four congruent sides and four right angles.



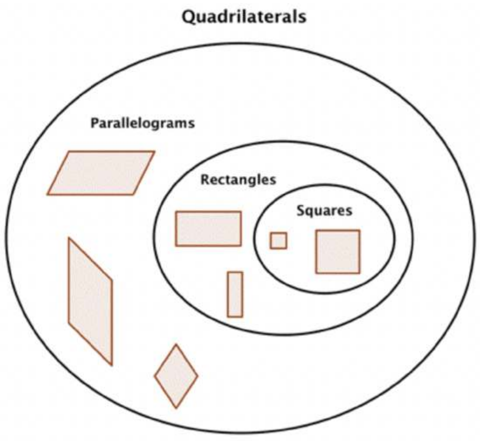
A square is also a rectangle because it has two sets of parallel sides and four right angles. A square is also a parallelogram because its opposite sides are parallel. So, a square can be classified in any of these three ways, with “parallelogram” being the least specific description and “square,” the most descriptive.

Another quadrilateral that you might see is called a **rhombus**. All four sides of a rhombus are congruent. Its properties include that each pair of opposite sides is parallel, also making it a parallelogram.



In summary, all squares are rectangles, but not all rectangles are squares. All rectangles are parallelograms, but not all parallelograms are rectangles. And *all* of these shapes are quadrilaterals.

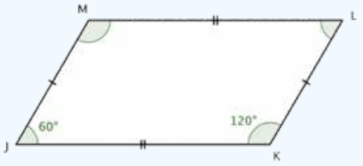
The diagram below illustrates the relationship between the different types of quadrilaterals.



You can use the properties of parallelograms to solve problems. Consider the example that follows.

✓ Example

Determine the measures of $\angle M$ and $\angle L$.



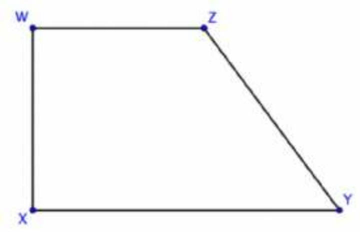
Solution

$\angle L$ is opposite $\angle J$ $\angle M$ is opposite $\angle K$	Identify opposite angles.
$\angle L \cong \angle J$ $\angle M \cong \angle K$	A property of parallelograms is that opposite angles are congruent.
$m\angle J = 60^\circ, \text{ so } m\angle L = 60^\circ$ $m\angle K = 120^\circ, \text{ so } m\angle M = 120^\circ$	Use the given angle measurements to determine measures of opposite angles.

$m\angle L = 60^\circ \text{ and } m\angle M = 120^\circ$

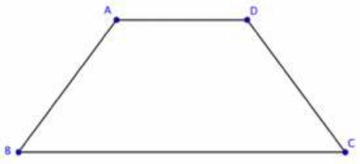
Trapezoids

There is another special type of quadrilateral. This quadrilateral has the property of having *only one* pair of opposite sides that are parallel. Here is one example of a **trapezoid**.



Notice that $\overline{XY} \| \overline{WZ}$, and that $\overline{WX}$ and $\overline{ZY}$ are not parallel. You can easily imagine that if you extended sides $\overline{WX}$ and $\overline{ZY}$, they would intersect above the figure.

If the non-parallel sides of a trapezoid are congruent, the trapezoid is called an **isosceles trapezoid**. Like the similarly named triangle that has two sides of equal length, the isosceles trapezoid has a pair of opposite sides of equal length. The other pair of opposite sides is parallel. Below is an example of an isosceles trapezoid.



In this trapezoid $ABCD, \overline{BC} \| \overline{AD}$ and $\overline{AB} \cong \overline{CD}$.

Exercise

Which of the following statements is true?

A. Some trapezoids are parallelograms.
B. All trapezoids are quadrilaterals.
C. All rectangles are squares.
D. A shape cannot be a parallelogram and a quadrilateral.

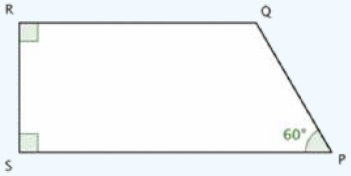
Answer

A. Incorrect. Trapezoids have only one pair of parallel sides; parallelograms have two pairs of parallel sides. A trapezoid can never be a parallelogram. The correct answer is that all trapezoids are quadrilaterals.
B. Correct. Trapezoids are four-sided polygons, so they are all quadrilaterals.
C. Incorrect. Some rectangles may be squares, but not *all* rectangles have four congruent sides. All squares are rectangles however. The correct answer is that all trapezoids are quadrilaterals.
D. Incorrect. All parallelograms are quadrilaterals, so if it is a parallelogram, it is also a quadrilateral. The correct answer is that all trapezoids are quadrilaterals.

You can use the properties of quadrilaterals to solve problems involving trapezoids. Consider the example below.

Example

Find the measure of $\angle Q$.

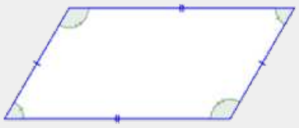
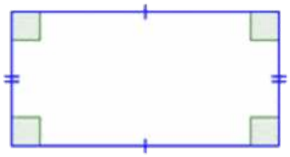
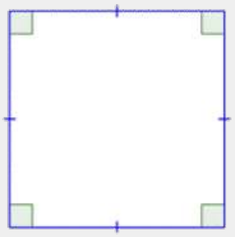
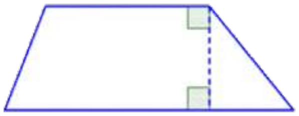


Solution

$m\angle P + m\angle Q + m\angle R + m\angle S = 360^\circ$	The sum of the measures of the interior angles of a quadrilateral is 360^{o}.
$m\angle R = 90^\circ$ $m\angle S = 90^\circ$	The square symbol indicates a right angle.
$60^\circ + m\angle Q + 90^\circ + 90^\circ = 360^\circ$	Since three of the four angle measures are given, you can find the fourth angle measurement.
$m\angle Q + 240^\circ = 360^\circ$ $m\angle Q = 120^\circ$	Now, for $\angle Q$, calculate the measurement. From the image, you can see that it is an obtuse angle, so its measure must be greater than 90^{o}.

$m\angle Q = 120^\circ$

The table below summarizes the special types of quadrilaterals and some of their properties.

Name of Quadrilateral	**Quadrilateral**	**Description**
Parallelogram		2 pairs of parallel sides. Opposite sides and opposite angles are congruent.
Rectangle		2 pairs of parallel sides. 4 right angles (90º). Opposite sides are parallel and congruent. All angles are congruent.
Square		4 congruent sides. 4 right angles (90º). Opposite sides are parallel. All angles are congruent.
Trapezoid		Only one pair of opposite sides is parallel.

Summary

A quadrilateral is a mathematical name for a four-sided polygon. Parallelograms, squares, rectangles, and trapezoids are all examples of quadrilaterals. These quadrilaterals earn their distinction based on their properties, including the number of pairs of parallel sides they have and their angle and side measurements.

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7.2.2: Perimeter and Area

Learning Objectives

- Find the perimeter of a polygon.
- Find the area of a polygon.
- Find the area and perimeter of non-standard polygons.

Introduction

Perimeter and **area** are two important and fundamental mathematical topics. They help you to quantify physical space and also provide a foundation for more advanced mathematics found in algebra, trigonometry, and calculus. Perimeter is a measurement of the distance around a shape and area gives us an idea of how much surface the shape covers.

Knowledge of area and perimeter is applied practically by people on a daily basis, such as architects, engineers, and graphic designers, and is math that is very much needed by people in general. Understanding how much space you have and learning how to fit shapes together exactly will help you when you paint a room, buy a home, remodel a kitchen, or build a deck.

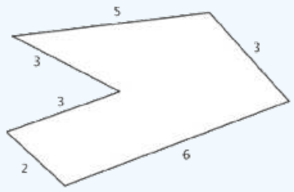
Perimeter

The perimeter of a two-dimensional shape is the distance around the shape. You can think of wrapping a string around a triangle. The length of this string would be the perimeter of the triangle. Or walking around the outside of a park, you walk the distance of the park's perimeter. Some people find it useful to think "peRIMeter" because the edge of an object is its rim and peRIMeter has the word "rim" in it.

If the shape is a **polygon**, then you can add up all the lengths of the sides to find the perimeter. Be careful to make sure that all the lengths are measured in the same units. You measure perimeter in linear units, which is one dimensional. Examples of units of measure for length are inches, centimeters, or feet.

Example

Find the perimeter of the given figure. All measurements indicated are inches.



Solution

$P = 5+3+6+2+3+3$	Since all the sides are measured in inches, just add the lengths of all six sides to get the perimeter.
$P = 22$ inches	Remember to include units.

This means that a tightly wrapped string running the entire distance around the polygon would measure 22 inches long.

Example

Find the perimeter of a triangle with sides measuring 6 cm, 8 cm, and 12 cm.

Solution

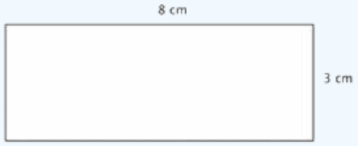
$P = 6+8+12$	Since all the sides are measured in centimeters, just add the lengths of all three sides to get the perimeter.

$P = 26$ centimeters

Sometimes, you need to use what you know about a polygon in order to find the perimeter. Let’s look at the rectangle in the next example.

✓ Example

A rectangle has a length of 8 centimeters and a width of 3 centimeters. Find the perimeter.



Solution

$P=3+3+8+8$	Since this is a rectangle, the opposite sides have the same lengths, 3 cm. and 8 cm. Add up the lengths of all four sides to find the perimeter.

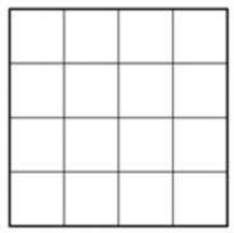
$P=22\text{ cm}$

Notice that the perimeter of a rectangle always has two pairs of equal length sides. In the above example you could have also written $P=2(3)+2(8)=6+16=22\text{ cm}$. The formula for the perimeter of a rectangle is often written as $P=2l+2w$, where l is the length of the rectangle and w is the width of the rectangle.

Area of Parallelograms

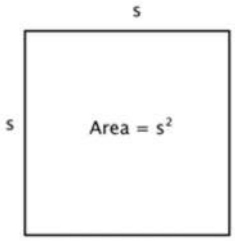
The area of a two-dimensional figure describes the amount of surface the shape covers. You measure area in square units of a fixed size. Examples of square units of measure are square inches, square centimeters, or square miles. When finding the area of a polygon, you count how many squares of a certain size will cover the region inside the polygon.

Let’s look at a 4 by 4 square.



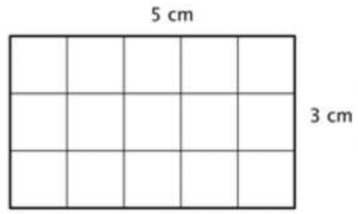
You can count that there are 16 squares, so the area is 16 square units. Counting out 16 squares doesn’t take too long, but what about finding the area if this is a larger square or the units are smaller? It could take a long time to count.

Fortunately, you can use multiplication. Since there are 4 rows of 4 squares, you can multiply $4\cdot 4$ to get 16 squares! And this can be generalized to a formula for finding the area of a square with any length, s: $\text{Area }=s\cdot s=s^2$.

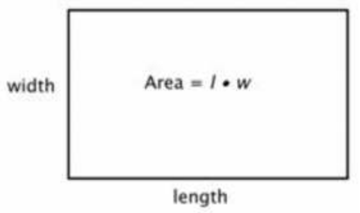


You can write “ in^2” for square inches and “ ft^2” for square feet.

To help you find the area of the many different categories of polygons, mathematicians have developed formulas. These formulas help you find the measurement more quickly than by simply counting. The formulas you are going to look at are all developed from the understanding that you are counting the number of square units *inside* the polygon. Let's look at a rectangle.



You can count the squares individually, but it is much easier to multiply 3 times 5 to find the number more quickly. And, more generally, the area of any rectangle can be found by multiplying *length* times *width*.



Example

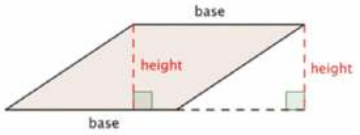
A rectangle has a length of 8 centimeters and a width of 3 centimeters. Find the area.

Solution

$A=l\cdot w$	Start with the formula for the area of a rectangle, which multiplies the length times the width.
$A=8\cdot 3$	Substitute 8 for the length and 3 for the width.
$A=24\text{ cm}^2$	Be sure to include the units, in this case square centimeters.

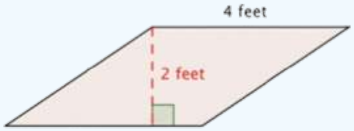
It would take 24 squares, each measuring 1 centimeter on a side, to cover this rectangle.

The formula for the area of any parallelogram (remember, a rectangle is a type of parallelogram) is the same as that of a rectangle: $\text{Area }=l\cdot w$. Notice in a rectangle, the length and the width are perpendicular. This should also be true for all parallelograms. *Base* (b for the length (of the base), and *height* (h) for the width of the line perpendicular to the base is often used. So the formula for a parallelogram is generally written, $A=b\cdot h$.



Example

Find the area of the parallelogram.



Solution

$A=b\cdot h$	Start with the formula for the area of a parallelogram:
$A=4\cdot 2$	$\text{Area} = \text{base} \cdot \text{height}$. Substitute the values into the formula.
$A=8$	Multiply.

The area of the parallelogram is 8 ft^2.

? Exercise

Find the area of a parallelogram with a height of 12 feet and a base of 9 feet.

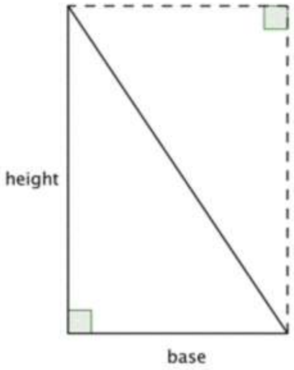
A. 21ft^2
B. 54ft^2
C. 42ft
D. 108ft^2

Answer

A. Incorrect. It looks like you added the dimensions; remember that to find the area, you multiply the base by the height. The correct answer is 108ft^2.
B. Incorrect. It looks like you multiplied the base by the height and then divided by 2. To find the area of a parallelogram, you multiply the base by the height. The correct answer is 108ft^2.
C. Incorrect. It looks like you added $12+12+9+9$. This would give you the perimeter of a 12 by 9 rectangle. To find the area of a parallelogram, you multiply the base by the height. The correct answer is 108ft^2.
D. Correct. The height of the parallelogram is 12 and the base of the parallelogram is 9; the area is 12 times 9, or 108ft^2.

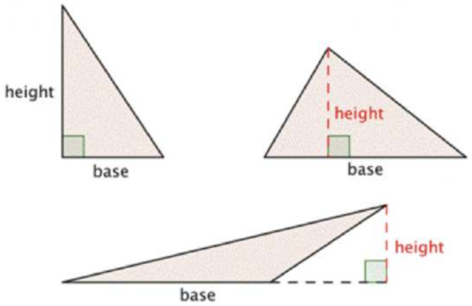
Area of Triangles and Trapezoids

The formula for the area of a triangle can be explained by looking at a right triangle. In the image below is a rectangle with the same height and base as the original triangle. The area of the triangle is one half of the rectangle!



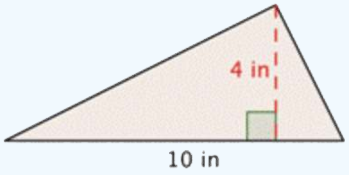
Since the area of two congruent triangles is the same as the area of a rectangle, you can come up with the formula $\text{Area} = \frac{1}{2}b\cdot h$ to find the area of a triangle.

When you use the formula for a triangle to find its area, it is important to identify a base and its corresponding height, which is perpendicular to the base.



Example

A triangle has a height of 4 inches and a base of 10 inches. Find the area.



Solution

$A=\frac{1}{2}bh$	Start with the formula for the area of a triangle.
$A=\frac{1}{2}\cdot 10\cdot 4$	Substitute 10 for the base and 4 for the height.
$A=\frac{1}{2}\cdot 40$ $A=20$	Multiply.

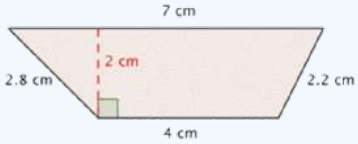
$A=20\text{ in }^2$

Now let’s look at the trapezoid. To find the area of a trapezoid, take the average length of the two parallel bases and multiply that length by the height: $A=\frac{(b_1+b_2)}{2}h$.

An example is provided below. Notice that the height of a trapezoid will always be perpendicular to the bases (just like when you find the height of a parallelogram).

Example

Find the area of the trapezoid.



Solution

$A=\frac{(b_1+b_2)}{2}h$	Start with the formula for the area of a trapezoid.
$A=\frac{(4+7)}{2}\cdot 2$ $A=\frac{11}{2}\cdot 2$ $A=11$	Substitute 4 and 7 for the bases and 2 for the height, and find A.

The area of the trapezoid is 11 cm^2.

Area Formulas

Use the following formulas to find the areas of different shapes.

square: $A = s^2$

rectangle: $A = l \cdot w$

parallelogram: $A = b \cdot h$

triangle: $A = \frac{1}{2} b \cdot h$

trapezoid: $A = \frac{(b_1 + b_2)}{2} h$

Working with Perimeter and Area

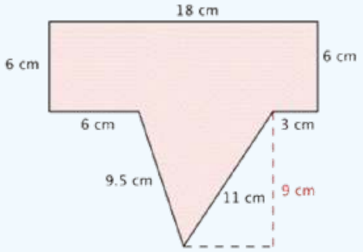
Often you need to find the area or perimeter of a shape that is not a standard polygon. Artists and architects, for example, usually deal with complex shapes. However, even complex shapes can be thought of as being composed of smaller, less complicated shapes, like rectangles, trapezoids, and triangles.

To find the perimeter of non-standard shapes, you still find the distance around the shape by adding together the length of each side.

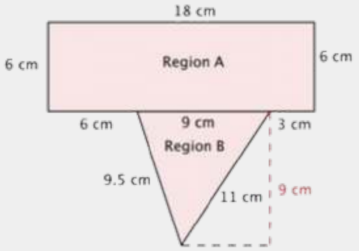
Finding the area of non-standard shapes is a bit different. You need to create regions *within* the shape for which you can find the area, and add these areas together. Have a look at how this is done below.

✓ Example

Find the area and perimeter of the polygon.



Solution

$P = 18 + 6 + 3 + 11 + 9.5 + 6 + 6$ $P = 59.5 \text{ cm}$	To find the perimeter, add together the lengths of the sides. Start at the top and work clockwise around the shape.
 18 cm 6 cm Region A 6 cm 6 cm 9 cm 3 cm Region B 9.5 cm 11 cm 9 cm *Area of Polygon* = (*Area of A*) + (*Area of B*)	To find the area, divide the polygon into two separate, simpler regions. The area of the entire polygon will equal the sum of the areas of the two regions.
$\text{Area of Region } A = l \cdot w$ $= 18 \cdot 6$ $= 108$	Region A is a rectangle. To find the area, multiply the length (18) by the width (6). The area of Region A is 108 cm^2.
$\text{Area of Region B} = \frac{1}{2} b \cdot h$ $= \frac{1}{2} \cdot 9 \cdot 9$ $= \frac{1}{2} \cdot 81$ $= 40.5$	Region B is a triangle. To find the area, use the formula $\frac{1}{2} bh$, where the base is 9 and the height is 9.

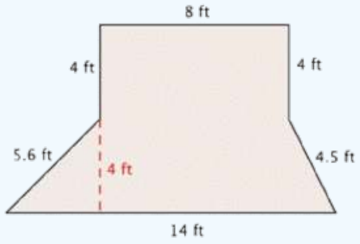
$108\text{ cm}^2 + 40.5\text{ cm}^2 = 148.5\text{ cm}^2$	The area of Region B is 40.5 cm^2. Add the regions together.

$$\begin{aligned}\text{Perimeter } &= 59.5\text{ cm}\\ \text{Area } &= 148.5\text{ cm}^2\end{aligned}$$

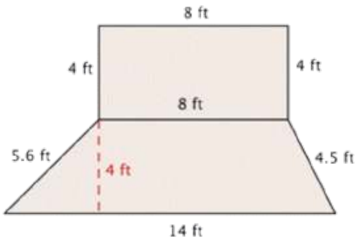
You also can use what you know about perimeter and area to help solve problems about situations like buying fencing or paint, or determining how big a rug is needed in the living room. Here’s a fencing example.

✓ Example

Rosie is planting a garden with the dimensions shown below. She wants to put a thin, even layer of mulch over the entire surface of the garden. The mulch costs \$3 a square foot. How much money will she have to spend on mulch?



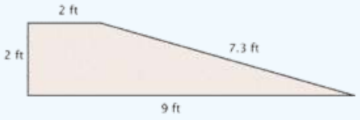
Solution

 8 ft 4 ft 4 ft 8 ft 5.6 ft 4 ft 4.5 ft 14 ft	This shape is a combination of two simpler shapes: a rectangle and a trapezoid. Find the area of each.
$A = l \cdot w$ $A = 8 \cdot 4$ $A = 32\text{ft}^2$	Find the area of the rectangle.
$A = \frac{(b_1+b_2)}{2}h$ $A = \frac{(14+8)}{2} \cdot 4$ $A = \frac{22}{2} \cdot 4$ $A = 11 \cdot 4$ $A = 44\text{ft}^2$	Find the area of the trapezoid.
$32\text{ft}^2 + 44\text{ft}^2 = 76\text{ft}^2$	Add the measurements.
$76\text{ft}^2 \cdot \$3 = \$228$	Multiply by \$3 to find out how much Rosie will have to spend.

Rosie will spend \$228 to cover her garden with mulch.

? Exercise

Find the area of the shape shown below.



A. 11ft^2
B. 18ft^2
C. 20.3ft
D. 262.8ft^2

Answer

A. Correct. This shape is a trapezoid, so you can use the formula $A=\frac{(b_1+b_2)}{2}h$ to find the area: $A=\frac{(2+9)}{2}\cdot 2$.
B. Incorrect. It looks like you multiplied 2 by 9 to get 18ft^2; this would work if the shape was a rectangle. This shape is a trapezoid, though, so use the formula $A=\frac{(b_1+b_2)}{2}h$. The correct answer is 11ft^2.
C. Incorrect. It looks like you added all the dimensions together. This would give you the perimeter. To find the area of a trapezoid, use the formula $A=\frac{(b_1+b_2)}{2}h$. The correct answer is 11ft^2.
D. Incorrect. It looks like you multiplied all of the dimensions together. This shape is a trapezoid, so you use the formula $A=\frac{(b_1+b_2)}{2}h$. The correct answer is 11ft^2.

Summary

The perimeter of a two-dimensional shape is the distance around the shape. It is found by adding up all the sides (as long as they are all the same unit). The area of a two-dimensional shape is found by counting the number of squares that cover the shape. Many formulas have been developed to quickly find the area of standard polygons, like triangles and parallelograms.

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7.2.3: Circles

Learning Objectives

- Identify properties of circles.
- Find the circumference of a circle.
- Find the area of a circle.
- Find the area and perimeter of composite geometric figures.

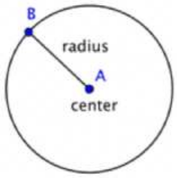
Introduction

Circles are a common shape. You see them all over: wheels on a car, Frisbees passing through the air, compact discs delivering data. These are all circles.

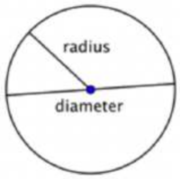
A circle is a two-dimensional figure just like polygons and quadrilaterals. However, circles are measured differently than these other shapes. You even have to use some different terms to describe them. Let's take a look at this interesting shape.

Properties of Circles

A circle represents a set of points, all of which are the same distance away from a fixed, middle point. This fixed point is called the center. The distance from the center of the circle (point A) to any point on the circle (point B) is called the **radius**. A circle is named by the point at its center, so this circle would be called Circle A.



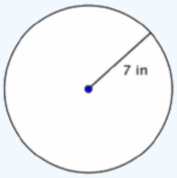
When two radii (the plural of radius) are put together to form a line segment across the circle, you have a **diameter**. The diameter of a circle passes through the center of the circle and has its endpoints on the circle itself.



The diameter of any circle is two times the length of that circle's radius. It can be represented by the expression $2r$, or "two times the radius." So if you know a circle's radius, you can multiply it by 2 to find the diameter; this also means that if you know a circle's diameter, you can divide by 2 to find the radius.

Example

Find the diameter of the circle.



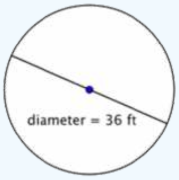
Solution

$d = 2r$ $d = 2(7)$ $d = 14$	The diameter is two times the radius, or $2r$. The radius of this circle is 7 inches, so the diameter is $2(7) = 14$ inches.

The diameter is 14 inches.

Example

Find the radius of the circle.



Solution

$r=\frac{1}{2}d$ $r=\frac{1}{2}(36)$ $r=18$	The radius is half the diameter, or $\frac{1}{2}d$. The diameter of this circle is 36 feet, so the radius is $\frac{1}{2}(36)=18$ feet.

The radius is 18 feet.

Circumference

The distance around a circle is called the **circumference**. (Recall, the distance around a polygon is the perimeter.)

One interesting property about circles is that the ratio of a circle's circumference and its diameter is the same for all circles. No matter the size of the circle, the ratio of the circumference and diameter will be the same.

Some actual measurements of different items are provided below. The measurements are accurate to the nearest millimeter or quarter inch (depending on the unit of measurement used). Look at the ratio of the circumference to the diameter for each one. Although the items are different, the ratio for each is approximately the same.

Item	**Circumference (C) (rounded to nearest hundredth)**	**Diameter (d)**	**Ratio $\frac{C}{d}$**
Cup	253 mm	79 mm	$\frac{253}{79}=3.2025\ldots$
Quarter	84 mm	27 mm	$\frac{84}{27}=3.1111\ldots$
Bowl	37.25 inches	11.75 inches	$\frac{37.25}{11.75}=3.1702\ldots$

The circumference and the diameter are approximate measurements, since there is no precise way to measure these dimensions exactly. If you were able to measure them more precisely, however, you would find that the ratio $\frac{C}{d}$ would move towards 3.14 for each of the items given. The mathematical name for the ratio $\frac{C}{d}$ is **pi**, and is represented by the Greek letter π.

π is a non-terminating, non-repeating decimal, so it is impossible to write it out completely. The first 10 digits of π are 3.141592653; it is often rounded to 3.14 or estimated as the fraction $\frac{22}{7}$. Note that both 3.14 and $\frac{22}{7}$ are *approximations* of π, and are used in calculations where it is not important to be precise.

Since you know that the ratio of circumference to diameter (or π) is consistent for all circles, you can use this number to find the circumference of a circle if you know its diameter.

$$\frac{C}{d}=\pi,\ \text{so } C=\pi d$$

$$\text{Also, since } d=2r,\ \text{then } C=\pi d=\pi(2r)=2\pi r.$$

Circumference of a Circle

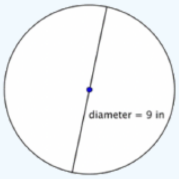
To find the circumference (C) of a circle, use one of the following formulas:

If you know the diameter (d) of a circle: $C=\pi d$

If you know the radius (r) of a circle: $C = 2\pi r$

Example

Find the circumference of the circle.



Solution

$C = \pi d$ $C = \pi \cdot 9$ $C \approx 3.14 \cdot 9$ $C \approx 28.26$	To calculate the circumference given a diameter of 9 inches, use the formula $C = \pi d$. Use 3.14 as an approximation for π.
	Since you are using an approximation for π, you cannot give an exact measurement of the circumference. Instead, you use the symbol $\approx$ to indicate "approximately equal to."

The circumference is 9 π or approximately 28.26 inches.

Example

Find the circumference of a circle with a radius of 2.5 yards.

Solution

$C = 2\pi r$ $C = 2\pi \cdot 2.5$ $C = \pi \cdot 5$ $C \approx 3.14 \cdot 5$ $C \approx 15.7$	To calculate the circumference of a circle given a radius of 2.5 yards, use the formula $C = 2\pi r$. Use 3.14 as an approximation for π.

The circumference is 5 π or approximately 15.7 yards.

Exercise

A circle has a radius of 8 inches. What is its circumference, rounded to the nearest inch?

A. 25 inches
B. 50 inches
C. 64 inches^2
D. 201 inches

Answer

A. Incorrect. You multiplied the radius times π; the correct formula for circumference when the radius is given is $C = 2\pi r$. The correct answer is 50 inches.
B. Correct. If the radius is 8 inches, the correct formula for circumference when the radius is given is $C = 2\pi r$. The correct answer is 50 inches.
C.
D. Incorrect. It looks like you squared 8 and then multiplied 64 by π to arrive at this answer. Remember that the formula for circumference when the radius is given is $C = 2\pi r$. The correct answer is 50 inches.

Area

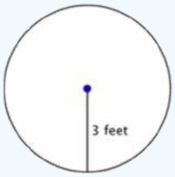
π is an important number in geometry. You have already used it to calculate the circumference of a circle. You use π when you are figuring out the *area* of a circle, too.

Area of a Circle

To find the area (A) of a circle, use the formula: $A=\pi r^2$

Example

Find the area of the circle.



Solution

$A=\pi r^2$ $A=\pi\cdot 3^2$ $A=\pi\cdot 9$ $A\approx 3.14\cdot 9$ $A\approx 28.26$	To find the area of this circle, use the formula $A=\pi r^2$. Remember to write the answer in terms of square units, since you are finding the area.

The area is 9 π or approximately 28.26 feet^2.

Exercise

A button has a diameter of 20 millimeters. What is the area of the button? Use 3.14 as an approximation of π.

A. 62.8 mm
B. 314 mm^2
C. 400 mm^2
D. 1256 mm^2

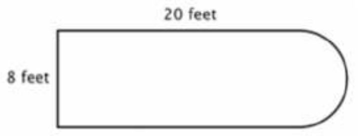
Answer

A. Incorrect. You found the circumference of the button: $20\cdot 3.14=62.8$. To find the area, use the formula $A=\pi r^2$. The correct answer is 314 mm^2
B.
C.
D.

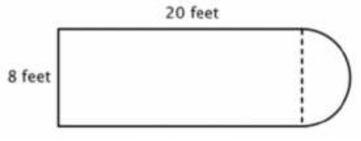
Composite Figures

Now that you know how to calculate the circumference and area of a circle, you can use this knowledge to find the perimeter and area of composite figures. The trick to figuring out these types of problems is to identify shapes (and parts of shapes) within the composite figure, calculate their individual dimensions, and then add them together.

For example, look at the image below. Is it possible to find the perimeter?

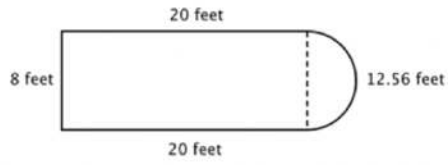


The first step is to identify simpler figures within this composite figure. You can break it down into a rectangle and a semicircle, as shown below.



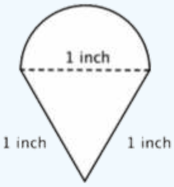
You know how to find the perimeter of a rectangle, and you know how to find the circumference of a circle. Here, the perimeter of the three solid sides of the rectangle is $8+20+20=48 \text{ feet}$. (Note that only three sides of the rectangle will add into the perimeter of the composite figure because the other side is not at an edge; it is covered by the semicircle!)

To find the circumference of the semicircle, use the formula $C=\pi d$ with a diameter of 8 feet, then take half of the result. The circumference of the semicircle is 4 π, or approximately 12.56 feet, so the total perimeter is about 60.56 feet.

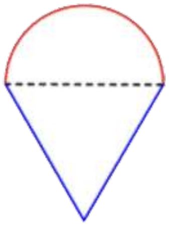
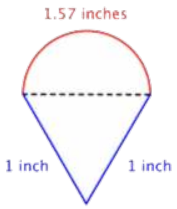


✓ Example

Find the perimeter (to the nearest hundredth) of the composite figure made up of a semi-circle and a triangle.



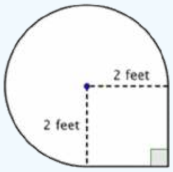
Solution

	Identify smaller shapes within the composite figure. This figure contains a semicircle and a triangle.
Diameter $(d)=1$ $C=\pi d$ $C=\pi(1)$ $C=\pi$	Find the circumference of the circle. Then divide by 2 to find the circumference of the semi-circle.
Circumference of semicircle $=\frac{1}{2}\pi$ or approximately 1.57 inches $1+1+\frac{1}{2}\pi \approx 3.57 \text{ inches}$ 	Find the total perimeter by adding the circumference of the semicircle and the lengths of the two legs. Since our measurement of the semi-circle's circumference is approximate, the perimeter will be an approximation also.

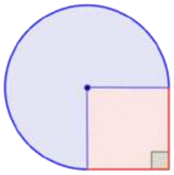
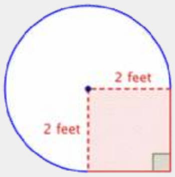
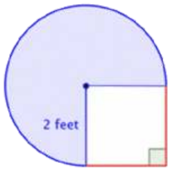
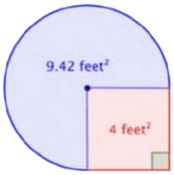
Approximately 3.57 inches

✓ Example

Find the area of the composite figure, made up of three-quarters of a circle and a square, to the nearest hundredth.



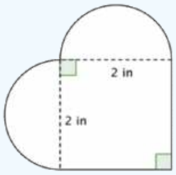
Solution

	Identify smaller shapes within the composite figure. This figure contains a circular region and a square. If you find the area of each, you can find the area of the entire figure.
 Area of square $= s^2$ $= (2)^2$ $= 4\text{ft}^2$	Find the area of the square.
 Area of full circle $= \pi r^2$ $= \pi(2)^2$ $= 4\pi\text{ft}^2$	Find the area of the circular region. The radius is 2 feet.
Area of region $= \frac{3}{4} \cdot 4\pi$ $= 3\pi$ $\approx 3 \cdot 3.14 \text{ ft}^2$ This is approximately 9.42 feet^2	Note that the region is $\frac{3}{4}$ of a whole circle, so you need to multiply the area of the circle by $\frac{3}{4}$. Use 3.14 as an approximation for π.
 $4 \text{ feet}^2 + 3\pi \text{ feet}^2 =$ approximately 13.42 feet^2	Add the two regions together. Since your measurement of the circular's area is approximate, the area of the figure will be an approximation also.

The area is approximately 13.42 feet^2.

? Exercise

What is the area (to the nearest hundredth) of the figure shown below? (Both rounded regions are semi-circles.)



A. 16.56 in^2
B. 7.14 in^2
C. 4 in^2
D. 3.14 in^2

Answer

A.
B.
C.
D.

Summary

Circles are an important geometric shape. The distance around a circle is called the circumference, and the interior space of a circle is called the area. Calculating the circumference and area of a circle requires a number called pi (π), which is a non-terminating, non-repeating decimal. Pi is often approximated by the values 3.14 and $\frac{22}{7}$. You can find the perimeter or area of composite shapes, including shapes that contain circular sections, by applying the circumference and area formulas where appropriate.

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SECTION OVERVIEW

7.3: Volume of Geometric Solids

Topic hierarchy

7.3.1: Solids

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7.3.1: Solids

Learning Objectives

- Identify geometric solids.
- Find the volume of geometric solids.
- Find the volume of a composite geometric solid.

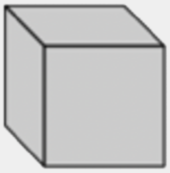
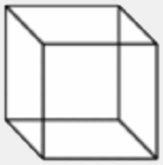
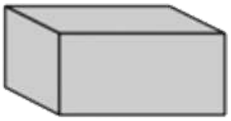
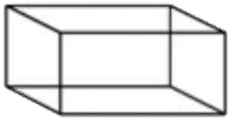
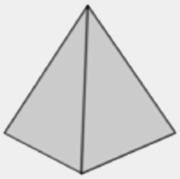
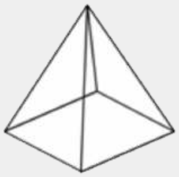
Introduction

Living in a two-dimensional world would be pretty boring. Thankfully, all of the physical objects that you see and use every day, such as computers, phones, cars, and shoes, exist in three dimensions. They all have length, width, and height. (Even very thin objects like a piece of paper are three-dimensional. The thickness of a piece of paper may be a fraction of a millimeter, but it does exist.)

In the world of geometry, it is common to see three-dimensional figures. In mathematics, a flat side of a three-dimensional figure is called a **face**. **Polyhedrons** are shapes that have four or more faces, each one being a polygon. These include cubes, prisms, and pyramids. Sometimes you may even see single figures that are composites of two of these figures. Let’s take a look at some common polyhedrons.

Identifying Solids

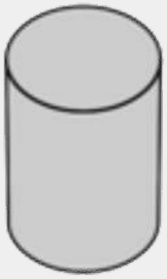
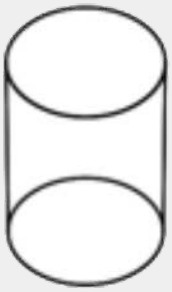
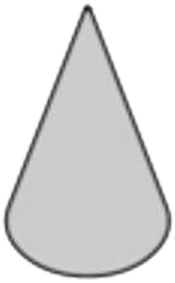
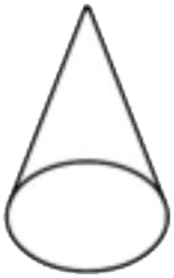
The first set of solids contains rectangular bases. Have a look at the table below, which shows each figure in both solid and transparent form.

Name	**Definition**	**Solid Form**	**Transparent Form**
Cube	A six-sided polyhedron that has congruent squares as faces.		
Rectangular prism	A polyhedron that has three pairs of congruent, rectangular, parallel faces.		
Pyramid	A polyhedron with a polygonal base and a collection of triangular faces that meet at a point.		

Notice the different names that are used for these figures. A **cube** is different than a square, although they are sometimes confused with each other; a cube has three dimensions, while a square only has two. Likewise, you would describe a shoebox as a **rectangular prism** (not simply a rectangle), and the ancient **pyramids** of Egypt as…well, as pyramids (not triangles)!

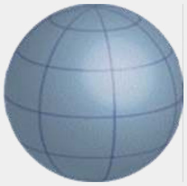
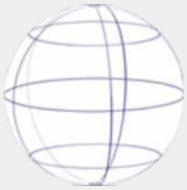
In this next set of solids, each figure has a circular base.

Name	**Definition**	**Solid Form**	**Transparent Form**

Cylinder	A solid figure with a pair of circular, parallel bases and a round, smooth face between them.		
			

Take a moment to compare a pyramid and a **cone**. Notice that a pyramid has a rectangular base and flat, triangular faces; a cone has a circular base and a smooth, rounded body.

Finally, let’s look at a shape that is unique: a **sphere**.

Name	**Definition**	**Solid Form**	**Transparent Form**
Sphere	A solid, round figure where every point on the surface is the same distance from the center.		

There are many spherical objects all around you: soccer balls, tennis balls, and baseballs being three common items. While they may not be perfectly spherical, they are generally referred to as spheres.

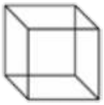
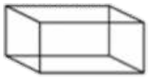
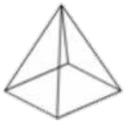
✓ Example

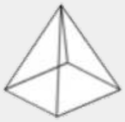
A three-dimensional figure has the following properties:

- **It has a rectangular base.**
- **It has four triangular faces.**

What kind of a solid is it?

Solution

  	A rectangular base indicates that it must be a cube, rectangular prism, or pyramid.

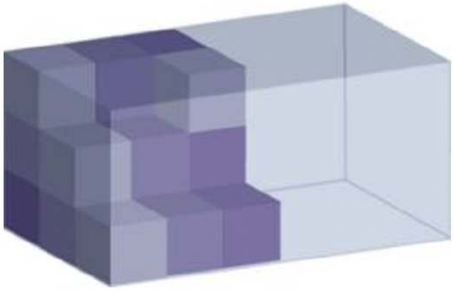
	Since the faces are triangular, it must be a pyramid.

The solid is a pyramid.

Volume

Recall that perimeter measures one dimension (length), and area measures two dimensions (length and width). To measure the amount of space a three-dimensional figure takes up, you use another measurement called **volume**.

To visualize what “volume” measures, look back at the transparent image of the rectangular prism mentioned earlier (or just think of an empty shoebox). Imagine stacking identical cubes inside that box so that there are no gaps between any of the cubes. Imagine filling up the entire box in this manner. If you counted the number of cubes that fit inside that rectangular prism, you would have its volume.

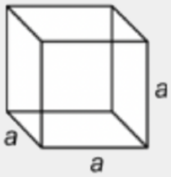
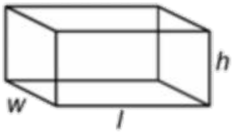


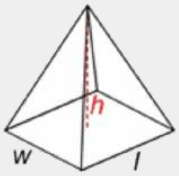
Volume is measured in cubic units. The shoebox illustrated above may be measured in cubic inches (usually represented as in^3 or inches^3), while the Great Pyramid of Egypt would be more appropriately measured in cubic meters (m^3 or meters^3).

To find the volume of a geometric solid, you could create a transparent version of the solid, create a bunch of 1 by 1 by 1 cubes, and then stack them carefully inside. However, that would take a long time! A much easier way to find the volume is to become familiar with some geometric formulas, and to use those instead.

Let’s go through the geometric solids once more and list the volume formula for each.

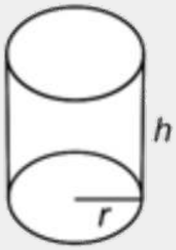
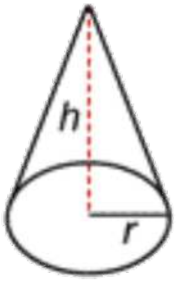
As you look through the list below, you may notice that some of the volume formulas look similar to their area formulas. To find the volume of a rectangular prism, you find the area of the base and then multiply that by the height.

Name	**Transparent Form**	**Volume Formula**
Cube		$V = a \cdot a \cdot a = a^3$ $a =$ the length of one side
Rectangular prism		$V = l \cdot w \cdot h$ l = length w = width h = height

		$V = \frac{l\cdot w\cdot h}{3}$ $l =$ length $w =$ width $h =$ height

Remember that all cubes are rectangular prisms, so the formula for finding the volume of a cube is the area of the base of the cube times the height.

Now let’s look at solids that have a circular base.

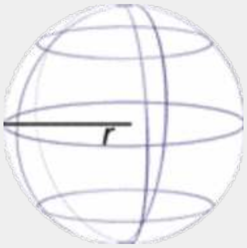
Name	**Transparent Form**	**Volume Formula**
Cylinder		$V = \pi\cdot r^2\cdot h$ $r =$ radius $h =$ height
Cone		$V = \frac{\pi\cdot r^2\cdot h}{3}$ $r =$ radius $h =$ height

Here you see the number π again.

The volume of a **cylinder** is the area of its base, πr^2, times its height, h.

Compare the formula for the volume of a cone ($V = \frac{\pi\cdot r^2\cdot h}{3}$) with the formula for the volume of a pyramid ($V = \frac{l\cdot w\cdot h}{3}$). The numerator of the cone formula is the volume formula for a cylinder, and the numerator of the pyramid formula is the volume formula for a rectangular prism. Then divide each by 3 to find the volume of the cone and the pyramid. Looking for patterns and similarities in the formulas can help you remember which formula refers to a given solid.

Finally, the formula for a sphere is provided below. Notice that the radius is cubed, not squared and that the quantity πr^3 is multiplied by $\frac{4}{3}$.

Name	**Wireframe Form**	**Volume Formula**
Sphere		$V = \frac{4}{3}\pi r^3$ $r =$ radius

Applying the Formulas

You know how to identify the solids, and you also know the volume formulas for these solids. To calculate the actual volume of a given shape, all you need to do is substitute the solid's dimensions into the formula and calculate.

In the examples below, notice that cubic units (meters^3, inches^3, feet^3) are used.

Example

Find the volume of a cube with side lengths of 6 meters.

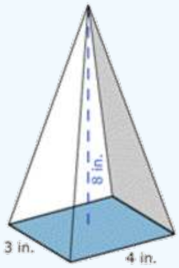
Solution

$V = a \cdot a \cdot a = a^3$ $a =$ side length	Identify the proper formula to use.
$V = 6 \cdot 6 \cdot 6 = 6^3$	Substitute $a = 6$ into the formula.
$6 \cdot 6 \cdot 6 = 216$	Calculate the volume.

Volume $= 216$ meters ^{3}

Example

Find the volume of the shape shown below.



Solution

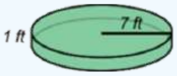
Pyramid	Identify the shape. It has a rectangular base and rises to a point, so it is a pyramid.
$V = \frac{l \cdot w \cdot h}{3}$	Identify the proper formula to use.
$l =$ length $w =$ width $h =$ height $4 =$ length $3 =$ width $8 =$ height	Use the image to identify the dimensions. Then substitute $l = 4, w = 3,$ and $h = 8$ into the formula.
$V = \frac{4 \cdot 3 \cdot 8}{3}$ $V = \frac{96}{3}$ $= 32$	Calculate the volume.

The volume of the pyramid is 32 inches ^{3}.

Example

Find the volume of the shape shown below.

Use 3.14 for π, and round the answer to the nearest hundredth.



Solution

Cylinder	Identify the shape. It has a circular base and has uniform thickness (or height), so it is a cylinder.
$V=\pi\cdot r^2\cdot h$	Identify the proper formula to use.
$V=\pi\cdot 7^2\cdot 1$	Use the image to identify the dimensions. Then substitute $r=7$ and $h=1$ into the formula.
$V=\pi\cdot 49\cdot 1$ $=49\pi$ ≈ 153.86	Calculate the volume, using 3.14 as an approximation for π.

The volume is 49 π or approximately 153.86 feet ^3.

? Exercise

Find the volume of a rectangular prism that is 8 inches long, 3 inches wide, and 10 inches tall.

A. 24 inches^2
B. 30 inches^2
C. 240 inches^3
D. 720 inches^3

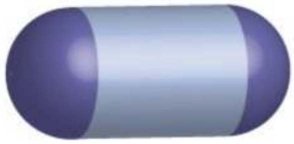
Answer

A. Incorrect. It looks like you multiplied 8 inches by 3 inches to arrive at 24 inches^2. This is the area of the base, not the volume of the rectangular prism. The correct answer is 240 inches^3.
B. Incorrect. It looks like you multiplied 3 inches by 10 inches to arrive at 30 inches^2. This is the area of one face of the rectangular prism, not the volume of the rectangular prism. The correct answer is 240 inches^3.
C. Correct. To find the volume of the rectangular prism, use the formula $V=l\cdot w\cdot h$, and then substitute in the values for the length, width, and height. $8\text{ inches }\cdot 3\text{ inches }\cdot 10\text{ inches }=240\text{ inches}^3$.
D. Incorrect. Check your multiplication again. To find the volume of the rectangular prism, use the formula $V=l\cdot w\cdot h$ and then substitute in the values for the length, width, and height. The correct answer is 240 inches^3.

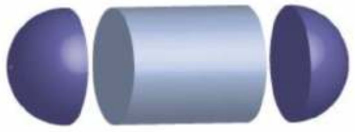
Composite Solids

Composite geometric solids are made from two or more geometric solids. You can find the volume of these solids as well, as long as you are able to figure out the individual solids that make up the composite shape.

Look at the image of a capsule below. Each end is a half-sphere. You can find the volume of the solid by taking it apart. What solids can you break this shape into?



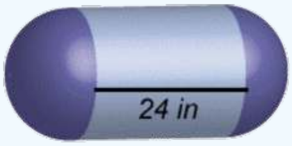
You can break it into a cylinder and two half-spheres.



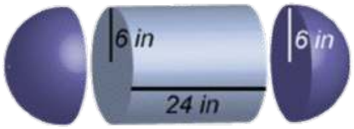
Two half-spheres form a whole one, so if you know the volume formulas for a cylinder and a sphere, you can find the volume of this capsule.

✓ Example

If the radius of the spherical ends is 6 inches, find the volume of the solid below. Use 3.14 for π. Round your final answer to the nearest whole number.



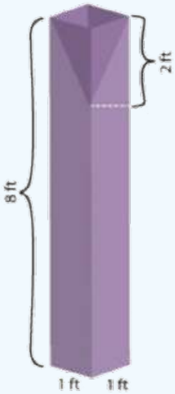
Solution

	Identify the composite solids. This capsule can be thought of as a cylinder with a half-sphere on each end.
Volume of a cylinder: $\pi \cdot r^2 \cdot h$ Volume of a sphere: $\frac{4}{3}\pi r^3$	Identify the proper formulas to use.
Volume of a cylinder: $\pi \cdot 6^2 \cdot 24$ Volume of a sphere: $\frac{4}{3}\pi \cdot 6^3$	Substitute the dimensions into the formulas. The height of a cylinder refers to the section between the two circular bases. This dimension is given as 24 inches, so $h=24$.
Volume of the cylinder: $V = \pi \cdot 36 \cdot 24$ $= 864 \cdot \pi$ ≈ 2712.96 Volume of the sphere: $V = \frac{4}{3}\pi \cdot 216$ $= 288 \cdot \pi$ ≈ 904.32	The radius of the sphere is 6 inches. You can use $r=6$ in both formulas. Calculate the volume of the cylinder and the sphere.
Volume of capsule: $2712.96 + 904.32 \approx 3617.28$	Add the volumes.

The volume of the capsule is $1,152\pi$ or approximately 3617 inches^3.

✓ Example

A sculptor carves a rectangular prism out of a solid piece of wood. Then, at the top, she hollows out an inverted pyramid. The solid, and its dimensions, are shown at right. What is the volume of the finished piece?



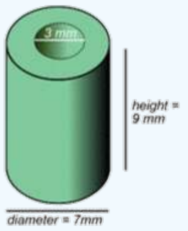
Solution

	Identify the composite solids. This sculpture can be thought of as a rectangular prism with a pyramid removed.
Volume of rectangular prism: $l\cdot w\cdot h$	Identify the proper formulas to use.
Volume of pyramid: $\frac{l\cdot w\cdot h}{3}$	
Volume of rectangular prism: $1\cdot 1\cdot 8=8$ Volume of pyramid: $\frac{1\cdot 1\cdot 2}{3}=\frac{2}{3}$	Substitute the dimensions into the formulas, and calculate.
Volume of sculpture: $V=8-\frac{2}{3}=7\frac{1}{3}$	Subtract the volume of the pyramid from the volume of the rectangular prism.

The volume of the sculpture is $7\frac{1}{3}$ feet ^{3}.

? Exercise

A machine takes a solid cylinder with a height of 9mm and a diameter of 7mm, and bores a hole all the way through it. The hole that it creates has a diameter of 3mm. Which of the following expressions would correctly find the volume of the solid?



A. $(\pi\cdot 7^2\cdot 9)-(\pi\cdot 3^2\cdot 9)$
B. $(\pi\cdot 3.5^2\cdot 9)-(\pi\cdot 1.5^2\cdot 9)$
C. $(\pi\cdot 7^2\cdot 9)+(\pi\cdot 3^2\cdot 9)$
D. $(\pi\cdot 3.5^2\cdot 9)+(\pi\cdot 1.5^2\cdot 9)$

Answer

A. Incorrect. Remember that the formula for the volume of a cylinder is πr^2h, not πd^2h. The correct answer is $(\pi\cdot 3.5^2\cdot 9)-(\pi\cdot 1.5^2\cdot 9)$.
B. Correct. You find the volume of the entire cylinder by multiplying $\pi\cdot 3.5^2\cdot 9$, then subtract the empty cylinder in the middle, which is found by multiplying $\pi\cdot 1.5^2\cdot 9$.
C. Incorrect. Remember that the formula for the volume of a cylinder is πr^2h, not πd^2h. Also, you should subtract the volume of the inner cylinder from the volume of the outer cylinder, not add it. The correct answer is $(\pi\cdot 3.5^2\cdot 9)-(\pi\cdot 1.5^2\cdot 9)$.
D. Incorrect. You have calculated the correct volumes for the outer and inner cylinders, but you should subtract the inner from the outer instead of adding them. The correct answer is $(\pi\cdot 3.5^2\cdot 9)-(\pi\cdot 1.5^2\cdot 9)$.

Summary

Three-dimensional solids have length, width, and height. You use a measurement called volume to figure out the amount of space that these solids take up. To find the volume of a specific geometric solid, you can use a volume formula that is specific to that solid. Sometimes, you will encounter composite geometric solids. These are solids that combine two or more basic solids. To find the volume of these, identify the simpler solids that make up the composite figure, find the volumes of those solids, and combine them as needed.

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CHAPTER OVERVIEW

8: Concepts in Statistics

8.1: Statistical Graphs and Tables
  8.1.1: Graphing Data
  8.1.2: Other Types of Graphs
8.2: Measures of Center
  8.2.1: Measures of Center
8.3: Graphical Representations
  8.3.1: Use and Misuse of Graphical Representations
8.4: Probability
  8.4.1: Probability

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SECTION OVERVIEW

8.1: Statistical Graphs and Tables

Topic hierarchy

8.1.1: Graphing Data

8.1.2: Other Types of Graphs

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8.1.1: Graphing Data

Learning Objectives

- Read and interpret data from tables and pictographs.
- Read and interpret data from bar graphs and histograms.

Introduction

A nurse is collecting blood type data from her patients. When a new patient is checked in, the nurse does a simple finger-prick test to see whether the patient's blood is type A, B, AB, or O. (These are the four possible blood types. Each one also carries a + or - to represent the RH factor, but for our purposes, let's just track the type, not the + or -.) She tracks her results by creating a two-column table with the patient's name and blood type.

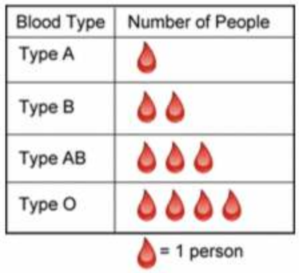
Name	Blood Type
Dominique	A
Ilya	O
Raul	AB
Madison	O
Philip	AB
Samuel	B
Josefine	O
Brett	O
Paula	B
Leticia	AB

The information in this table is an example of **data**, or information. In this case, the nurse has gathered a fair amount of data about her patients' blood types. By analyzing the data, she can learn more about the range of patients that she serves.

Data helps us make many kinds of decisions. Organizing data into graphs can help us get a clear picture of a situation and can often help us make decisions based on the picture. So how do you take data and make a picture out of it? Let's take a look.

Pictures of Data

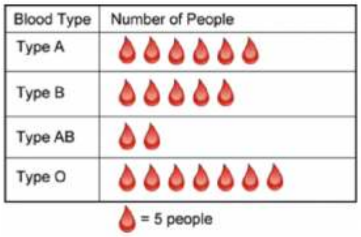
Let's return to the data set used previously. If the nurse wanted to represent the data visually, she could use a **pictograph**. Pictographs represent data using images. This visual presentation helps illustrate that for the data in her table, Type O blood is the most common and Type A blood is the least common.



Blood Type	Number of People
Type A	💧
Type B	💧💧
Type AB	💧💧💧
Type O	💧💧💧💧

💧 = 1 person

Interested by the results of this small survey, the nurse continues to document the blood types of her patients until she has surveyed 100 people. She puts all of this data in a table, but she finds that it is hard for her to quickly identify what the data is telling her. She decides to make another pictograph using a different scale.



To read this pictograph, all you need is the scale, which, in this case, is the number of people that each blood drop symbol represents. In this graph, each blood drop represents 5 people. There are six drops next to Type A, so $5 \cdot 6 = 30$ people had Type A blood. The table below shows the rest of the information.

Blood Type	Number of People
Type A	$6 \text{ drops } \cdot 5 \text{ people } = 30 \text{ people}$
Type B	$5 \text{ drops } \cdot 5 \text{ people } = 25 \text{ people}$
Type AB	$2 \text{ drops } \cdot 5 \text{ people } = 10 \text{ people}$
Type O	$7 \text{ drops } \cdot 5 \text{ people } = 35 \text{ people}$

Example

The pictograph below shows the number of medals earned at an international competition. How many medals did Japan earn?

Country	Medals
Japan	**MMMMM**
Argentina	**M**
Germany	**MMMMMM**
Egypt	**MMMM**
	M = 4 medals

Solution

	Look at the scale of the pictograph. Each **M** represents 4 medals.
$5 \cdot 4 = 20$	Japan has **5M** symbols, so the total number of medals is $5 \cdot 4 = 20$.

Japan earned 20 medals.

Exercise

Which table accurately represents the data shown in the pictograph below?

Employee	Hourly wage
Wayne	$ $ $ $ $
Sarah	$ $ $ $ $ $ $
Leigh	$ $ $ $
	$ = $4

A.

Employee	Hourly wage
Wayne	$5
Sarah	$7
Leigh	$4

B.

Employee	Hourly wage
Wayne	$20
Sarah	$22
Leigh	$19

C.

Employee	Hourly wage
Wayne	$10
Sarah	$14
Leigh	$8

D.

Employee	Hourly wage
Wayne	$20
Sarah	$28
Leigh	$16

Answer

A. You identified the correct number of symbols, but it looks like you forgot to factor in the scale. Each dollar sign represents $4, so multiply the number of dollar signs in a row by 4. The correct answer is Table D.
B. You correctly identified that Wayne earns $20, since the table indicates that he has 5 dollar signs, and each dollar sign represents $4. Sarah's row shows that she earns seven dollar sign symbols, so she earns $28, not $22. The correct answer is Table D.
C. You identified the correct number of symbols, but it looks like you used the wrong scale. Each dollar sign represents $4, not $2. Multiply the number of dollar signs in a row by 4. The correct answer is Table D.
D. Each dollar sign symbol represents $4, so if you multiply the number of dollar signs in a row by $4, you will find that Wayne earns $20, Sarah earns $28, and Leigh earns $16.

Bar Graphs

Representing data as pictures doesn't always make sense either. **Bar graphs** are an alternative (and popular) way to represent data sets, especially those with large amounts of data or which do not lend themselves well to individual symbols. In a bar graph, the number of items in a data category is represented by the height or length of bars.

As when reading pictographs, paying attention to the scale is essential: small differences in the height of two bars can sometimes represent thousands of dollars, for example!

Let's consider an example. Here is some information about the average life span of five animals in the wild, presented in a table.

Source: National Geographic, accessed July 2011

Animal	Average Life Span in the Wild (Years)
Zebra	25

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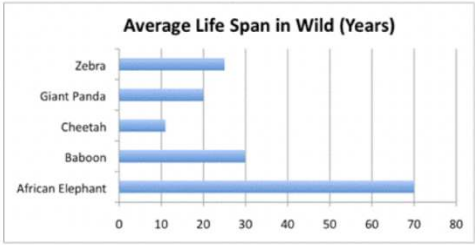
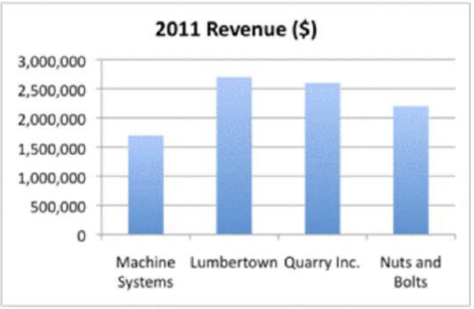
Giant Panda	20
Cheetah	11
Baboon	30
African Elephant	70

This data is fine in a table, but presenting it as a bar graph helps the viewer compare the different life spans more easily. Look at the bar graph below. In this example, the animals are listed on the left side of the graph (also called the **y-axis**), and the life span in years is listed on the bottom (the **x-axis**). The graph shows the information by the length of the bar associated with each animal name.



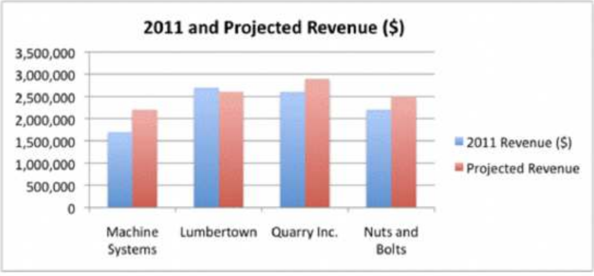
Bar graphs are generally used to compare quantities, not to determine exact quantities, especially when the scale is large, as in the next graph.

The bar graph below shows total revenue for four fictional companies in 2011. Notice that the scale, on the y-axis, is very large; each horizontal line represents an increase of $500,000. For this reason, it is difficult to tell exactly how much money each company made in 2011. However, comparing the bars is straightforward. Glancing at the data, you can tell that Lumbertown earned the most (a little over $2,500,000), while Machine Systems earned the least (about one million less, at just over $1,500,000).



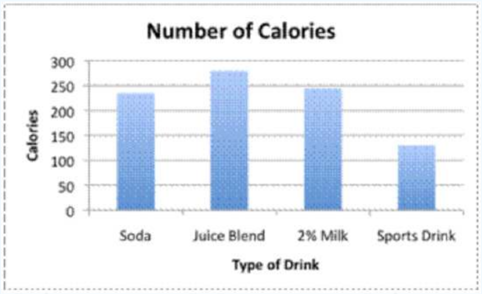
You can also use bar graphs to showcase multiple pieces of information about a specific situation. For example, let's show the next year's projected revenue for each company on the graph that you just looked at. You can leave the existing bars in the graph and just add four more.

The blue columns remain, but now they are accompanied by four new red columns that represent the projected revenue for these companies. Again, this data could be expressed in a table. With a bar graph, you gain ease of quick comparison, but lose the detail of the exact values. Looking at this graph tells you that while Lumbertown has the highest revenue for 2011, it is projected to decrease. Conversely, Machine Systems is projected to increase its revenue. Seeing data visually can help you understand the story that the data is telling about a situation.



Example

Use the graph to list the drinks from the most number of calories to the least number of calories (serving size: 16 ounces).



Solution

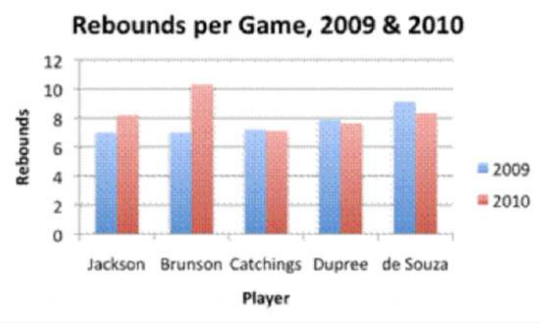
	The y-axis shows total calories, and the x-axis shows the drink. The taller the bar, the more calories the drink has.
$\text{Juice Blend } \approx 275$	The juice blend contains over 250 calories, so it has the most calories per serving.
$2\% \text{ Milk } \approx 245$ $\text{Soda } \approx 230$	Soda and 2% milk are both between 200 and 250 calories, but the bar for 2% milk is taller, so it must contain more calories.
$\text{Sports Drink } \approx 125$	Sports drink has the shortest bar; it contains about 125 calories.

From most to least number of calories per serving:

Juice Blend, 2% Milk, Soda, Sports Drink

Example

Based on the graph below, which player’s rebounding increased the most from 2009 to 2010?



Source: WNBA.com, accessed July 2011

Solution

Jackson and Brunson	The y-axis shows rebounds per game, and the x-axis shows the player’s name. A taller bar represents more rebounds per game by the player. This graph shows two sets of data: one for 2009, in blue, and one for 2010, in red. To compare the data from one year to the next, compare the heights of the two bars for each player. Two players had higher rebound averages in 2010 than they did in 2009. This is indicated by the red bar being taller than the blue bar. The other players’ red bars are shorter, so their rebounds decreased. Comparing the sizes of the increases, you can tell that Brunson increased her per game rebounding *more* than Jackson did.

The player whose rebounding increased the most from 2009 to 2010 was Brunson.

Histograms

The data sets you have looked at so far have shown **categorical data**. Categorical data is data that can be separated into categories. For instance, data about eye color (brown, blue, green), blood type (O, A, B, AB), and the type of computer you use at work or home (PC or Mac) are all categorical.

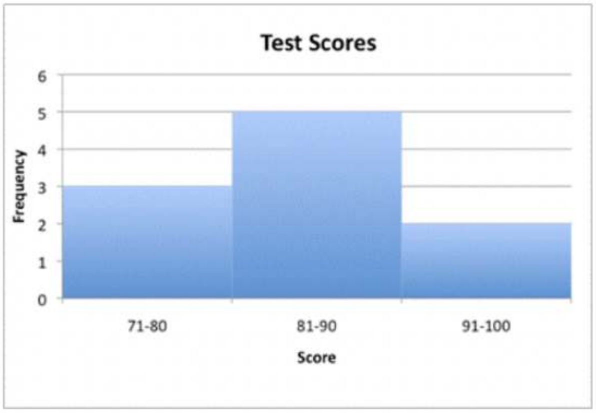
Quantitative data is sometimes called *numerical data*, because it is data that is represented by numbers. Quantitative data sets consist of quantities such as age (1, 2, 23, 34, 77...), test scores (90, 95, 100, 72...), and height (55 inches, 50 inches, 68 inches, ...). Notice that all of the data here are numbers.

Continuous quantitative data can be graphed using a **histogram**. A histogram resembles a bar graph, but instead of having categories along the axis, it has numbers listed in order and usually grouped in intervals (such as 0, to 10, 11 to 20, and so on). While the bars in bar graphs can have space between the bars, the bars of a histogram are usually touching, as the data is continuous.

Let’s look at how you could display the set of test score data listed below.

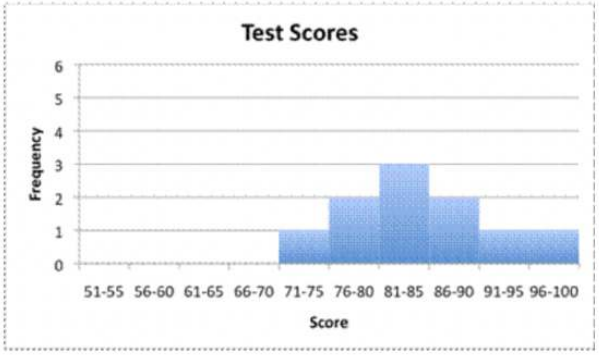
Name	Score (0 to 100)
Alex	81
Beatriz	73
Celia	79
Donnie	91
Erykah	87
Fred	79
Gigi	81
Helene	84
Irma	88
Joelle	96

Looking at the scores, you can tell that some people scored in the 70s, some scored in the 80s, and some scored in the 90s. You can organize the data in a histogram with the continuous data from 0 to 100, by using the intervals to 71 to 80, 81 to 90, and 91 to 100.



There are only three intervals for this histogram and ten data points. For a histogram to be meaningful, it should include 100 data points or more and 7 intervals or more. (For this reason, many histograms are created using a variety of technical tools.) The histogram then takes on interesting shapes that can provide a lot of information.

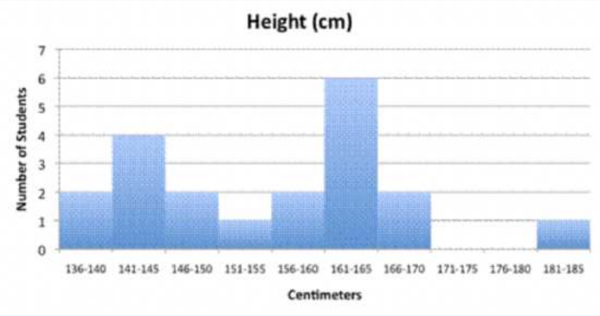
Even with only 10 data points, you find a much more meaningful histogram by using smaller intervals (such as 71 to 75, 76 to 80, 81 to 85, 86 to 90, 91 to 95, and 96 to 100).



This second graph shows a bit of a curve to the data. Nobody scored 70 or lower, and the data peaks in the 81 to 85 range. Increasing the number of intervals has an effect on the shape of the graph, and helps us see some trends that are in the data.

✓ Example

Based on the histogram below, how many students are between 151 and 180 centimeters tall?



Solution

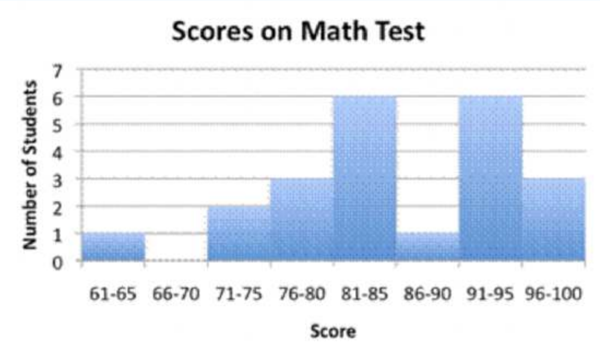
Number of students in each interval:	

$151 \text{ to } 155,\ 1$ $156 \text{ to } 160,\ 2$ $161 \text{ to } 165,\ 6$ $166 \text{ to } 170,\ 2$ $171 \text{ to } 175,\ 0$ $176 \text{ to } 180,\ 0$	Each interval in this histogram is 5 centimeters.
$1+2+6+2+0+0=11$	Add the number of students.

11 students are between 151 and 180 centimeters tall.

Exercise

A teacher made this histogram to track scores on a recent math test. How many students scored in the 91 to 100 range?



A. 3
B. 6
C. 9
D. 22

Answer

A. Incorrect. Three students scored between 96 to 100 on the test, but if the teacher wants to find out how many students scored between 91 to 100, then she also has to consider the number of students who scored between 91 to 95. The correct answer is 9.
B. Incorrect. Six students scored between 91 to 95 on the test, but if the teacher wants to find out how many students scored between 91 to 100, then she also has to consider the number of students who scored between 96 to 100. The correct answer is 9.
C. Correct. Six students scored between 91 to 95 on the test, and 3 students scored between 96 to 100. The total number of students who scored between 91 to 100 is 9.
D. Incorrect. It looks like you calculated the total number of students who took the test. To find the number who scored between 91 to 100, look at the intervals for 91 to 95 and 96 to 100. The correct answer is 9.

Summary

Data is mathematical information. Mathematical data is often recorded in tables to organize, or spreadsheets to organize and sort. Graphs can help you see the data visually, which can help you to better understand the data. A pictograph is a graph that uses symbols to represent data. Bar graphs show the frequency of categorical data, using bars instead of symbols. Histograms are similar to bar graphs in that they are constructed out of a series of bars. However, histograms represent continuous quantitative (numerical) data, which can illustrate trends as well as other more advanced attributes.

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8.1.2: Other Types of Graphs

Learning Objectives

- Read and interpret data from circle graphs (pie charts).
- Read and interpret data from simple line graphs.
- Read and interpret data from stem and leaf displays.

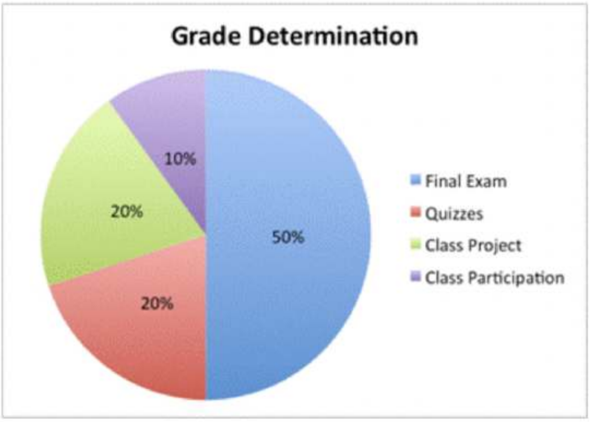
Introduction

Different graphs tell different stories. While a bar graph might be appropriate for comparing some types of data, there are a number of other types of graphs that can present data in a different way. You might see them in news stories or reports, so it's helpful to know how to read and interpret them.

Circle Graphs

Sometimes you will see categorical data presented in a **circle graph**, or pie chart. In these types of graphs, individual pieces of data are represented as sections of the circle (or "pieces of the pie"). Circle graphs are often used to show how a whole set of data is broken down into individual components.

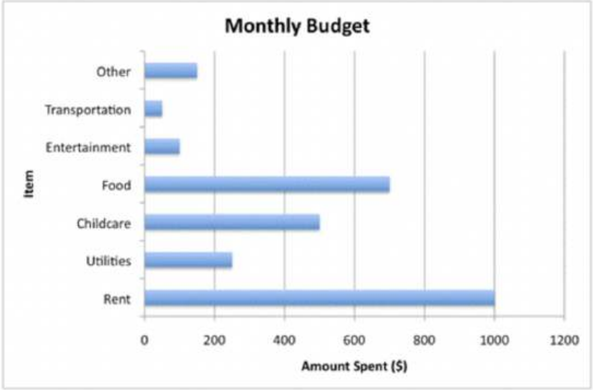
Here's an example. At the beginning of a semester, a teacher talks about how she will determine student grades. She says, "Half your grade will be based on the final exam and 20% will be determined by quizzes. A class project will also be worth 20% and class participation will count for 10%." In addition to telling the class this information, she could also create a circle graph.



This graph is useful because it relates each part (the final exam, the quizzes, the class project, and the class participation) to the whole. It is easy to see that students in this class had better study for the final exam!

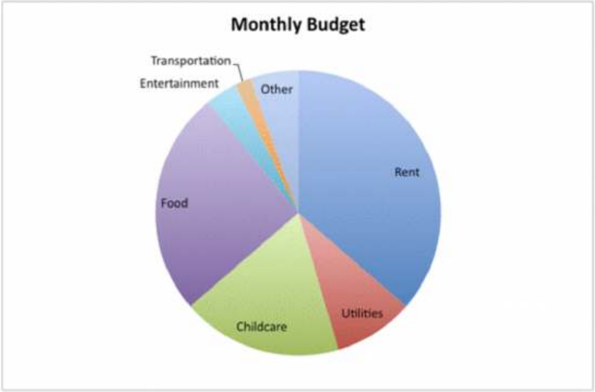
Because circle graphs relate individual parts and a whole, they are often used for budgets and other financial purposes. A sample family budget follows. It has been graphed two ways: first using a bar graph, and then using a circle graph. Each representation illustrates the information a little differently.

The bar graph shows the amounts of money spent on each item during one month. Using this data, you could figure out how much the family needs to earn every month to make this budget work.



The bar graph above focuses on the amount spent for each category. The circle graph below shows how each piece of the budget relates to the other pieces of the budget. This makes it easier to see where the greatest amounts of money are going, and how much of the whole budget these pieces take up. Rent and food are the greatest expenses here, with childcare also taking up a sizeable portion.

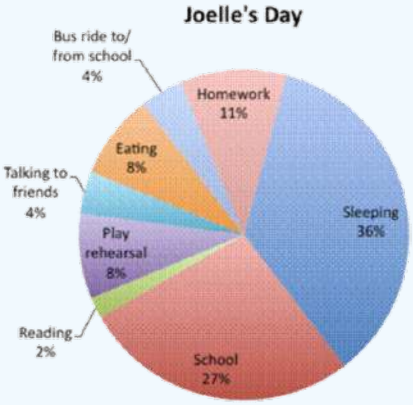
If you look closely at the circle graph, you can see that the sections for food, childcare, and utilities take up almost exactly half of the circle: this means that these three items represent half the budget! This kind of analysis is harder to do with bar graphs because each item is represented as its own entity, and is not part of a larger whole.



Circle graphs often show the relationship of each piece to the whole using percentages, as in the next example.

Example

The circle graph below shows how Joelle spent her day. Did she spend more time sleeping or doing school-related work (school, homework, and play rehearsal)?



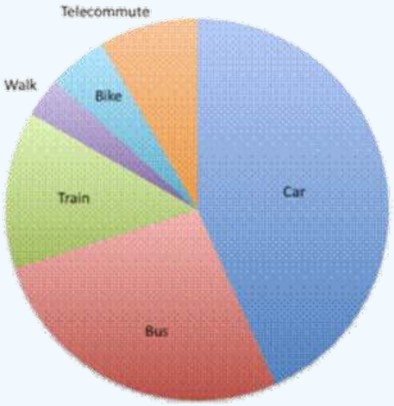
Solution

Sleeping: 36% School-related: $27\% + 8\% + 11\% = 46\%$	Look at the circle graph. The section labeled “Sleeping” is a little larger than the section named “School” (and notice that the percentage of time sleeping is greater than the percentage of time at school!) “Homework” and “Play rehearsal” are both smaller, but when the percentages of time are added to “School,” they add up to a larger portion of the day.

Joelle spent more time doing school-related work.

? Exercise

The graph below shows data about how people in one company commute to work each day.



Which statement is true?

A. Everyone takes a car, bus, or train to work.
B. Taking the bus is more popular than walking or biking.
C. More people take the train than take the bus.
D. Telecommuting is the least popular method of commuting to work.

Answer

A. Incorrect. While most people commute by car, bus, or train, some commute by other means, as shown in the other sections of the graph. The correct answer is that taking the bus is more popular than walking or biking.
B. Correct. The graph shows that about one-fourth of the company takes the bus to work, but only a small portion of people walk or bike.
C. Incorrect. The “Train” section of the circle graph is smaller than the “Bus” section, so the reverse is correct: more people take the bus than take the train. The correct answer is that taking the bus is more popular than walking or biking.
D. Incorrect. The least popular method of transportation is walking, as it has the smallest slice of the pie. The correct answer is that taking the bus is more popular than walking or biking.

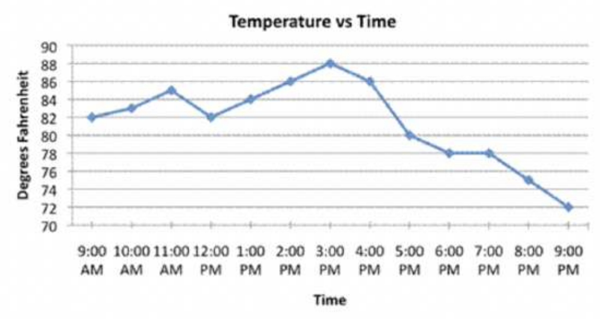
Line Graphs

Unlike circle graphs, **line graphs** are usually used to relate data over a period of time. In a line graph, the data is shown as individual points on a grid; a trend line connects all data points.

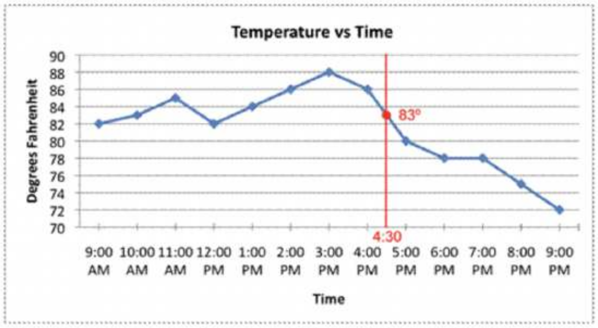
A typical use of a line graph involves the mapping of temperature over time. One example is provided below. Look at how the temperature is mapped on the y-axis and the time is mapped on the x-axis.

Each point on the grid shows a specific relationship between the temperature and the time. At 9AM, the temperature was 82^{o}. It rose to 83^{o} at 10 AM, and then again to 85^{o} at 11AM. It cooled off a bit by noon, as the temperature fell to 82^{o}. What happened the rest of the day?

The data on this graph shows that the temperature peaked at 88^{o} at 3PM. By 9PM that evening, it was down to 72^{o}.



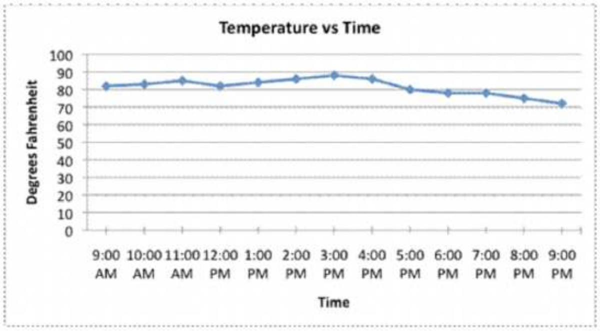
The line segments connecting each data point are important to consider, too. While this graph only provides data points for each hour, you could track the temperature each minute (or second!) if you wanted. The line segments connecting the data points indicate that the temperature vs. time relationship is continuous, meaning it can be read at any point. The line segments also provide an estimate for what the temperature would be if the temperature were measured at any point between two existing readings. For example, if you wanted to estimate the temperature at 4:30, you could find 4:30 on the x-axis and draw a vertical line that passes through the trend line; the place where it intersects the graph will be the temperature estimate at that time.



Note that this is just an estimate based on the data. There are many different possible temperature fluctuations between 4PM and 5PM. For example, the temperature could have held steady at 86^{o} for most of the hour, and then dropped sharply to 80^{o} just before 5PM. Alternatively, the temperature could have dropped to 76^{o} due to a sudden storm, and then climbed back up to 80^{o} once the storm passed. In either of these cases, our estimate of 83^{o} would be incorrect! Based on the data, though, 83^{o} seems like a reasonable prediction for 4:30PM.

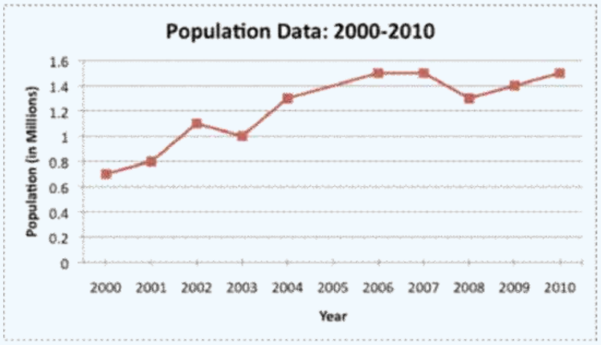
Finally, a quick word about the scale in this graph. Notice the y-axis, which is the vertical line where the Degrees Fahrenheit are listed. Notice that it starts at 70^{o}, and then increases in increments of 2^{o} each time. Since the scale is small and the graph begins at 70^{o}, the temperature data looks pretty volatile, as if the temperature went from being warm to hot to very cold! Look at the same data set when plotted on a line graph that begins at 0^{o} and has a scale of 10^{o}.

As you can see, changing the scale of the graph can affect how a viewer perceives the data within the graph.



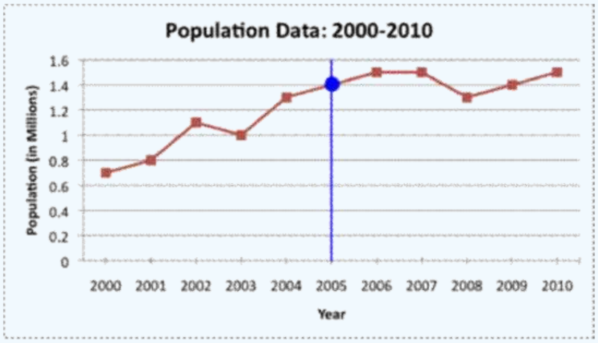
Example

Population data for a fictional city is given below. Estimate the city's population in 2005.



Solution

Look at the line graph. The population starts at about 0.7 million (or 700,000) in 2000, rises to 0.8 million in 2001, and then again to 1.1 million in 2002. To find the population in 2005, find 2005 on the x-axis and draw a vertical line that intersects the trend line.



The lines intersect at 1.4, so million (or 1,400,000) would be a good estimate.

The population in 2005 was about 1.4 million.

Stem-and-Leaf Plots

A **stem-and-leaf plot** provides yet another way to visualize quantitative data. They retain the original data (unlike pictographs), and allow easy identification of the largest and smallest values, clusters, and gaps.

Here's an example. An economist is wondering about the amount of spare change that people carry, and whether the amount is greater in the morning or in the evening. To collect this data, she heads to a subway station downtown and, one morning, randomly asks people to count the amount of change they have in their pocket or purse. She stops when she has asked 30

people how much change they are carrying. The results are shown here in the table.

Pocket Change (in cents)

0	85	95
50	44	81
12	25	25
15	10	5
0	76	0
59	75	73

4	94	62
42	0	60
75	15	25
31	38	50

The economist wants to get a better sense of this data, so she creates a stem-and-leaf plot. In this graph, the “stem” is the ten’s digit for each piece of data and the “leaf” is the one’s digit. If she had to graph a three-digit number, the stem would be the first two digits and the leaf would be the ones digit. The key point in a stem-and-leaf plot is that the leaves do not contain more than the 10 digits from 0 through 9.

She begins by listing the stem values, which are all the possible tens digits: the numbers from 0 to 9.

Stem	Leaf
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Then she starts going through her data. The first person said that he was carrying 0 cents. The stem here is 0, and the leaf is also 0 (it is important to include this 0 to show that data was collected).

Stem	Leaf
0	0
1	
2	
3	
4	
5	
6	
7	
8	
9	

Looking down the table, she sees the next two pieces of data: 50 and 12. She enters that data in the graph.

Stem	Leaf

0	0
1	2
2	
3	
4	
5	0
6	
7	
8	
9	

The next piece of data is 15. The stem is 1 and the leaf is 5, so she puts the 5 next to the 2 that is already in the leaf section of the graph.

Stem	**Leaf**
0	0
1	2 5
2	
3	
4	
5	0
6	
7	
8	
9	

She continues to add pieces of data until she has the following stem-and-leaf plot.

Stem	**Leaf**
0	000045
1	0255
2	555
3	18
4	24
5	009
6	02
7	3556
8	15

9	45

From this plot, she can see that half the people carried less than 40 cents (since there are 30 pieces of data, the median number is the average of the 15th and 16th items. The 15th item is 38 and the 16th is 42, so the median is 40.) The data is fairly spread out, too; there does not seem to be a discernible pattern about how much change people are carrying.

That afternoon, the economist gathers change data from 30 more people. Instead of making a new stem-and-leaf plot, though, she rearranges her original graph and adds on the new data by adding a new set of leaves, as shown here.

Leaf (Morning)	Stem	Leaf (Afternoon)
000045	0	0005
0255	1	05
555	2	2558
18	3	02
24	4	55
009	5	5
02	6	
3556	7	0056
15	8	11368
45	9	058889

She notices that the shape of the afternoon data is different than the shape of the morning data. In the morning, many people were carrying less than 40 cents. In the afternoon, however, people seemed to be carrying more change, with half the respondents carrying between 70 cents and 99 cents.

✓ Example

A biologist uses a stem-and-leaf plot to record the ages (in years) of ten trees in a forest. How old were the trees that she surveyed?

Stem	Leaf
5	244
6	6
7	
8	0146
9	59

Solution

Stem	Leaf	To read a stem-and-leaf plot, look at the stem, then the leaf. The first piece of data has a stem of 5 and a leaf of 2. This is the number 52.
5	2 44	
6	6	
7		
8	0146	
9	59	

Stem	Leaf	The second piece of data also has a stem of 5 but a leaf of 4. This is the number 54.
5	2 44	
6	6	
7		
8	0146	
9	59	

Continue reading the stems and leaves in the graph.

The ages of the trees are

52, 54, 54, 66, 80, 81, 84, 86, 95, and 99.

? Exercise

Which set of numbers does this stem-and-leaf plot represent?

Stem	Leaf
0	09
1	336
2	0
3	5779

A. 0, 9, 13, 13, 16, 20, 35, 37, 37, 39
B. 0, 0, 3, 3, 5, 6, 7, 7, 9, 9
C. 0, 9, 13, 16, 20, 35, 37, 39
D. 0, 2, 31, 31, 53, 61, 73, 73, 90, 93

Answer

A. Correct. To find the original set of numbers from which this stem-and-leaf plot was created, combine the stem with each of its leaves. The stem will be the tens digit of the number, and the leaf will be the ones digit.
B. Incorrect. It looks like you just looked at the leaves and ignored the stems. To find the original set of numbers from which this stem-and-leaf plot was created, combine the stem with each of its leaves: the stem will be the tens digit of the number, and the leaf will be the ones digit. The correct answer is 0, 9, 13, 13, 16, 20, 35, 37, 37, 39.
C. Incorrect. It looks like you know how to create the data set from the stem-and-leaf plot, but you mistakenly took out the repeated values in the data set. All data points, even repeated ones, are necessary. The correct answer is 0, 9, 13, 13, 16, 20, 35, 37, 37, 39.
D. Incorrect. It looks like you reversed the stem and leaf! To find the original set of numbers from which this stem-and-leaf plot was created, combine the stem with each of its leaves: the stem will be the tens digit of the number, and the leaf will be the ones digit. The correct answer is 0, 9, 13, 13, 16, 20, 35, 37, 37, 39.

Summary

Bar graphs are only the beginning of the story when it comes to visualizing data sets. Circle graphs show how a set of data is divided up into sections, and they help the viewer visualize how each section relates to the whole. By contrast, line graphs are usually used to relate continuous data over a period of time. A third type of graph, the stem-and-leaf plot, provides another way to organize quantitative data. Stem-and-leaf plots are useful for getting a quick picture of the smallest and largest values, clusters, and gaps of the data within a set.

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SECTION OVERVIEW

8.2: Measures of Center

Topic hierarchy

8.2.1: Measures of Center

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8.2.1: Measures of Center

Learning Objectives

- Find the mean, median, and mode of a set of numbers.
- Find the range and midrange of a data set.
- Read and interpret data from box-and-whisker plots.
- Solve application problems that require the calculation of the mean, median, range, or midrange.

Introduction

Mean, **median**, and **mode** are important tools in the statistician's toolbox. These measures of center all use data points to approximate and understand a "middle value" or "average" of a given data set. Two more measures of interest are the **range** and **midrange**, which use the greatest and least values of the data set to help describe the spread of the data.

So why would you need to find out the middle of a data set? And why do you need three measures instead of just one? Let's look closely at these measures of center and learn how they can help us understand sets of data.

Mean, Median, and Mode

"Mean" is a mathematical term for "average" which you may already know. Also referred to as the "arithmetic mean," it is found by adding together all the data values in a set and dividing that sum by the number of data items.

You can often find the average of two familiar numbers, such as 10 and 16, in your head without much calculation. What number lies half way between them? 13. A mathematical way to solve this, though, is to add 10 and 16 (which gives you 26) and then divide by 2 (since there are 2 numbers in the data set). $26 \div 2 = 13$

Knowing the process helps when you need to find the mean of more than two numbers. For example, if you are asked to find the mean of the numbers 2, 5, 3, 4, 5, and 5, first find the sum: 2 + 5 + 3 + 4 + 5 + 5 = 24. Then, divide this sum by the number of numbers in the set, which is 6. So the mean of the data is $24 \div 6$, or 4.

In the previous data set, notice that the mean was 4 and that the set also contained a value of 4. This does not always occur. In the example that follows, the mean is 18, although 18 is not in the data set at all.

Example

Find the mean of the set: 4, 7, 28, 33.

Solution

$4+7+28+33=72$	Add all the values.
$\frac{72}{4}=18$	Divide by 4, the number of values.

The mean is 18.

Next, let's look at the "median." The median is the middle value when the data is ordered. If there are two middle values, the median is the average of the two middle values.

To calculate the median, you first put your data into numerical order from least to greatest. Then identify the middle value(s).

For example, let's look at the following values: 4, 5, 1, 3, 2, 7, 6. To find the median of this set, you would put it in order from least to greatest.

$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$

Then identify the middle value. There are three values to the right of four and three values to the left of four. The middle value is 4, so 4 is the median.

If there is an even number of data items, however, the median will be the mean of the two center data items.

✓ Example

Find the median of the set: 2, 5, 3, 4, 5, 5

Solution

$2,3,4,5,5,5$	Arrange the values from least to greatest.
$2,3,\mathbf{4},\mathbf{5},5,5$ $\frac{9}{2}=4.5$	The set has 2 middle values, 4 and 5. So take the mean (average) of the two values.

The median is 4.5.

Finally, let’s consider the “mode.” The mode is found by looking for the data value that appears most often. If there is a two-way tie for most often, the data is bimodal and you use both data values as the modes. Sometimes there is no mode. This happens when there is no data value that occurs most often. In our example data set (2, 3, 4, **5**, **5**, **5**), the number 5 appears 3 times and all other numbers appear once, so the mode is 5.

✓ Example

Find the mode of the set: 12, 4, 12, 5, 5, 8, 12, 0, 1, 12

Solution

0, 1, 4, 5, 5, 8, 12, 12, 12, 12	Arrange the values from least to greatest (although this is not a necessary step, it sometimes helps to find the mode if the numbers are arranged in ascending order).
0, 1, 4, 5, 5, 8, **12**, **12**, **12**, **12**	0, 1, 4, 5, 5, 8, 12, 12, 12, 12

The mode is 12.

Let’s look at an example with some relevant data.

✓ Example

Carlos received the following scores on his mathematics exams: 84, 92, 74, 98, and 82. Find the mean, median, and mode of his scores.

Solution

$\frac{84+92+74+98+82}{5}$	To find the mean, add all the test scores together and divide by the number of tests.
$\frac{430}{5}=86$	The mean is 86.
$74,82,84,92,98$	To find the median, order the test scores from least to greatest.
84	There are five scores, so the middle test score is the third in the ordered list. This is the median.
$74,82,84,92,98$	Since each number appears exactly one time, there is no mode.

The mean is 86.

The median is 84.

There is no mode.

What can be learned from the mean, median, and mode of Carlos’ test scores? Notice that these values are not the same.

Both the mean and the median give us a picture of how Carlos is doing. Looking at these measures, you notice that the middle of the data set is in the mid-80s: the mean value is 86, and the median value is 84. That's all you are really after when using median and mean: finding the center, or middle, of the data. Notice, also, that there is no mode, since Carlos did not score the same on two tests. In the case of test taking, the mode is often meaningless, unless there are a lot of 0s, which could mean that the student didn't do his homework, or really doesn't know what's going on!

✓ Example

Find the mean, median, and mode of the following set of numbers: 12, 11, 13, 11, 12, 10, 10, 11, 13, 14

Solution

$\frac{12+11+13+11+12+10+10+11+13+14}{10}=\frac{117}{10}=11.7$	To find the mean, add together all the numbers and divide that sum by the number of numbers.
$10, 10, 11, 11, 11, 12, 12, 13, 13, 14$	To find the median, first order the numbers from least to greatest.
$\frac{11+12}{2}=\frac{23}{2}=11.5$	Since there are 10 numbers (an even number) the median is the mean of the middle two numbers (the 5th and 6th), or halfway between 11 and 12.
$10, 10, \mathbf{11}, \mathbf{11}, \mathbf{11}, 12, 12, 13, 13, 14$	To find the mode, look for the number that appears most often.
11	The number 11 appears most often, 3 times.

The mean is 11.7.

The median is 11.5.

The mode is 11.

In this case, the mean, median, and mode are very close in value. This shows some consistency in the data, with a middle (average) value of about 11. If this data represented the ages of students on a chess team, for example, you would have a good idea that everyone on the team was about 11 years old, with a few older and younger members.

? Exercise

During a seven-day period in July, a meteorologist recorded that the median daily high temperature was 91º.

Which of the following are true statements?

1. The high temperature was exactly 91^{o} on each of the seven days.
2. The high temperature was never lower than 92^{o}.
3. Half the high temperatures were above 91^{o} and half were below 91^{o}.

A. 1 only
B. 2 only
C. 3 only
D. 1, 2, and 3

Answer

A. Incorrect. Just because the median high was 91^{o} does not mean that the temperature reached 91^{o} on each day. The correct answer is statement 3 only.
B. Incorrect. You know the median is 91^{o}, so 91^{o} is a member of the data set, meaning that the temperature had to have been lower than 92^{o} at least one time during the week. The correct answer is statement 3 only.
C. Correct. Half of the high temperatures were above 91^{o} and half were below 91^{o} since the median will always represent the value where half the data is higher and half the data is lower.

D. Incorrect. Statement 1 is incorrect because a median high of 91^o does not necessarily mean that the temperature reached 91^o on each day. Statement 2 is incorrect because you know that the median is 91^o, so 91^o is a member of the data set, meaning that the temperature had to have been lower than 92^o at least one time during the week. The correct answer is statement 3 only.

Range and Midrange

There are other useful measures other than mean, median, and mode to help you analyze a data set. When looking at data, you often want to understand the spread of the data: the gap between the greatest number and the least number. This is the range of the data. To find the range, subtract the least value of the data set from the greatest value. For example, in the data of 2, 5, 3, 4, 5, and 5, the least value is 2 and the greatest value is 5, so the range is 5-2, or 3.

It is also useful to know what number is mid-way between the least value and the greatest value of the data set. This number is called the midrange. To find the midrange, add together the least and greatest values and divide by two, or in other words, find the mean of the least and greatest values.

The midrange of 2, 5, 3, 4, 5, and 5 is $\frac{2+5}{2}=\frac{7}{2}=3.5$.

Let’s look at a couple of examples.

✓ Example

Find the range and midrange for the following set of numbers: 2, 4, 7, 10, 14, 35.

Solution

range: $35-2=33$	Subtract the least value from the greatest value to find the range.
midrange: $\frac{35+2}{2}=\frac{37}{2}=18.5$	Add together the greatest value and the least value and divide by 2.

The range is 33.

The midrange is 18.5.

✓ Example

Find the range and midrange for the following set of numbers: 62, 88, 20, 145, 37, 105, 93, 22.

Solution

least number: 20 greatest number: 145	Since the data set isn’t ordered from least to greatest, identify the least and greatest numbers.
range: 145-20=125	Subtract the least value from the greatest value to find the range.
midrange: $\frac{145+20}{2}=\frac{165}{2}=82.5$	Add together the greatest value and the least value and divide by 2.

The range is 125.

The midrange is 82.5.

Box-and Whisker-Plots

Another type of graph that you might see is called a **box-and-whisker plot**. These graphs provide a visual way of understanding both the range and the middle of a data set.

Here is a sample set of 15 numbers to get us started.

12, 5, 18, 20, 11, 9, 3, 5, 7, 18, 12, 15, 6, 10, 11

Creating a box-and-whisker plot from this data requires finding the median of the set. To do this, order the data.

3, 5, 5, 6, 7, 9, 10, **11**, 11, 12, 12, 15, 18, 18, 20

This data set has 15 numbers, so the median will be the 8th number in the set: 11.

Finding the median of the data set essentially divides it into two: a set of numbers below the median, and a set of numbers above the median. A box-and-whisker plot requires you to find the median of these numbers as well!

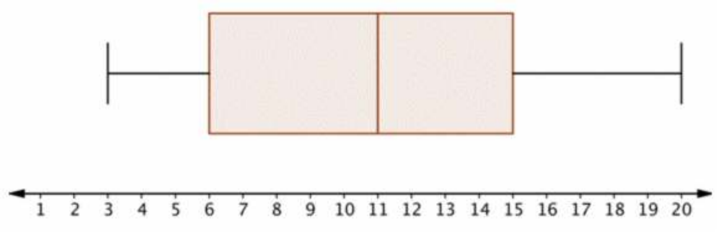
Lower set: 3, 5, 5, 6, 7, 9, 10. Median: 6

Upper set: 11, 12, 12, 15, 18, 18, 20. Median: 15

So, the median of the set is 11, the median of the lower half is 6, and the median of the upper half is 15.

3, 5, 5, **6**, 7, 9, 10, **11**, 11, 12, 12, **15**, 18, 18, 20

A box-and-whisker plot for this data set is shown here. Do you notice any similarities between the numbers above and the location of the box?



Notice that one "box" (rectangle section) begins at 6 (the median of the lower set) and goes to 11 (the median of the full set), and the other box goes from 11 to 15 (the median of the upper set).

The "whiskers" are the line segments on either end. One stretches from 3 (the least value in the set) to 6, and the other goes from 15 to 20 (the greatest value in the set).

The box-and-whisker plot essentially divides the data set into four sections (or **quartiles**): whisker, box, box, whisker. The size of the quartiles may be different, but *the number of data points* in each quartile is the same.

You can use a box-and-whisker plot to analyze how data in a set are distributed. You can also use the box-and-whisker plots to compare two sets of data.

Using Measures of Center to Solve Problems

Using mean, median, and mode, as well as range and midrange can help you to analyze situations and make decisions about things like which is the best, whether it is more reliable to walk or take the bus to school, or even whether to buy or sell a particular stock on the stock market.

Let's look at an example of how analyzing data using measures of center can help you to make choices (and even get to school on time!).

✔ Example

Below is a table listing the amount of time it took Marta to get to school by either riding the bus or by walking, on 12 separate days. The times are door to door, meaning the clock starts when she leaves her front door and ends when she enters school.

Bus	Walking
16 minutes	22 minutes
14 minutes	19 minutes
15 minutes	21 minutes
14 minutes	20 minutes
31 minutes	21 minutes

15 minutes	20 minutes

- **Which method of travel is faster?**
- **If she leaves her house 25 minutes before school starts, should she walk or take the bus to be assured of arriving at school on time?**

Solution

bus: $\frac{16+14+15+14+31+15}{6} = 17.5$ walking: $\frac{22+19+21+20+21+20}{6} = 20.5$	Determine the mean of each travel method.
bus: $31 - 14 = 17$ walking: $22 - 19 = 3$	Determine the range each travel method.
bus: $14, 14, 15, 15, 16, 31$ $\frac{15+15}{2} = 15$ walking: $19, 20, 20, 21, 21, 22$ $\frac{20+21}{2} = 20.5$	Determine the median for each travel method.
bus: $14, 15$ walking: $20, 21$	Determine the mode for each travel method.

	Bus	**Walking**
Mean	17.5	20.5
Median	15	20.5
Mode	14, 15	20, 21
Range	17	3

Looking at the mean, median, and the mode, the faster way to school is riding the bus. The data also shows that the bus is the most variable, with a range of 17, so if Marta wants to be *sure* that she gets to school on time, she should walk.

In the previous example, riding the bus is, *on average*, a faster way to school than walking. This is revealed in the mean of each method, which shows that the bus is 3 minutes faster. The mode and median show an even greater time advantage to riding the bus, and this is due to the one time high value of 31 minutes that isn't really accounted for in these measures. Notice the difference in the mean (17.5) and the median (15) for riding the bus, which lets you know there is some variance in the data.

As far as getting to school on time is concerned, while not being the fastest method, walking is the most reliable, with consistent values for mean, median, and mode, and a low value for the range, meaning that the spread of the data is very small.

Let's look at another example.

✓ Example

The top three men's tennis players during July in the years 2007 through 2011 were (in no particular order), Roger Federer, Rafael Nadal, and Novak Djokovic. Based on their July rankings, who has been the top performer over this time span?

July ATP Rankings, 2007 through 2011

July	**Nadal**	**Federer**	**Djokovic**
2011	2	3	1
2010	1	3	2
2009	2	1	4

2008	2	1	3
2007	2	1	3

Data taken from ATP World Tour, August 2011

Solution

Nadal: $\frac{2+1+2+2+2}{5}=\frac{9}{5}=1.8$ Federer: $\frac{3+3+1+1+1}{5}=\frac{9}{5}=1.8$ Djokovic: $\frac{1+2+4+3+3}{5}=\frac{13}{5}=2.6$	Find the mean of each player’s ranking.
Nadal: $1,2,2,2,2=2$ *Federer*: $1,1,1,3,3=1$ *Djokovic*: $1,2,3,3,4=3$	Find the median of each player’s ranking.
Nadal: $1,2,2,2,2=2$ *Federer*: $1,1,1,3,3=1$ *Djokovic*: $1,2,3,3,4=3$	Find the mode of each player’s ranking.
Nadal: $2-1=1$ *Federer*: $3-1=2$ *Djokovic*: $4-1=3$	Find the range of each player’s ranking.

It appears that Nadal and Federer are virtually tied for the best performer over those 5 years, with Djokovic in third. Nadal’s and Federer’s mean ranking is 1.8, so they are even there. And while Federer’s rankings have a median of 1 and a mode of 1 (beating out Nadal both times), he also has a wider range: Federer has spent some time in 3rd place, which Nadal has not.

Depending on your viewpoint (and your preference of player!), you could make an argument that either Federer or Nadal is better than the other, but the data seems to show that they are nearly equal. Clearly, over time, the performance of Nadal and Federer is very close, with the mean of their rankings identical. The mean probably gives the best evaluation of overall performance, but does not tell us the whole story. Using the range, Nadal has been more consistent than Federer, although he has fewer number one rankings. Not shown in any of these measures is how Djokovic’s ranking has been moving upward and Federer’s ranking has been moving downward. It is also worth noting that it is hard to make a conclusion with such a small amount of data; using other measures, like tournaments won and average ranking of opponents may also help you understand who the best player was over this time.

Summary

Measures of center help you to analyze numerical data. The mean (or arithmetic mean) is often called the “average,” and is found by dividing the sum of the data items by the number of items. The median is the number that is in the middle when the data is ordered from least to greatest, and the mode is the number that appears most often. The range is the difference between the least number and the greatest number, and the midrange is the mean of the greatest value and the least value. Box-and-whisker plots use the median and range to help you to interpret the data visually.

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SECTION OVERVIEW

8.3: Graphical Representations

Topic hierarchy

8.3.1: Use and Misuse of Graphical Representations

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8.3.1: Use and Misuse of Graphical Representations

Learning Objectives

- Identify which type of graph best represents the data for a given situation.
- Explain how graphs can lead to misinterpretation of data.

Introduction

In addition to **bar graphs**, **histograms**, and **circle graphs** (pie charts), there are other graphs that statisticians use to represent data and analyze what it shows. But you have to be careful when creating and reading graphs. If they are not carefully created, they can be misleading, and sometimes people purposefully make them misleading.

Choosing a Graph

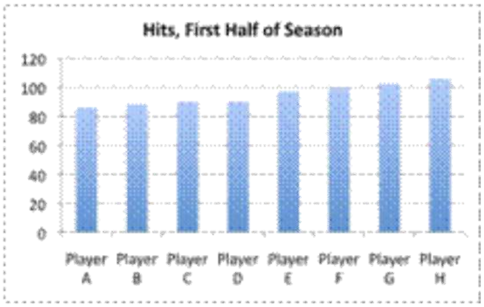
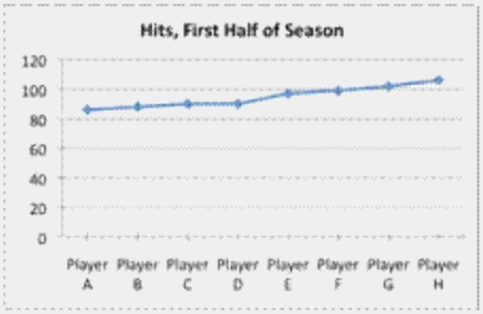
Choosing what type of graph to use to represent a specific data set takes some trial and error. And, sometimes, there is more than one appropriate type of graph you can use. What you choose depends on the way you want to present your data, as well as your own personal preferences. Modern spreadsheet programs like Excel are very flexible at creating different types of graphs; with only a couple clicks, you can view data represented as a bar graph, line graph, or circle graph. From there, you can choose which one best paints the picture you want to show.

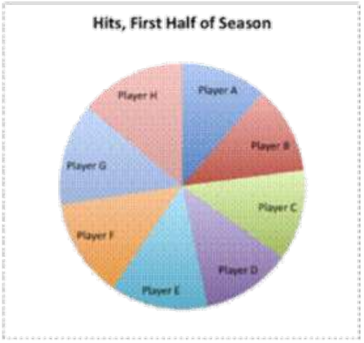
Since there is often more than one way to graph a data set, let's look at some examples and think about the different possibilities that are available to us.

Example

A baseball writer wants to create a graph showing the total hits for the players with the greatest number of hits in the first half of the baseball season. These players have the following number of hits: 86, 88, 90, 90, 97, 99, 102, and 106. What type of graph should the writer use to represent the data?

Solution

	The data set contains information about the hit totals of 8 players. Either a bar graph or a pictograph could help show the total number of hits that each player had compared to the other players.
	A line graph would not be appropriate, since the data is not continuous. There is no data "in between" each player's hit totals.

	A circle graph does not make sense either, unless the writer wants to show the percentage of hits that each player has of the total number of hits. But that data wouldn't be too helpful if the writer just wants to show the total amount of hits.

A bar graph or pictograph would work best here. (A pictograph may be preferable for a small amount of data, and a bar graph may be preferable for a lot of data.)

The writer could use a stem-and-leaf plot to show the distribution of numerical data, but this kind of graph is not as effective at showing the relationship between each player and the number of hits that he has. A box-and-whisker plot, which shows the middle of a data set, is not useful here since the writer is interested in the hit totals, not the average number of hits or the spread of the data.

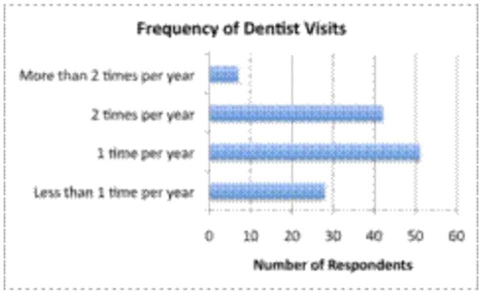
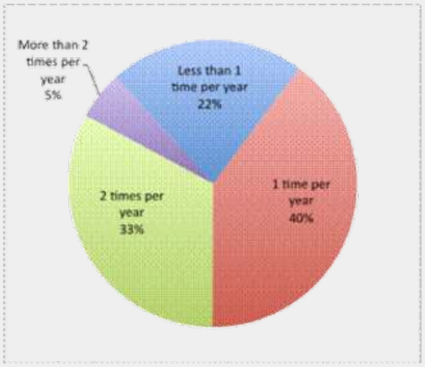
✓ Example

A statistician is collecting data on the frequency with which adults go to the dentist. She surveys 128 people and finds the following information.

- Less than 1 time per year: 28 respondents
- 1 time per year: 51 respondents
- 2 times per year: 42 respondents
- More than 2 times per year: 7 respondents

In a presentation to dentists, she especially wants to highlight the population that visits the dentist less than 1 time per year. What type of graph should she use to represent the data?

Solution

	To show her findings, the statistician could use a couple of different graphs. A bar graph would be fine to use here since the data is categorical. She has grouped her findings into 4 categories.
	A circle graph may be even better. The statistician is interested in the *percentage* or *share* of people who responded to each question. A circle graph allows for easy comparison among the categories surveyed.

A circle graph is best, but a bar graph would also be acceptable.

As with the first example, stem-and-leaf plots and box-and-whisker plots are not useful here. The statistician is not interested in the average amount of times that a person goes to the dentist each year. A line graph would not be appropriate either, as the data is not continuous.

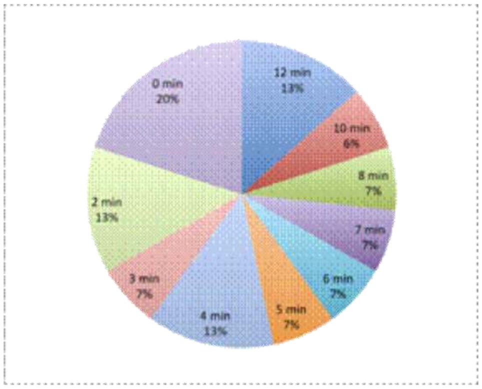
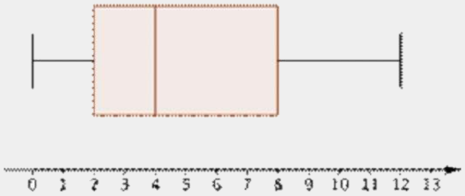
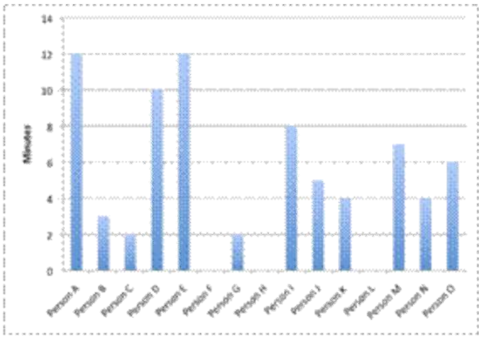
Example

An amusement park planner wants to better understand the distribution of wait times that people experience while waiting for a popular ride. At the park one day, he asks 15 random people about the length of time they had to wait (in minutes).

$$12, 3, 2, 10, 12, 0, 2, 0, 8, 5, 4, 0, 7, 4, 6$$

What type of graph provides the best visual representation of this set of data: a circle graph, a box-and-whisker plot, or a bar graph?

Solution

	The key idea here is that the planner wants to understand the distribution of wait times. A circle graph does not show distribution. The planner could create a circle graph (like the one at left) that shows the percentage of people who waited for different amounts of time, but this does not help him understand the distribution of the data.
	Creating a box-and-whisker plot would be more helpful. This type of graph would show the distribution more effectively, as seen at left. Half the people waited between 2 and 8 minutes for the ride.
	A bar graph can show the length that each person waited, but it does not show much about the distribution of wait times.
A box-and-whisker plot is best.	Ultimately, the box-and-whisker plot gives the most information about distribution, so it is the most useful.

Exercise

An oceanographer wants to make a graph that shows the height (in centimeters) of a specific coral over the period of 2 years. Which type of graph is the most appropriate?

A. Circle graph
B. Box-and-whisker plot
C. Line graph
D. Stem-and-leaf plot

Answer

A. Circle graph

Incorrect. A circle graph is often used to show parts of a whole, not changes over time. The correct answer is a line graph.

B. Box-and-whisker plot

Incorrect. A box-and-whisker plot is used to show the middle of a data set; it does not reveal much about growth over time. The correct answer is a line graph.

C. Line graph

Correct. A line graph mapping height along the y-axis and time along the x-axis is the most appropriate type of graph for this situation.

D. Stem-and-leaf plot

Incorrect. A stem-and-leaf plot is used to show the spread of a data set; it does not reveal much about growth over time. The correct answer is a line graph.

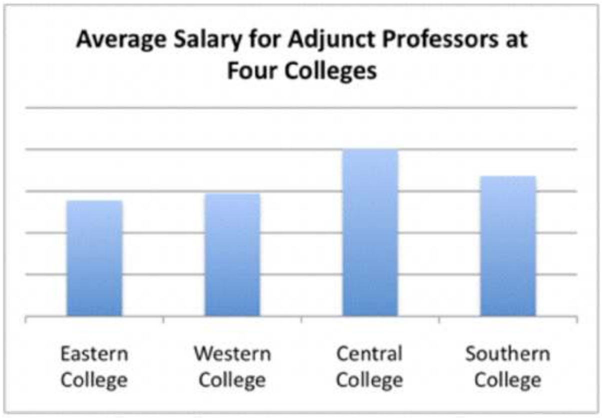
Misleading Graphs

As you have seen, graphs provide a visual way to represent data sets. Pictures can be misleading, though, so you also need to know how to identify graphs that seem to show something different than what the data says. This may be due to carelessness or it may be done on purpose. Below are some general questions to keep in mind as you read graphs.

Questions to Consider when Reading Graphs

- Are the graphs labeled sufficiently?
- What is the scale?
- Does the graph show a full picture of the data, or only a select picture?

Look at the graph that follows. The title states “Average Salary for Adjunct Professors at Four Colleges,” and four bars appear on the graph. You can tell which colleges are being compared, but you are given no information about the scale that is being used. The graph makes it appear that the average salary for Adjunct Professors at Central College is much higher than that at Eastern College, but without a scale, you cannot know for certain. (You do know that the salary is higher; you just do not know how much higher.) To make this graph less misleading, a y-axis with salary information should be included.

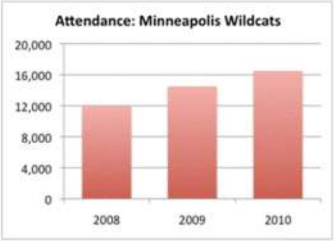


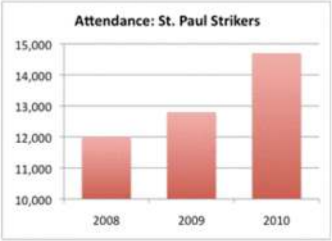
Even when both axes are present and labeled correctly, graphical representations of data can be misleading. This is shown in the set of attendance graphs that follow.

In the graph on the left, the scale begins at 0 and goes to 20,000. The graph itself shows that attendance at Minneapolis Wildcats games has steadily increased each year since 2008, topping out in 2010 at just over 16,000.

Now look at the graph on the right. It appears to show that attendance at St. Paul Strikers' games has increased even more dramatically: the bar for 2010 is more than twice as tall as that in 2008. From looking at these two graphs, you may conclude that the Strikers have been the more popular team recently, as the height of the bars seems to indicate that their attendance has grown faster than that of the Wildcats.

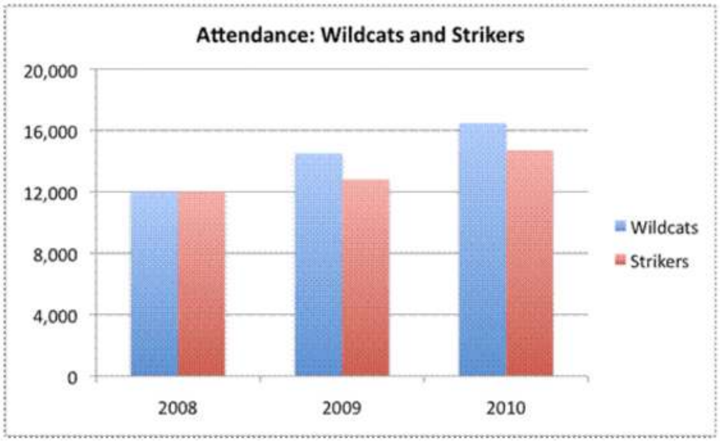
But notice something interesting: the scale of the Strikers graph is very different. It begins at 10,000! This paints a skewed picture of the data when compared with the Wildcats graph, which starts at 0. And by examining the actual data (the actual attendance, not just the height of the bars), you can tell that attendance is actually greater at the Wildcats games. In 2010, for instance, Wildcats attendance is a little over 16,000, while attendance at Strikers games is below 15,000.





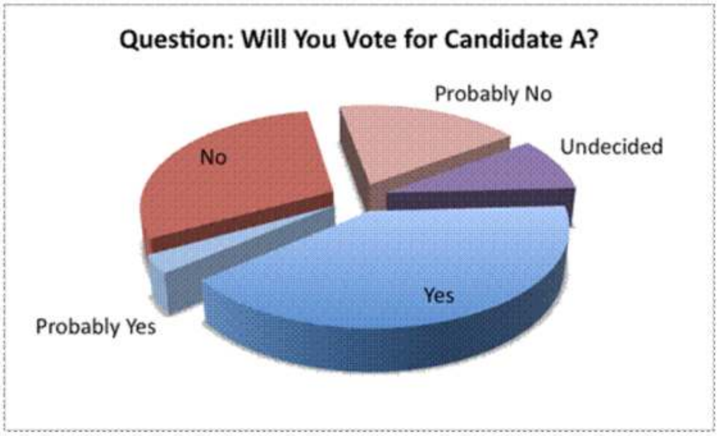
This brings up an important point. When you are using graphs to compare data sets, the scales need to be consistent; otherwise, it is very difficult to compare the data itself. As you can tell from the two previous graphs, changing the scale of a graph can dramatically change the way it looks and the impression the graph makes.

A more honest representation of the attendance data can be found in a double-bar graph, where the attendance figures from both teams is mapped side-by-side using the same scale. Look at the results below. Now it is clear that the attendance for the Wildcats is greater than the attendance for the Strikers.

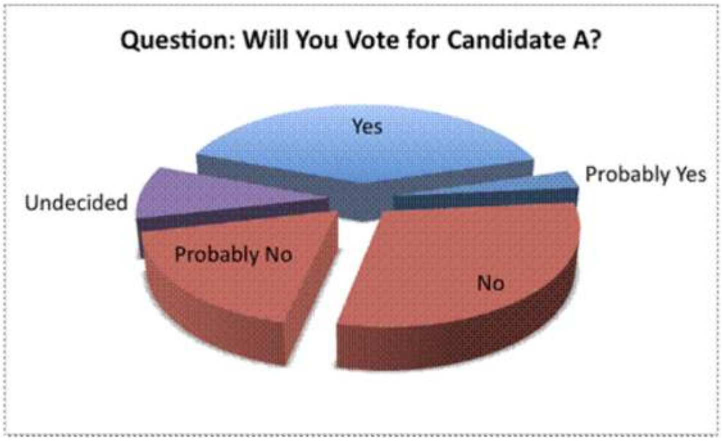


The circle graph here is another example of a misleading representation. The actual percentages of people who responded to each question are not available, and the viewer has to interpret the data based on the size of the sections. At first glance, this graph seems to be showing that a lot of voters seem to be favoring Candidate A, as the “Yes” section is very large.

Part of the reason why this section appears large is because the graph has been created so that it *looks* large. The circle graph is presented in three-dimensional form, and the data that is foremost in the graph (the “Yes” slice) appears the most prominent. The creator of this graph is hoping that this graph will make you think that Candidate A is very popular!

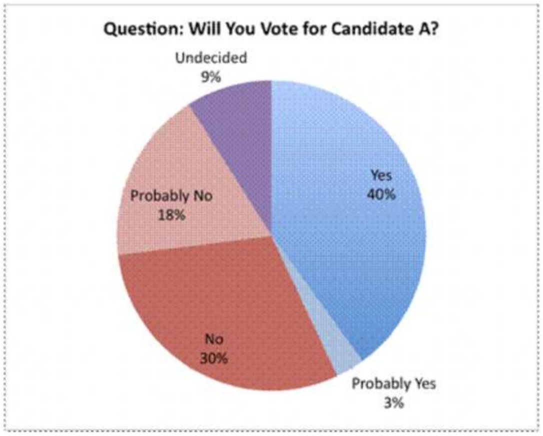


On closer inspection, though, the data does not seem to support this contention. Combining the “Yes” and “Probably Yes” sections is roughly equal to combining the “No” and “Probably No” sections, which means that the candidate is not as popular as this representation would suggest. In fact, someone who did not want this candidate to appear favorable could have represented the data using the next graph. Notice the different positions of the “No” and “Probably No” sections, as well as the consistent colors.



Notice how perspective and color make a difference in viewing and analyzing data!

Next is a more honest way of representing this data. In this graph, the circle graph is shown from above, and the actual percentages are included.



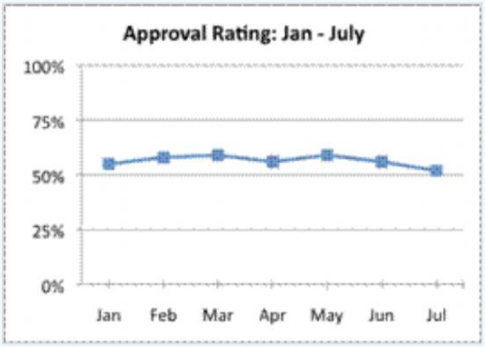
? Exercise

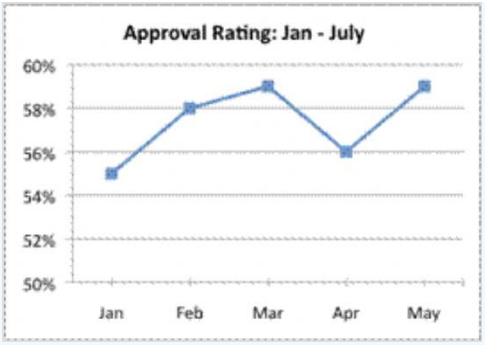
Results from a poll measuring a politician's approval rating are shown in the table below.

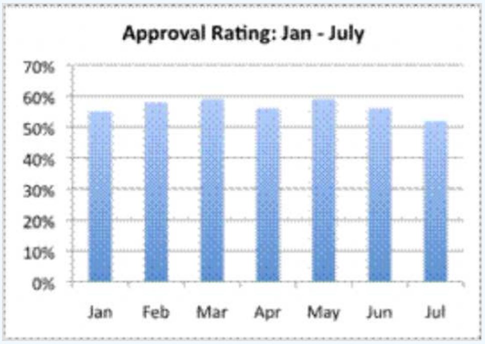
Date	**Approval Rating (%)**
January	55%
February	58%
March	59%
April	56%
May	59%
June	56%
July	52%

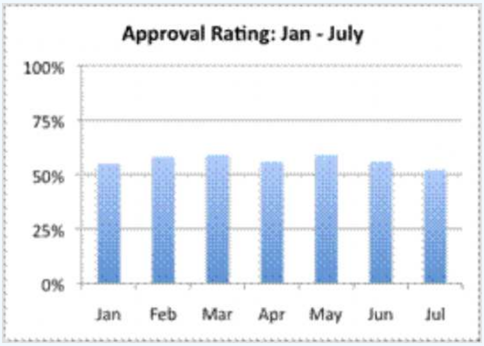
Which of the following graphs is *most* misleading?

A.



B. 

C. 

D. 

Answer

A. Incorrect.

This line graph accurately displays the data; the axes are labeled appropriately, and the scale is from 0% to 100%. The graph shows that although there has been some up and down variation, the politician's approval rating has stayed in the mid-50's. The correct answer is Graph B.

B. Correct.

This graph uses a very small scale (10%, from 50% to 60%) and has eliminated the final two data points. This graph represents only a part of the data, and is designed to make the reader think that the politician is rated more favorably than he really is.

C. Incorrect.

This bar graph accurately displays the data; the axes are labeled appropriately, and the scale is from 0% to 70%. The politician's approval ratings look a bit higher in this graph than they do in Graph D, but there is nothing dishonest about this graph. The graph shows that although there has been some up and down variation, the politician's approval rating has stayed in the mid-50's. The correct answer is Graph B.

D. Incorrect.

This bar graph accurately displays the data; the axes are labeled appropriately, and the scale is from 0% to 100%, with 25% increments. It looks different than Graph C, but the data itself is not misleading: it is the scale of the data that is different. The correct answer is Graph B.

Summary

Graphs have a big impact on how you understand a set of data. Use an appropriate type of graph and you can communicate your data effectively; use the wrong type of graph, though, and your viewers may misunderstand the story you are trying to tell. When reading graphs in newspapers and online, be sure to look at the axes, the scale, and the presentation of the data itself. These can all help you identify if the graph is representing a data set fairly or unfairly.

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SECTION OVERVIEW

8.4: Probability

Topic hierarchy

8.4.1: Probability

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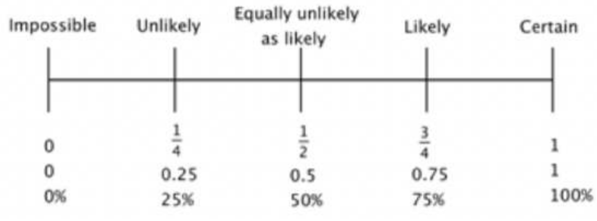
8.4.1: Probability

Learning Objectives

- Define event, outcome, trial, simple event, and sample space, and calculate the probability that an event will occur.
- Calculate the probability of events for more complex outcomes.
- Solve applications involving probabilities.

Introduction

Probability provides a measure of how likely it is that something will occur. It is a number between and including the numbers and 1. It can be written as a fraction, a decimal, or a percent.



Picking numbers randomly means that there is no specific order in which they are chosen. Many games use dice or spinners to generate numbers randomly. If you understand how to calculate probabilities, you can make thoughtful decisions about how to play these games by knowing the likelihood of various outcomes.

Definitions

First you need to know some terms related to probability. When working with probability, a random action or series of actions is called a **trial**. An **outcome** is the result of a trial, and an **event** is a particular collection of outcomes. Events are usually described using a common characteristic of the outcomes.

Let's apply this language to see how the terms work in practice. Some games require rolling a die with six sides, numbered from 1 to 6. (Dice is the plural of die.) The chart below illustrates the use of trial, outcome, and event for such a game:

Trial	Outcomes	Examples of Events
Rolling a die	There are 6 possible outcomes: {1, 2, 3, 4, 5, 6}	Rolling an even number: {2, 4, 6} Rolling a 3: {3} Rolling a 1 or a 3: {1, 3} Rolling a 1 *and* a 3: {} (Only one number can be rolled, so this outcome is impossible. The event has no outcomes in it.)

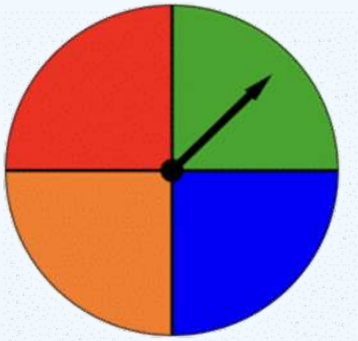
Notice that a collection of outcomes is put in braces and separated by commas.

A **simple event** is an event with only *one* outcome. Rolling a 1 would be a simple event, because there is only one outcome that works: 1! Rolling more than a 5 would also be a simple event, because the event includes only 6 as a valid outcome. A **compound event** is an event with more than one outcome. For example, in rolling one six-sided die, rolling an even number could occur with one of three outcomes: 2, 4, and 6.

When you roll a six-sided die many times, you should not expect any outcome to happen more often than another (assuming that it is a fair die). The outcomes in a situation like this are said to be **equally likely**. It’s very important to recognize when outcomes are equally likely when calculating probability. Since each outcome in the die-rolling trial is equally likely, you would expect to get each outcome $\frac{1}{6}$ of the rolls. That is, you'd expect $\frac{1}{6}$ of the rolls to be 1, $\frac{1}{6}$ of the rolls to be 2, $\frac{1}{6}$ of the rolls to be 3, and so on.

Exercise

A spinner is divided into four equal parts, each colored with a different color as shown below. When this spinner is spun, the arrow points to one of the colors. Are the outcomes equally likely?



A. Yes, they are equally likely.
B. No, they are not equally likely.

Answer

All the outcomes are equally likely. Each color provides a different outcome, and each color takes up $\frac{1}{4}$ of the circle. You would expect the arrow to point to each color $\frac{1}{4}$ of the time.

Probability of Events

The probability of an event is how often it is expected to occur. It is the ratio of the size of the **event space** to the size of the **sample space**.

First, you need to determine the size of the sample space. The size of the sample space is the total number of possible outcomes. For example, when you roll 1 die, the sample space is 1, 2, 3, 4, 5, or 6. So the size of the sample space is 6.

Then you need to determine the size of the event space. The event space is the number of outcomes in the event you are interested in. The event space for rolling a number less than three is 1 or 2. So the size of the event space is 2.

For equally likely outcomes, the probability of an event *E* can be written *P(E)*.

$$P(E)=\frac{\text{size of the event space}}{\text{size of the sample space}}=\frac{\text{number of outcomes in the event}}{\text{total number of possible outcomes}}$$

Example

A game requires rolling a six-sided die numbered from 1 to 6. What is the probability of rolling an even number?

Solution

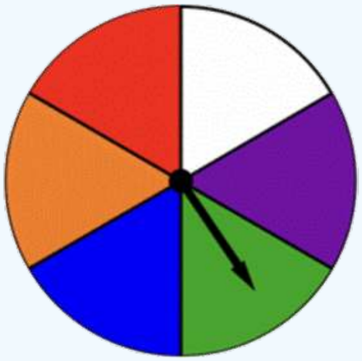
$\begin{aligned}\text{Sample space } &= \{1,2,3,4,5,6\}\\ \text{Event space } &= \{2,4,6\}\end{aligned}$	First, find the sample space and the event space. The sample space is all the possible outcomes, and the event space is the outcomes in the event. In this case, the event is “rolling an even number.”
$P(\text{ even number })=P(\{2,4,6\})=\frac{3}{6}=\frac{1}{2}$	Since the outcomes are equally likely, the probability of the event is the ratio of event space to sample space.

$P(\text{ even number })=\frac{1}{2}$

It is a common practice with probabilities, as with fractions in general, to simplify a probability into lowest terms since that makes it easier for most people to get a sense of how great it is. Unless there is reason not to do so, express all final probabilities in lowest terms.

Exercise

A spinner is divided into equal parts, each colored with a different color as shown below. Find the probability of spinning blue or green on this spinner:



A. $\frac{1}{6}$
B. $\frac{1}{3}$
C. 2
D. 6

Answer

A. Incorrect. There are 6 equally likely outcomes, so the probability of each outcome is $\frac{1}{6}$. However, the event has two outcomes in it, {blue, green}. The probability is $\frac{\text{size of event space}}{\text{size of sample space}} = \frac{2}{6}$. The correct answer is $\frac{1}{3}$.
B. Correct. There are 6 equally likely outcomes, and the event has two outcomes in it, {blue, green}. The probability is $\frac{\text{size of event space}}{\text{size of sample space}} = \frac{2}{6} = \frac{1}{3}$.
C. Incorrect. The highest number that a probability value can be is 1! There are 2 outcomes in the event space, but the probability of the event is the ratio of outcomes in the event space to the total number of equally likely outcomes. There are 6 equally likely outcomes, so the probability is $\frac{\text{size of event space}}{\text{size of sample space}} = \frac{2}{6} = \frac{1}{3}$.
D. Incorrect. The highest number that a probability value can be is 1! There are 6 equally likely outcomes in the sample space, but the probability of the event is the ratio of outcomes in the event space to the total number of equally likely outcomes. The probability is $\frac{\text{size of event space}}{\text{size of sample space}} = \frac{2}{6} = \frac{1}{3}$.

Counting Methods to find Sample Spaces

The most difficult thing for calculating a probability can be finding the size of the sample space, especially if there are two or more trials. There are several counting methods that can help.

The first one to look at is making a chart. In the example below, Tori is flipping *two* coins. So you need to determine the sample space carefully. A chart such as the one shown in the example that follows is a good approach.

Example

Tori is flipping a pair of coins and noting how many flips of "heads" she gets. What is the probability that she flips 2 heads? What is the probability that she flips only 1 head?

Solution

<table>
<tr><td>Outcomes:
<table>
<tr><th>First coin</th><th>Second coin</th><th>outcome</th></tr>
<tr><td>H</td><td>H</td><td>HH</td></tr>
<tr><td>H</td><td>T</td><td>HT</td></tr>
<tr><td>T</td><td>H</td><td>TH</td></tr>
<tr><td>T</td><td>T</td><td>TT</td></tr>
</table>
sample space: {HH, HT, TH, TT}
event space for 2 heads: {HH}
event space for 1 head: {HT, TH}</td><td>Create a chart to record the results of flipping the first coin, followed by the result of flipping the second coin.</td></tr>
<tr><td>$P(2 \text{ heads }) = P(\{\mathrm{HH}\}) = \frac{1}{4}$
$P(1 \text{ head }) = P(\{\mathrm{HT}, \mathrm{TH}\}) = \frac{2}{4} = \frac{1}{2}$</td><td>Since the outcomes are equally likely, the probability of the event is the ratio of event space to sample space.</td></tr>
</table>

$P(2 \text{ heads }) = \frac{1}{4}$

$P(1 \text{ head }) = \frac{1}{2}$

In the example below, the sample space for Tori is simple as only one die is being rolled. However, since James is rolling two die, a chart helps to organize the information.

✓ Example

Tori rolled a six-sided die and wanted to get a result of either 1 or 4. James rolled two six-sided dice, one blue and one red, and wanted to get a result of both a 1 *and* a 3, at the same time. Which event has a greater probability?

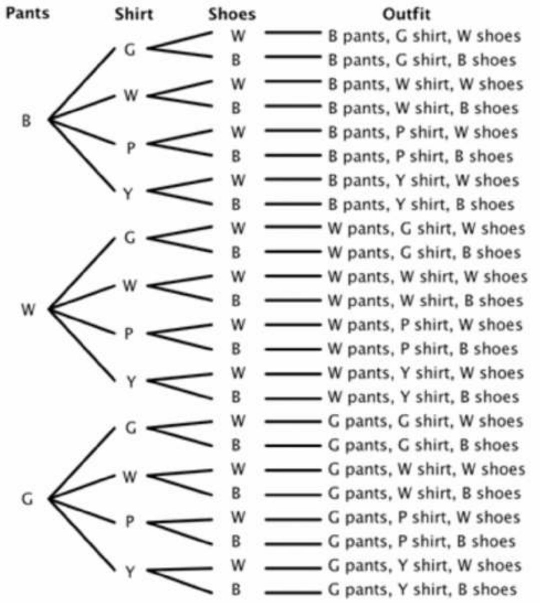
Solution

<table>
<tr><td>Tori's sample space: {1, 2, 3, 4, 5, 6}
Tori's event space: {1, 4}</td><td>First, find the sample space and the event space for the two trials. For Tori's trial, this is straightforward.</td></tr>
<tr><td>$\text{Tori: } P(1 \text{ or } 4) = \frac{2}{6} = \frac{1}{3}$</td><td>Since the outcomes are equally likely, the probability of the event is the ratio of event space to sample space.</td></tr>
<tr><td>
<table>
<tr><td></td><td></td><th colspan="6">Red die</th></tr>
<tr><td></td><td></td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
<tr><td rowspan="6">Blue die</td><td>1</td><td>1, 1</td><td>1, 2</td><td>1, 3</td><td>1, 4</td><td>1, 5</td><td>1, 6</td></tr>
<tr><td>2</td><td>2, 1</td><td>2, 2</td><td>2, 3</td><td>2, 4</td><td>2, 5</td><td>2, 6</td></tr>
<tr><td>3</td><td>3, 1</td><td>3, 2</td><td>3, 3</td><td>3, 4</td><td>3, 5</td><td>3, 6</td></tr>
<tr><td>4</td><td>4, 1</td><td>4, 2</td><td>4, 3</td><td>4, 4</td><td>4, 5</td><td>4, 6</td></tr>
<tr><td>5</td><td>5, 1</td><td>5, 2</td><td>5, 3</td><td>5, 4</td><td>5, 5</td><td>5, 6</td></tr>
<tr><td>6</td><td>6, 1</td><td>6, 2</td><td>6, 3</td><td>6, 4</td><td>6, 5</td><td>6, 6</td></tr>
</table>
</td><td>It's not so obvious for James’ trial, since he is rolling two dice. He can create and use a chart to find the possibilities.</td></tr>
<tr><td>James' sample space has 36 outcomes.
James' event space has 2 outcomes.</td><td>There are 36 outcomes. Of these, there are 2 that have both 1 and 3.</td></tr>
<tr><td>$\text{James: } P(1 \text{ and } 3) = \frac{2}{36} = \frac{1}{18}$</td><td>Since the outcomes are equally likely, the probability of the event is the ratio of event space to sample space.</td></tr>
</table>

Tori's event has a greater probability.

You can also use a **tree diagram** to determine the sample space. A tree diagram has a branch for every possible outcome for each event.

Suppose a closet has three pairs of pants (black, white, and green), four shirts (green, white, purple, and yellow), and two pairs of shoes (black and white). How many different outfits can be made? There are 3 choices for pants, 4 choices for shirts, and 2 choices for shoes. For our tree diagram, let's use B for black, W for white, G for green, P for purple, and Y for yellow.


Pants
Shirt
Shoes
Outfit
B
G
W
P
Y
W
B
B pants, G shirt, W shoes
B pants, G shirt, B shoes
B pants, W shirt, W shoes
B pants, W shirt, B shoes
B pants, P shirt, W shoes
B pants, P shirt, B shoes
B pants, Y shirt, W shoes
B pants, Y shirt, B shoes
W pants, G shirt, W shoes
W pants, G shirt, B shoes
W pants, W shirt, W shoes
W pants, W shirt, B shoes
W pants, P shirt, W shoes
W pants, P shirt, B shoes
W pants, Y shirt, W shoes
W pants, Y shirt, B shoes
G pants, G shirt, W shoes
G pants, G shirt, B shoes
G pants, W shirt, W shoes
G pants, W shirt, B shoes
G pants, P shirt, W shoes
G pants, P shirt, B shoes
G pants, Y shirt, W shoes
G pants, Y shirt, B shoes

You can see from the tree diagram that there are 24 possible outfits (some perhaps not great choices) in the sample space.

Now you could fairly easily solve some probability problems. For example, what is the probability that if you close your eyes and choose randomly you would choose pants and shoes with the same color? In the chart there are 8 outfits where the pants and the shoes match.

$P(\text{ same color })=\frac{8}{24}=\frac{1}{3}$

As you've seen, when a trial involves more than one random element, such as flipping more than one coin or rolling more than one die, you don't always need to identify every outcome in the sample space to calculate a probability. You only need the *number* of outcomes.

The **Fundamental Counting Principle** is a way to find the number of outcomes without listing and counting every one of them.

The Fundamental Counting Principle

If one event has p possible outcomes, and another event has m possible outcomes, then there are a total of $p \cdot m$ possible outcomes for the two events.

Examples

- Rolling two six-sided dice: Each die has 6 equally likely outcomes, so the sample space is $6 \cdot 6$ or 36 equally likely outcomes.
- Flipping three coins: Each coin has 2 equally likely outcomes, so the sample space is $2 \cdot 2 \cdot 2$ or 8 equally likely outcomes.
- Rolling a six-sided die and flipping a coin: The sample space is $6 \cdot 2$ or 12 equally likely outcomes.

So you could use the Fundamental Counting Principle to find out how many outfits there are in the previous example. There are 3 choices for pants, 4 choices for shirts, and 2 choices for shoes. Using the Fundamental Counting Principle, you have $4 \cdot 3 \cdot 2=24$ different outfits.

✓ Example

Barry volunteers at a charity walk to make lunches for all the other volunteers. In each bag he puts:

- **one of two sandwiches (peanut butter and jelly, or turkey and cheese),**
- **one of three chips (regular potato chips, baked potato chips, or corn chips),**
- **one piece of fruit (an apple or an orange).**

He forgot to mark what was in the bags. Assuming that each choice is equally likely, what is the probability that the bag Therese gets holds a peanut butter and jelly sandwich and an apple?

Solution

Size of sample space: Multiply the (number of sandwich choices) by the (number of chip choices) by the (number of fruit choices) This gives you: $2 \cdot 3 \cdot 2 = 12$	First, use the Fundamental Counting Principle to find the *size* of the sample space.
Size of event space: Multiply the (number of sandwich choices in the event) by the (number of chip choices in the event) by the (number of fruit choices in the event) This gives you: $1 \cdot 3 \cdot 1 = 3$	For the event space, follow the same principle. In this case, there is only one sandwich and one piece of fruit of interest, but any of the three types of chips are acceptable.
$P(\text{ PB\& J and apple }) = \frac{\text{size of event space}}{\text{size of sample space}} = \frac{3}{12} = \frac{1}{4}$	Use the ratio to find the probability.

? Exercise

Carrie flips four coins and counts the number of tails. There are four ways to get exactly one tail: HHHT, HHTH, HTHH, and THHH. What is the probability that Carrie gets exactly one tail?

A. $\frac{1}{16}$
B. $\frac{1}{8}$
C. $\frac{1}{4}$
D. $\frac{1}{2}$

Answer

A. Incorrect. Since there are two outcomes for each coin, there are 16 possible outcomes since $2 \cdot 2 \cdot 2 \cdot 2 = 16$. However, there are four outcomes in the event, so the probability is $\frac{4}{16}$, or $\frac{1}{4}$.
B. Incorrect. Since there are two outcomes for each coin, there are 16 possible outcomes since $2 \cdot 2 \cdot 2 \cdot 2 = 16$. There are four outcomes in the event, so the probability is $\frac{4}{16}$, or $\frac{1}{4}$.
C. Correct. Since there are two outcomes for each coin, there are 16 possible outcomes since $2 \cdot 2 \cdot 2 \cdot 2 = 16$. There are four outcomes in the event, so the probability is $\frac{4}{16}$, or $\frac{1}{4}$.
D. Incorrect. There are two outcomes for each coin, but there are 4 coins. That means there are 16 possible outcomes since $2 \cdot 2 \cdot 2 \cdot 2 = 16$. There are four outcomes in the event, so the probability is $\frac{4}{16}$, or $\frac{1}{4}$.

Summary

Probability helps you understand random, unpredictable situations where multiple outcomes are possible. It is a measure of the likelihood of an event, and it depends on the ratio of event and possible outcomes, if all those outcomes are equally likely.

$$P(E) = \frac{\text{size of the event space}}{\text{size of the sample space}} = \frac{\text{number of outcomes in the event}}{\text{total number of possible outcomes}}$$

The Fundamental Counting Principle is a shortcut to finding the size of the sample space when there are many trials and outcomes:

If one event has p possible outcomes, and another event has m possible outcomes, then there are a total of $p \cdot m$ possible outcomes for the two events.

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CHAPTER OVERVIEW

9: Real Numbers

9.1: Introduction to Real Numbers
  9.1.1: Variables and Expressions
  9.1.2: Integers
  9.1.3: Rational and Real Numbers
9.2: Operations with Real Numbers
  9.2.1: Adding Integers
  9.2.2: Adding Real Numbers
  9.2.3: Subtracting Real Numbers
  9.2.4: Multiplying and Dividing Real Numbers
9.3: Properties of Real Numbers
  9.3.1: Associative, Commutative, and Distributive Properties
9.4: Simplifying Expressions
  9.4.1: Order of Operations

Thumbnail: Real numbers can be thought of as points on an infinitely long number line. (Public Domai; Phrood~commonswiki via Wikipedia)

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SECTION OVERVIEW

9.1: Introduction to Real Numbers

Topic hierarchy

9.1.1: Variables and Expressions

9.1.2: Integers

9.1.3: Rational and Real Numbers

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9.1.1: Variables and Expressions

Learning Objectives

- Evaluate expressions with one variable for given values for the variable.
- Evaluate expressions with two variables for given values for the variables.

Introduction

Algebra involves the solution of problems using variables, expressions, and equations. This topic focuses on variables and expressions and you will learn about the types of expressions used in algebra.

Variables and Expressions

One thing that separates algebra from arithmetic is the **variable**. A variable is a letter or symbol used to represent a quantity that can change. Any letter can be used, but x and y are common. You may have seen variables used in formulas, like the area of a rectangle. To find the area of a rectangle, you multiply *length* times *width*, written using the two variables l and w.

$l \cdot w$

Here, the variable l represents the length of the rectangle. The variable w represents the width of the rectangle.

You may be familiar with the formula for the area of a triangle. It is $\frac{1}{2}b \cdot h$.

Here, the variable b represents the base of the triangle, and the variable h represents the height of the triangle. The $\frac{1}{2}$ in this formula is a **constant**. A constant, unlike a variable, is a quantity that does **not** change. A constant is often a number.

An **expression** is a mathematical phrase made up of a sequence of mathematical symbols. Those symbols can be numbers, variables, or operations $(+, -, \cdot, \div)$. Examples of expressions are $l \cdot w$ and $\frac{1}{2}b \cdot h$.

Example

Identify the constant and variable in the expression $24 - x$.

Solution

24 is the constant. x is the variable.	Since 24 cannot change its value, it is a constant. The variable is x, because it could be 0, or 2, or many other numbers.

Substitution and Evaluation

In arithmetic, you often evaluated, or simplified, expressions involving numbers.

$3 \cdot 25 + 4 \quad 25 \div 5 \quad 142 - \frac{12}{4} \quad \frac{3}{4} - \frac{1}{4} \quad 2.45 + 13$

In algebra, you will evaluate many expressions that contain variables.

$a + 10 \quad 48 \cdot c \quad 100 - x \quad l \cdot w \quad \frac{1}{2}b \cdot h$

To **evaluate** an expression means to find its value. If there are variables in the expression, you will be asked to evaluate the expression for a specified value for the variable.

The first step in evaluating an expression is to **substitute** the given value of a variable into the expression. Then you can finish evaluating the expression using arithmetic.

Example

Evaluate $24 - x$ when $x = 3$.

Solution

$24 - x$ $24 - 3$	Substitute 3 for the x in the expression.

$24-3=21$	Subtract to complete the evaluation.

21

When you have two variables, you substitute each given value for each variable.

Example

Evaluate $l \cdot w$ when $l=3$ and $w=8$.

Solution

$l \cdot w$ $3 \cdot 8$	Substitute 3 for l in the expression and 8 for w.
$3 \cdot 8=24$	Multiply.

24

When you multiply a variable by a constant number, you don't need to write the multiplication sign or use parentheses. For example, $3a$ is the same as $3 \cdot a$.

Notice that the sign $\cdot$ is used to represent multiplication. This is because the multiplication sign x looks a lot like the letter x, especially when hand written. Because of this, it's best to use parentheses or the $\cdot$ sign to indicate multiplication of numbers.

Example

Evaluate $4x-4$ when $x=10$.

Solution

$4x-4$ $4(10)-4$	Substitute 10 for x in the expression.
$40-4$ 36	Remember that you must multiply before you do the subtraction.

36

Since the variables are allowed to vary, there are times when you want to evaluate the same expression for different values for the variable.

Example

John is planning a rectangular garden that is 2 feet wide. He hasn't decided how long to make it, but he's considering 4 feet, 5 feet, and 6 feet. He wants to put a short fence around the garden. Using x to represent the length of the rectangular garden, he will need $x+x+2+2$, or $2x+4$, feet of fencing.

How much fencing will he need for each possible garden length? Evaluate the expression when $x=4$, $x=5$, and $x=6$ to find out.

Solution

$2x+4$ $2(4)+4$ $8+4$ 12	For $x=4$, substitute 4 for x in the expression. Evaluate by multiplying and adding.

$2x+4$ $2(5)+4$ $10+4$ 14	For $x=5$, substitute 5 for x. Evaluate by multiplying and adding.
$2x+4$ $2(6)+4$ $12+4$ 16	For $x=6$, substitute 6 for x and evaluate.

John needs 12 feet of fencing when $x=4$, 14 feet when $x=5$, and 16 feet when $x=6$.

Evaluating expressions for many different values for the variable is one of the powers of algebra. Computer programs are written to evaluate the same expression (usually a very complicated expression) for millions of different values for the variable(s).

? Exercise

Evaluate $8x-1$ when $x=2$.

A. 1
B. 7
C. 8
D. 15
E. 16

Answer

A. Incorrect. You may have forgotten to multiply the 8 by the 2. Substituting 2 for x gives $8(2)-1$. First multiply to get $16-1$, then subtract. The correct answer is 15.
B. Incorrect. You may have forgotten to substitute. Substituting 2 for x gives $8(2)-1$. First multiply to get $16-1$, then subtract. The correct answer is 15.
C. Incorrect. You may have forgotten the order of operations: multiply first, then subtract. Substituting 2 for x gives $8(2)-1$. First multiply to get $16-1$. Then subtract. The correct answer is 15.
D. Correct. Substituting 2 for x gives $8(2)-1$. First multiply to get $16-1$, then subtract to get 15.
E. Incorrect. You may have forgotten to subtract 1 after you multiplied. Substituting 2 for x gives $8(2)-1$. First multiply to get $16-1$, then subtract. The correct answer is 15.

Summary

Variables are an important part of algebra. Expressions made from variables, constants, and operations can represent a numerical value. You can evaluate an expression when you are provided with one or more values for the variables: substitute each variable’s value for the variable, then perform any necessary arithmetic.

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9.1.2: Integers

Learning Objectives

- Locate integers on a number line.
- Find the absolute value of a given number.
- Find the opposite of a given number.

Introduction

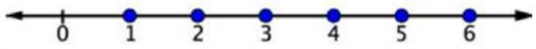
You've worked with numbers on a number line. You know how to graph numbers like 0, 1, 2, 3, etc. on the number line. There are other kinds of numbers that can be graphed on the number line, too**.** Let's see what they look like and where they are located on the number line.

Natural Numbers and Whole Numbers

In mathematics, it's sometimes helpful to talk about groups of things, which are called **sets**. Numbers can be grouped into sets, and a particular number can belong to more than one set.

You probably are familiar with the set of **natural numbers**, which are also called the **counting numbers**. These are the numbers 1, 2, 3, and so on—the numbers we use when counting.

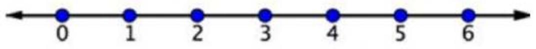
The following illustration shows the natural numbers graphed on a number line.



The number line continues in both directions. The set of natural numbers only continues to the right, so you can include 6, 7, and so on, all the way up into the hundreds, thousands, and beyond. You can only show so much on one picture!

When 0 is added to the set of 1, 2, 3, and so on, it forms the set of **whole numbers**. These are called “whole” because they have no fractional parts. (A trick to help you remember which are *natural* numbers and which are *whole* numbers is to think of a “hole,” which can be represented by 0. The whole ("hole") numbers include 0, the natural numbers do not.)

The following illustration shows the whole numbers graphed on the number line.

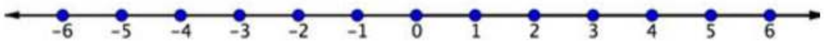


Integers

When you work with something like temperature, you sometimes want to use numbers that are less than zero, which are called **negative numbers**. Negative numbers are written using a negative sign in front, such as -1, -5, and -30. These are read negative one, negative five, and negative thirty. (The negative sign should not be read as minus; *minus* means subtraction.)

The numbers greater than 0 are called **positive numbers** and can be written with or without the “+” sign. Notice that 0 is neither positive nor negative!

Integers are the numbers: …, -3, -2, -1, 0, 1, 2, 3, ... Notice that all of the whole numbers are also integers. The following illustration shows the integers graphed on the number line. The integers include zero and continue to the right and to the left.



Exercise

The number 0 belongs to which of the following sets of numbers?

natural numbers

whole numbers

integers

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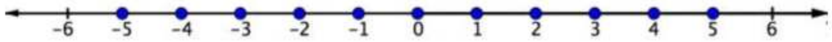
A. natural numbers only
B. whole numbers only
C. natural and whole numbers
D. integers only
E. whole numbers and integers

Answer

A. Incorrect. The natural numbers are 1, 2, 3, and so on. They don’t include 0. The correct answer is whole numbers and integers.
B. Incorrect. While the whole numbers include 0, so do the integers. The correct answer is whole numbers and integers.
C. Incorrect. The natural numbers are 1, 2, 3, and so on. They don’t include 0. The correct answer is whole numbers and integers.
D. Incorrect. While the integers include 0, so do the whole numbers. The correct answer is whole numbers and integers.
E. Correct. Both whole numbers and integers include 0, but the natural numbers do not.

Absolute Value and the Number Line

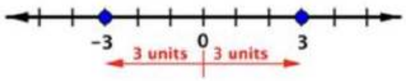
The number line below shows all the integers between and including -5 and 5. Notice that the positive integers go to the right: 1, 2, 3, and so on. The negative integers go to the left: -1, -2, -3, and so on.



The distance between a number’s place on the number line and 0 is called the number’s **absolute value**. To write the absolute value of a number, use short vertical lines (|) on either side of the number. For example, the absolute value of -3 is written as bar negative three bar.

Notice that distance is always positive or 0.

|-3| = 3, as -3 is 3 units away from 0 and |3|=3, as 3 is 3 units away from 0.



Here are some other examples.

$$
\begin{aligned}
|0| &= 0\\
|-23| &= 23\\
|6| &= 6\\
|817| &= 817\\
|-3,000| &= 3,000
\end{aligned}
$$

Example

Find $|x|$ when $x=-7$.

Solution

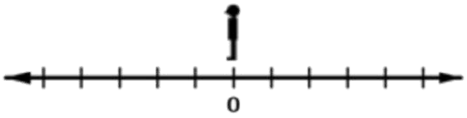
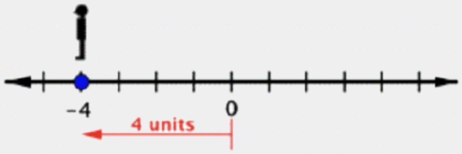
$\lvert x\rvert$ $\lvert -7\rvert$	Substitute -7 for x in the expression.
$\lvert -7\rvert = 7$	Since -7 is 7 units from 0, the absolute value is 7.

To locate an integer on the number line, imagine standing on the number line at 0. If the number is 0, you’re there. If the number is positive, face to the right toward numbers greater than 0. If the number is negative, face to the left toward numbers less than 0. Then, move forward the number of units equal to the absolute value of the number.

✓ Example

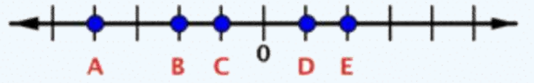
Find -4 on the number line. Then determine |-4|.

Solution

	Imagine standing at 0. Since -4 is negative, face to the left. Move 4 units from 0 in the negative direction.
	Draw a dot on the number line at that location, which is -4.
$\|-4\| = 4$	Direction moved does not affect absolute value, only the distance moved.

? Exercise

Which point represents -2 on this number line?



Answer

A. Incorrect. This point is 4 units to the left of 0, so it represents -4. The point that represents -2 is 2 units to the left of 0. Point B is correct.

B. Correct. Point B is 2 units to the left of 0, so it represents -2.

C. Incorrect. This point is 1 unit to the left of 0, so it represents -1. The point that represents -2 is 2 units to the left of 0. Point B is correct.

D. Incorrect. This point is 1 unit to the right of 0, so it represents 1. A negative number must be to the left of 0. The point that represents -2 is 2 units to the left of 0. Point B is correct.

E. Incorrect. This point is 2 units to the right of 0. A negative number must be to the left of 0. The point that represents -2 is 2 units to the left of 0. Point B is correct.

Opposites

You may have noticed that, except for 0, the integers come in pairs of positive and negative numbers: 1 and -1, 3 and -3, 72 and -3, 72 and -72, and so on. Each number is the **opposite** of the other number in the pair: 72 is the opposite of -72, and -72 is the opposite of 72.

A number and its opposite are the same distance from 0, so they have the same absolute value.

$|72| = 72, \text{ and } |-72| = 72$

The set of **integers** are all the whole numbers and their opposites.

? Exercise

What is the opposite of -29?

Answer

> The opposite of -29 is 29.

Summary

Some numbers are natural numbers (1, 2, 3, ...) or whole numbers (0, 1, 2, 3, ...). Whole numbers are also integers. There are other integers which are the opposites of the whole numbers (-1, -2, -3, ...). These negative numbers lie to the left of 0 on the number line. Integers are the whole numbers and their opposites. The absolute value of a number is its distance to 0 on the number line. Absolute values are always positive or 0.

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9.1.3: Rational and Real Numbers

Learning Objectives

- Identify the subset(s) of the real numbers that a given number belongs to.
- Locate points on a number line.
- Compare rational numbers.
- Identify rational and irrational numbers.

Introduction

You've worked with fractions and decimals, like 3.8 and $21\frac{2}{3}$. These numbers can be found between the integer numbers on a number line. There are other numbers that can be found on a number line, too. When you include all the numbers that can be put on a number line, you have the real number line. Let's dig deeper into the number line to see what those numbers look like and where they fall on the number line.

Rational Numbers

The fraction $\frac{16}{3}$, mixed number $5\frac{1}{3}$, and decimal 5.33... (or $5.\overline{3}$) all represent the same number. This number belongs to a set of numbers that mathematicians call **rational numbers**. Rational numbers are numbers that can be written as a ratio of two integers. Regardless of the form used, $5.\overline{3}$ is rational because this number *can* be written as the ratio of 16 over 3, or $\frac{16}{3}$.

Examples of rational numbers include the following.

0.5, as it can be written as $\frac{1}{2}$

$2\frac{3}{4}$, as it can be written as $\frac{11}{4}$

-1.6, as it can be written as $-1\frac{6}{10}=\frac{-16}{10}$

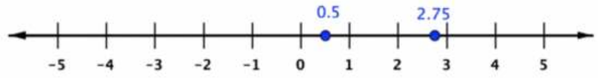
4, as it can be written as $\frac{4}{1}$

-10, as it can be written as $\frac{-10}{1}$

All of these numbers can be written as the ratio of two integers.

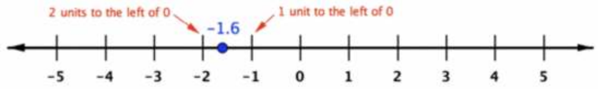
You can locate these points on the number line.

In the following illustration, points are shown for 0.5 or $\frac{1}{2}$, and for 2.75 or $2\frac{3}{4}=\frac{11}{4}$.



As you have seen, rational numbers can be negative. Each positive rational number has an opposite. The opposite of $5.\overline{3}$ is $-5.\overline{3}$, for example.

Be careful when placing **negative numbers** on a number line. The negative sign means the number is to the left of 0, and the absolute value of the number is the distance from 0. So to place -1.6 on a number line, you would find a point that is |-1.6| or 1.6 units to the left of 0. This is more than 1 unit away, but less than 2.



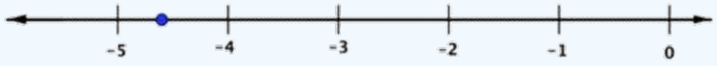
Example

Place $-\frac{23}{5}$ on a number line.

Solution

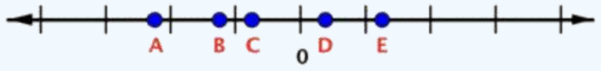
It's helpful to first write this improper fraction as a mixed number: 23 divided by 5 is 4 with a remainder of 3, so $-\frac{23}{5}$ is $-4\frac{3}{5}$.

Since the number is negative, you can think of it as moving $4\frac{3}{5}$ units to the *left* of 0. $-4\frac{3}{5}$ will be between -4 and -5.



Exercise

Which of the following points represents $-1\frac{1}{4}$?



Answer

A. Incorrect. This point is just over 2 units to the left of 0. The point should be 1.25 units to the left of 0. The correct answer is point B.
B. Correct. Negative numbers are to the left of 0, and $-1\frac{1}{4}$ should be 1.25 units to the left. Point B is the only point that’s more than 1 unit and less than 2 units to the left of 0.
C. Incorrect. Notice that this point is between 0 and and the first unit mark to the left of 0, so it represents a number between -1 and 0. The point for $-1\frac{1}{4}$ should be 1.25 units to the left of 0. You may have correctly found 1 unit to the left, but instead of continuing to the left another 0.25 unit, you moved right. The correct answer is point B.
D. Incorrect. Negative numbers are to the left of 0, not to the right. The point for $-1\frac{1}{4}$ should be 1.25 units to the left of 0. The correct answer is point B.
E. Incorrect. This point is 1.25 units to the right of 0, so it has the correct distance but in the wrong direction. Negative numbers are to the left of 0. The correct answer is point B.

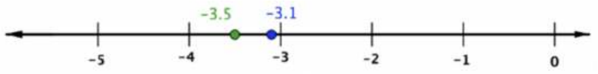
Comparing Rational Numbers

When two **whole numbers** are graphed on a number line, the number to the right on the number line is always greater than the number on the left.

The same is true when comparing two **integers** or rational numbers. The number to the right on the number line is always greater than the one on the left.

Here are some examples.

Numbers to Compare	**Comparison**	**Symbolic Expression**
-2 and -3	-2 is greater than -3 because -2 is to the right of -3	$-2 > -3$ or $-3 < -2$
2 and 3	3 is greater than 2 because 3 is to the right of 2	$3 > 2$ or $2 < 3$
-3.5 and -3.1	-3.1 is greater than -3.5 because -3.1 is to the right of -3.5 (see below)	$-3.1 > -3.5$ or $-3.5 < -3.1$



Exercise

Which of the following are true?

Option 1. $-4.1 > 3.2$

Option 2. $-3.2 > -4.1$

Option 3. $3.2 > 4.1$

Option $4.\quad -4.6 < -4.1$

A. Option 1 and Option 4
B. Option 1 and Option 2
C. Option 2 and Option 3
D. Option 2 and Option 4
E. Options 1, 2, and 3

Answer

A. Incorrect. -4.6 is to the left of -4.1, so -4.6<-4.1. However, positive numbers such as 3.2 are always to the right of negative numbers such as -4.1, so 3.2>-4.1 or -4.1<3.2. The correct answer is ii and iv, -3.2>-4.1 and -4.6<-4.1.
B. Incorrect. -3.2 is to the right of -4.1, so -3.2>-4.1. However, positive numbers such as 3.2 are always to the right of negative numbers such as -4.1, so 3.2>-4.1 or -4.1<3.2. The correct answer is ii and iv, -3.2>-4.1 and -4.6<-4.1.
C. Incorrect. -3.2 is to the right of -4.1, so -3.2>-4.1. However, 3.2 is to the left of 4.1, so 3.2<4.1. The correct answer is ii and iv, -3.2>-4.1 and -4.6<-4.1.
D. Correct. -3.2 is to the right of -4.1, so -3.2>-4.1. Also, -4.6 is to the left of -4.1, so -4.6<-4.1.
E. Incorrect. -3.2 is to the right of -4.1, so -3.2>-4.1. However, positive numbers such as 3.2 are always to the right of negative numbers such as -4.1, so 3.2>-4.1 or -4.1<3.2. Also, 3.2 is to the left of 4.1, so 3.2<4.1. The correct answer is ii and iv, -3.2>-4.1 and -4.6<-4.1.

Irrational and Real Numbers

There are also numbers that are not rational. **Irrational numbers** cannot be written as the ratio of two integers.

Any square root of a number that is not a perfect square, for example $\sqrt{2}$, is irrational. Irrational numbers are most commonly written in one of three ways: as a root (such as a square root), using a special symbol (such as π), or as a nonrepeating, nonterminating decimal.

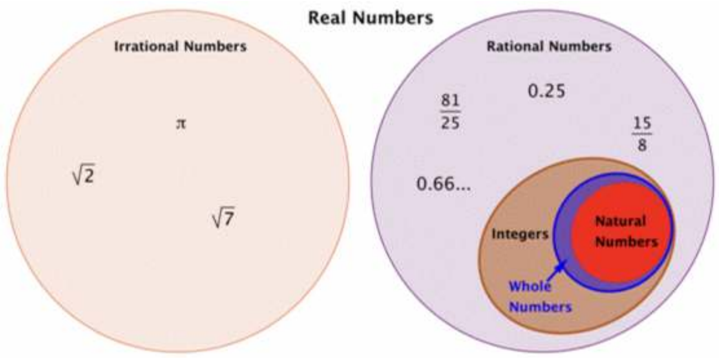
Numbers with a decimal part can either be **terminating decimals** or **nonterminating decimals**. Terminating means the digits stop eventually (although you can always write zeros at the end). For example, 1.3 is terminating, because there's a last digit. The decimal form of $\frac{1}{4}$ is 0.25. Terminating decimals are always rational.

Nonterminating decimals have digits (other than 0) that continue forever. For example, consider the decimal form of $\frac{1}{3}$, which is 0.3333 ... The 3s continue indefinitely. Or the decimal form of $\frac{1}{11}$, which is 0.090909 ...: the sequence "09" continues forever.

In addition to being nonterminating, these two numbers are also **repeating decimals**. Their decimal parts are made of a number or sequence of numbers that repeats again and again. A **nonrepeating decimal** has digits that never form a repeating pattern. The value of $\sqrt{2}$, for example, is 1.414213562... No matter how far you carry out the numbers, the digits will never repeat a previous sequence.

If a number is terminating or repeating, it must be rational; if it is both nonterminating *and* nonrepeating, the number is irrational.

Type of Decimal	Rational or Irrational	Examples
Terminating	Rational	$0.25\ (\text{or } \frac{1}{4})$ $1.3\ (\text{or } \frac{13}{10})$
Nonterminating and Repeating	Rational	$0.66\ldots\ (\text{or } \frac{2}{3})$ $3.242424\ldots\ (\text{or})$ $\frac{321}{99} = \frac{107}{33}$
Nonterminating and Nonrepeating	Irrational	π (or 3.14159...) $\sqrt{7}\ (\text{or } 2.6457\ldots)$



Example

Is -82.91 rational or irrational?

Solution

-82.91 is rational.	The number is rational, because it is a *terminating* decimal.

The **set** of **real numbers** is made by combining the set of rational numbers and the set of irrational numbers. The real numbers include **natural numbers** or **counting numbers**, whole numbers, integers, rational numbers (fractions and repeating or terminating decimals), and irrational numbers. The set of real numbers is all the numbers that have a location on the number line.

Sets of Numbers

Natural numbers 1, 2, 3, ...

Whole numbers 0, 1, 2, 3, ...

Integers ..., -3, -2, -1, 0, 1, 2, 3, ...

Rational numbers: numbers that can be written as a ratio of two integers—rational numbers are terminating or repeating when written in decimal form

Irrational numbers: numbers that cannot be written as a ratio of two integers—irrational numbers are nonterminating and nonrepeating when written in decimal form

Real numbers: any number that is rational or irrational

Example

What sets of numbers does 32 belong to?

Solution

The number 32 belongs to all these sets of numbers: Natural numbers Whole numbers Integers Rational numbers Real numbers	Every natural or counting number belongs to all of these sets!

Example

What sets of numbers does $382.\overline{3}$ **belong to?**

Solution

$382.\overline{3}$ belongs to these sets of numbers: Rational numbers Real numbers	The number is rational because it's a repeating decimal. It's equal to $382\frac{1}{3}$ or $\frac{1,147}{3}$, or $382.\overline{3}$.

✔ Example

What sets of numbers does $-\sqrt{5}$ belong to?

Solution

$-\sqrt{5}$ belongs to these sets of numbers: Irrational numbers Real numbers	The number is irrational because it can't be written as a ratio of two integers. Square roots that aren't perfect squares are always irrational.

? Exercise

Which of the following sets does $\frac{-33}{5}$ belong to?

whole numbers

integers

rational numbers

irrational numbers

real numbers

A. rational numbers only
B. irrational numbers only
C. rational and real numbers
D. irrational and real numbers
E. integers, rational numbers, and real numbers
F. whole numbers, integers, rational numbers, and real numbers

Answer

A. Incorrect. The number is rational (it's written as a ratio of two integers) but it's also real. All rational numbers are also real numbers. The correct answer is rational and real numbers, because all rational numbers are also real.
B. Incorrect. Irrational numbers can't be written as a ratio of two integers. The correct answer is rational and real numbers, because all rational numbers are also real.
C. Correct. The number is between integers, so it can't be an integer or a whole number. It's written as a ratio of two integers, so it's a rational number and not irrational. All rational numbers are real numbers, so this number is rational and real.
D. Incorrect. Irrational numbers can't be written as a ratio of two integers. The correct answer is rational and real numbers, because all rational numbers are also real.
E. Incorrect. The number is between integers, not an integer itself. The correct answer is rational and real numbers.
F. Incorrect. The number is between integers, so it can't be an integer or a whole number. The correct answer is rational and real numbers.

Summary

The set of real numbers is all numbers that can be shown on a number line. This includes natural or counting numbers, whole numbers, and integers. It also includes rational numbers, which are numbers that can be written as a ratio of two integers, and irrational numbers, which cannot be written as a the ratio of two integers. When comparing two numbers, the one with the greater value would appear on the number line to the right of the one with the lesser value.

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SECTION OVERVIEW

9.2: Operations with Real Numbers

Topic hierarchy

9.2.1: Adding Integers

9.2.2: Adding Real Numbers

9.2.3: Subtracting Real Numbers

9.2.4: Multiplying and Dividing Real Numbers

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9.2.1: Adding Integers

Learning Objectives

- Add two or more integers with the same sign.
- Add two or more integers with different signs.

Introduction

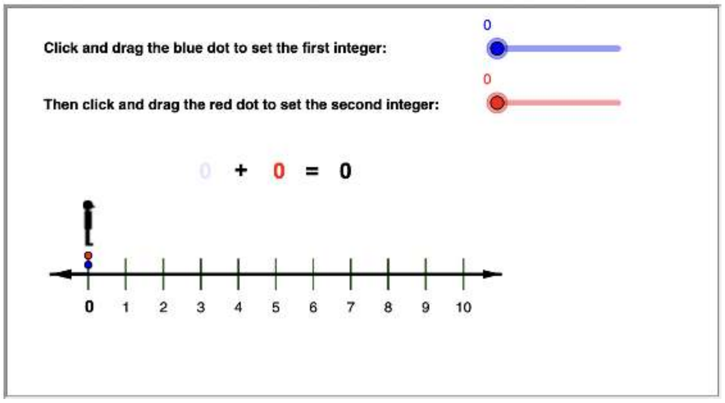
On an extremely cold day, the temperature may be -10. If the temperature rises 8 degrees, how will you find the new temperature? Knowing how to add integers is important here and in much of algebra.

Adding Integers with the Same Sign

Since positive **integers** are the same as natural numbers, adding two positive integers is the same as adding two natural numbers. To add positive integers on the number line, you move to the right (in the positive direction).

Supplemental Interactive Activity

Try out the interactive number line below. Choose a few pairs of positive integers to add. Click and drag the blue and red dots, and watch how the addition works.



Click and drag the blue dot to set the first integer:

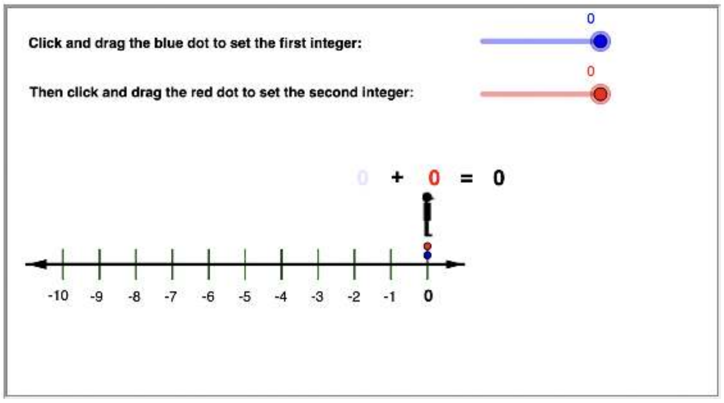
Then click and drag the red dot to set the second integer:

0 + 0 = 0

To add negative integers on a number line, you move to the left (in the negative direction).

Supplemental Interactive Activity

Try out the interactive number line below. Choose a few pairs of negative integers to add. Click and drag the blue and red dots, and watch how the subtraction works.



Click and drag the blue dot to set the first integer:

Then click and drag the red dot to set the second integer:

0 + 0 = 0

In both cases, the total number of units moved is the total distance moved. Since the distance of a number from 0 is the absolute value of that number, then the absolute value of the sum of the integers is the sum of the absolute values of the **addends**.

When both numbers are negative, you move left in a negative direction, and the sum is negative. When both numbers are positive, you move right in a positive direction, and the sum is positive.

To add two numbers with the same sign (both positive or both negative):

- Add their absolute values and give the sum the same sign.

Example

Find -23+(-16).

Solution

Both addends have the same sign (negative).

So, add their absolute values:

|-23|=23 and |-16|=16

The sum of those numbers is 23+16=39.

Since both addends are negative, the sum is negative.

-23+(-16)=-39

With more than two addends that have the same sign, use the same process with all addends.

Example

Find -27+(-138)+(-55).

Solution

All addends have the same sign (negative).

So, add their absolute values:

|-27|=27, |-138|=138, and |-55|=55

The sum of those numbers is 27+138+55=220.

Since all addends are negative, the sum is negative.

-27+(-138)+(-55)=-220

Exercise

Find -32+(-14).

A. 46
B. 18
C. -18
D. -46

Answer

A. Incorrect. You found the sum of the absolute values, but the sum must be negative, as you are adding two negative numbers. The correct answer -46
B. Incorrect. You subtracted 14 from 32. The sum is found by first adding the absolute values of the addends: |-32|+|-14|=32+14=46. Then you must give the sum the same sign as the two addends. The correct answer is -46.
C. Incorrect. The sum is found by first adding the absolute values of the addends: |-32|+|-14|=32+14=46. Then you must give the sum the same sign as the two addends. The correct answer is -46.
D. Correct. The sum is found by first adding the absolute values of the addends: |-32|+|-14|=32+14=46. Then you must give the sum the same sign as the two addends, so the answer is -46.

Adding Integers with Different Signs

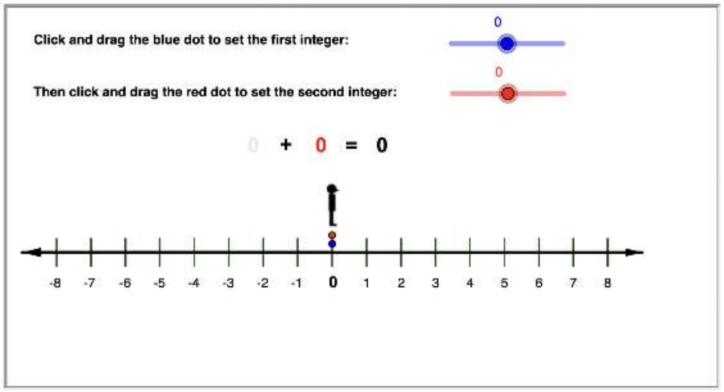
What happens when the addends have different signs, like in the temperature problem in the introduction? If it’s -10 degrees, and then the temperature rises 8 degrees, the new temperature is -10+8. How can you calculate the new temperature?

When you add a positive integer and a negative integer, you move in the positive (right) direction to the first number, and then move in the negative (left) direction to add the negative integer.

Since the distances overlap, the absolute value of the sum is the difference of their distances. So to add a positive number and a negative number, you subtract their absolute values (their distances from 0).

Supplemental Interactive Activity

Try adding integers with different signs with this interactive number line. Move in a positive direction (right) to add a positive number, and move in a negative direction (left) to add a negative number. See if you can find a rule for adding numbers without using the number line.

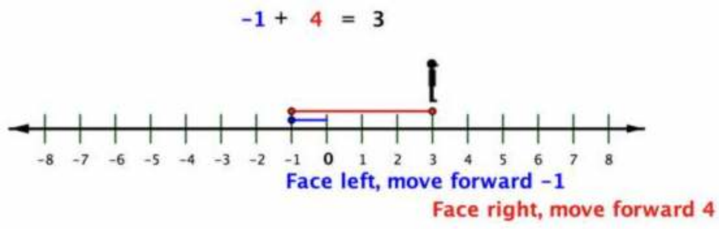


Click and drag the blue dot to set the first integer:

Then click and drag the red dot to set the second integer:

0 + 0 = 0

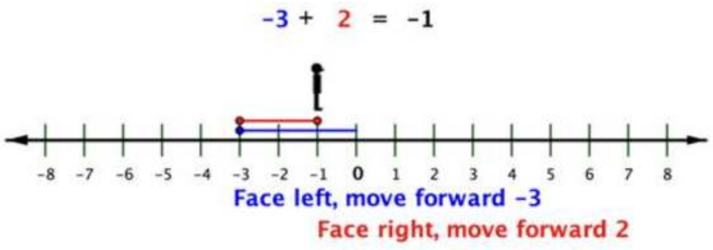
What is the sign of the sum of a positive and a negative integer? It’s pretty easy to figure out. If you moved further to the right than you did to the left, you ended to the right of 0, and the answer is positive; and if you move further to the left, the answer is negative.



If you didn’t have the number line to refer to, you could find the sum of -1+4 by

- subtracting the distances from zero (the absolute values) 4-1=3 and then
- applying the sign of the one furthest from zero (the largest absolute value). In this case, 4 is further from 0 than -1, so the answer is positive: -1+4=3

Look at the illustration below.



If you didn’t have the number line to refer to, you can find the sum of -3+2 by

- subtracting the distances from zero (the absolute values) 3-2=1 and then

- applying the sign of the one furthest from zero (the largest absolute value). In this case, |-3|>|2|, so the answer is negative: -3+2=-1

To add two numbers with different signs (one positive and one negative):

- Find the *difference* of their absolute values.
- Give the sum the same sign as the number with the greater absolute value.

Note that when you find the difference of the absolute values, you always subtract the lesser absolute value from the greater one. The example below shows you how to solve the temperature question that you considered earlier.

✓ Example

Find 8+(-10).

Solution

The addends have different signs.

So find the difference of their absolute values.

|-10|=10 and |8|=8

The difference of the absolute values is 10-8=2.

Since 10>8, the sum has the same sign as -10.

8+(-10)=-2

✓ Example

Evaluate $x+37$ **when** $x=-22$.

Solution

$x+37$ $-22+37$	Substitute -22 for x in the expression.
$\lvert-22\rvert=22 \text{ and } \lvert 37\rvert=37$ $37-22=15$	The addends have different signs. So find the difference of their absolute values. Since |37|>|-22|, the sum has the same sign as 37.

-22+37=15

With more than two addends, you can add the first two, then the next one, and so on.

✓ Example

Find -27+(-138)+55.

Solution

	Add two at a time, starting with -27+(-138).
$\lvert-27\rvert=27 \text{ and } \lvert-138\rvert=138$ $27+138=165$ $-27+-138=-165$	Since they have the same signs, you add their absolute values and use the same sign to get -165.
$-165+55$ $\lvert-165\rvert=165 \text{ and } \lvert 55\rvert=55$ $165-55=110$	Now add -165+55. Since -165 and 55 have different signs, you add them by *subtracting* their absolute values.
$-165+55=-110$	Since 165>55, the sign of the final sum is the same as the sign of -165.

$$-27+(-138)+55=-110$$

? Exercise

Find 32+(-14).

A. 46
B. 18
C. -18
D. -46

Answer

A. Incorrect. You found the sum of the absolute values. Since the addends have different signs, you must find the difference of the absolute values. |32|=32 and |-14|=14. The difference is 32-14=18. The sign of the sum is the same as the addend with the greater absolute value. Since |32|>|-14|, the sum is positive, the same sign as 32. The correct answer is 18.
B. Correct. Since the addends have different signs, you must find the difference of the absolute values. |32|=32 and |-14|=14. The difference is 32-14=18. The sign of the sum is the same as the addend with the greater absolute value. Since |32|>|-14|, the sum is positive.
C. Incorrect. You found the difference of the absolute values of the addends. However, because |32|>|-14|, the sign of the sum should be the same sign as 32. The correct answer is 18.
D. Incorrect. You added the absolute values of the addends, and gave the sum the wrong sign. Since the addends have different signs, you must find the difference of the absolute values. |32|=32 and |-14|=14. The difference is 32-14=18. The sign of the sum is the same as the addend with the greater absolute value. Since |32|>|-14|, the sum is positive, the same sign as 32. The correct answer is 18.

Summary

There are two cases to consider when adding integers. When the signs are the same, you add the absolute values of the addends and use the same sign. When the signs are different, you find the difference of the absolute values and use the same sign as the addend with the greater absolute value.

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9.2.2: Adding Real Numbers

Learning Objectives

- Add two or more real numbers with the same sign.
- Add two or more real numbers with different signs.
- Simplify by using the identity property of 0.
- Solve application problems requiring the addition of real numbers.

Introduction

Adding real numbers follows the same rules as adding integers. The number 0 has some special attributes that are very important in algebra. Knowing how to add these numbers can be helpful in real-world situations as well as algebraic situations.

Rules for Adding Real Numbers

The rules for adding **integers** apply to other real numbers, including **rational numbers**.

To add two numbers with the same sign (both positive or both negative)

- *Add* their absolute values.
- Give the sum the same sign.

To add two numbers with different signs (one positive and one negative)

- Find the *difference* of their absolute values. (Note that when you find the difference of the absolute values, you always subtract the lesser absolute value from the greater one.)
- Give the sum the same sign as the number with the greater absolute value.

Remember—to add fractions, you need them to have the same denominator. This is still true when one or more of the fractions are negative.

Example

Find $-\frac{3}{7}+(-\frac{6}{7})+\frac{2}{7}$

Solution

$\lvert -\frac{3}{7}\rvert = \frac{3}{7}$ and $\lvert -\frac{6}{7}\rvert = \frac{6}{7}$	This problem has three **addends**. Add the first two, and then add the third.
$\frac{3}{7}+\frac{6}{7}=\frac{9}{7}$	Since the signs of the first two are the same, find the sum of the absolute values of the fractions.
$-\frac{3}{7}+(-\frac{6}{7})=-\frac{9}{7}$	Since both addends are negative, the sum is negative.
$\lvert -\frac{9}{7}\rvert = \frac{9}{7}$ and $\lvert \frac{2}{7}\rvert = \frac{2}{7}$ $\frac{9}{7}-\frac{2}{7}=\frac{7}{7}$	Now add the third addend. The signs are different, so find the *difference* of their absolute values.
$-\frac{9}{7}+\frac{2}{7}=-\frac{7}{7}$	Since $\lvert -\frac{9}{7}\rvert > \lvert \frac{2}{7}\rvert$, the sign of the final sum is the same as the sign of $-\frac{9}{7}$.
$-\frac{3}{7}+(-\frac{6}{7})+\frac{2}{7}=-\frac{7}{7}$	The answer is $-\frac{7}{7}$, which can be simplified to -1.

Example

Find $-2\frac{3}{4}+\frac{7}{8}$.

Solution

$\left\lvert -2\frac{3}{4}\right\rvert = 2\frac{3}{4}$ and $\left\lvert \frac{7}{8}\right\rvert = \frac{7}{8}$ $2\frac{3}{4} - \frac{7}{8}$	The signs are different, so find the *difference* of their absolute values.
$2\frac{3}{4} = \frac{2(4)+3}{4} = \frac{11}{4}$	First rewrite $2\frac{3}{4}$ as an improper fraction, then rewrite the fraction using a common denominator.
$\frac{11}{4} = \frac{11\cdot 2}{4\cdot 2} = \frac{22}{8}$ $\frac{22}{8} - \frac{7}{8}$	Now substitute the rewritten fraction in the problem.
$\frac{22}{8} - \frac{7}{8} = \frac{15}{8}$	Subtract the numerators and keep the same denominator. Simplify to lowest terms, if possible.
$-2\frac{3}{4} + \frac{7}{8} = -\frac{15}{8}$	Since $\left\lvert -2\frac{3}{4}\right\rvert > \left\lvert \frac{7}{8}\right\rvert$, the sign of the final sum is the same as the sign of $-2\frac{3}{4}$. The answer $-\frac{15}{8}$ can be simplified to $-1\frac{7}{8}$.

When you add decimals, remember to line up the decimal points so you are adding tenths to tenths, hundredths to hundredths, and so on.

✓ Example

Find $27.832 + (-3.06)$**.**

Solution

Since the addends have different signs, subtract their absolute values.

$$|-3.06| = 3.06$$

$$\begin{array}{r} 27.832 \\ -3.06 \\ \hline 24.772 \end{array}$$

The sum has the same sign as 27.832, whose absolute value is greater.

$$27.832 + (-3.06) = 24.772$$

? Exercise

Find -32.22+124.3.

A. 19.79
B. 44.65
C. 92.08
D. 156.52

Answer

A. Incorrect. To find the sum of numbers with different signs, you must subtract their absolute values. When adding and subtracting decimals, you must pay attention to place value. You may have subtracted 32.22-12.43. instead of 124.3-32.22. The correct answer is 92.08.
B. Incorrect. To find the sum of numbers with different signs, you must subtract their absolute values. You may have added instead. Also, when adding and subtracting decimals, you must pay attention to place value. The correct answer is 92.08.
C. Correct. Since the addends have different signs, you must subtract their absolute values. 124.3-32.22 is 92.08. Since |124.3|>|-32.22|, the sum is positive.
D. Incorrect. To find the sum of numbers with different signs, you must subtract their absolute values. You added their absolute values. The correct answer is 92.08.

The Additive Identity

The rules for adding real numbers refer to the addends being positive or negative. But 0 is neither positive nor negative.

It should be no surprise that you add 0 the way you always have—adding 0 doesn't change the value.

$$\begin{array}{ll} 7+0=7 & -7+0=-7 \\ 0+3.6=3.6 & -\frac{2}{23}+0=-\frac{2}{23} \\ x+0=x & 0+x=x \end{array}$$

Notice that $x+0=x$ and $0+x=x$. This means that it doesn't matter which addend comes first.

The number 0 is called the **additive identity**. The **identity property of 0** states that adding 0 to other numbers doesn't change their value**.** You can think of it in this way: adding 0 lets the other number keep its identity.

? Exercise

What is 0+y, when y=3?

A. -3
B. 0
C. 3

Answer

A. Incorrect. The identity property says $0+\text{ any number }=\text{ that number }$. Substituting 3 for y gives $0+3$, and $0+3=3$.
B. Incorrect. The sum of a number and 0 is that number, not 0. Substituting 3 for y gives $0+3$, and $0+3=3$.
C. Correct. Substituting 3 for y gives $0+3$, and $0+3=3$.

Applications of Addition

There are many situations that use negative numbers. For example, temperatures colder than 0^{o} are usually described using negative numbers. In golf tournaments, players' scores are often reported as a number over or under par, instead of the total number of strokes it takes to hit the ball into the hole. (*Par* is the expected number of strokes needed to complete a hole.) A number under par is negative, and a number over par is positive.

The following examples show how addition of real numbers, including negative numbers, can be useful.

✓ Example

Boston is, on average, 7 degrees warmer than Bangor, Maine. The low temperature on one cold winter day in Bangor was -13^{o} Fahrenheit. About what low temperature would you expect Boston to have on that day?

Solution

If the temperature in Bangor is x, the temperature in Boston is $x+7$.	The phrase "7 degrees warmer" means you add 7 degrees to Bangor's temperature to estimate Boston's temperature.
$x=-13$	On that day, Bangor's low was -13^{o}.
Boston's temperature is $-13+7$	Substitute -13 for x to get Boston's temperature.
$-13+7=-6$	

You would expect Boston to have a temperature of -6 degrees.

✓ Example

Before Joanne could deposit her paycheck of $802.83, she overdrew her checking account. The balance was -$201.35. What was her balance after she deposited the paycheck?

Solution

<table>
<tr><td>$-201.35+802.83$</td><td>By depositing her paycheck, Joanne is adding money to her account. The new balance is the sum of the old (-201.35) and the paycheck amount.</td></tr>
<tr><td>$-201.35+802.83=601.83$</td><td>Since the numbers have different signs, find the difference of -201.35.
Since |802.83|>|-201.35|, the sum is positive.</td></tr>
</table>

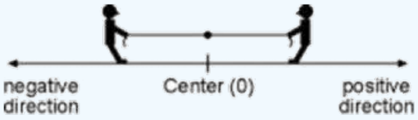
The new balance is $601.48.

When forces or objects are working in opposite directions, sometimes it's helpful to assign a negative value to one and a positive value to the other. This is done often in physics and engineering, but it could also be done in other contexts, such as football or a tug-of-war.

✓ Example

Two people are in a tug-of-war contest. They are facing each other, each holding the end of a rope. They both pull on the rope, trying to move the center toward themselves.

Here's an illustration of this situation. The person on the right is pulling in the *positive* direction, and the person on the left is pulling in the *negative* direction.



At one point in the competition, the person on the right was pulling with 122.8 pounds of force. The person on the left was pulling with 131.3 pounds of force. The forces on the center of the rope, then, were 122.8 lbs and -131.3 lbs.

a) What was the net (total sum) force on the center of the rope?

b) In which direction was it moving?

Solution

<table>
<tr><td>$\text{Net force } =122.8+(-131.3)$</td><td>The net force is the sum of the two forces on the rope.</td></tr>
<tr><td>$\text{Net force } =-8.5$</td><td>To find the sum, add the difference of the absolute values of the addends. Since |-131.3|>122.8, the sum is negative.</td></tr>
<tr><td>The net force is -8.5 lbs (or 8.5 lbs to the left). The center of the rope is moving to the left (the negative direction).</td><td>Notice that it makes sense that the rope was moving to the left, since that person was pulling with more force.</td></tr>
</table>

? Exercise

After Bangor reached a low temperature of -13^{o}, the temperature rose only 4 degrees higher for the rest of the day. What was the high temperature that day?

A. -17
B. -9
C. 9
D. 17

Answer

A. Incorrect. Although you used the correct sign, you found the sum of the absolute values. The temperature rose (added) 4 degrees from -13, so the high temperature is -13+4. Since the addends have different signs, you must find the difference of the absolute values. |-13|=13 and |4|=4. The difference is 13-4=9. The sign of the sum is the same as the addend with the greater absolute value. Since |-13|>|4|, the sum is negative. The correct answer is -9.

B. Correct. The temperature rose (added) 4 degrees from -13, so the high temperature is -13+4. Since the addends have different signs, you must find the difference of the absolute values. |-13|=13 and |4|=4. The difference is 13-4=9. The sign of the sum is the same as the addend with the greater absolute value. Since |-13|>|4|, the sum is -9.
C. Incorrect. You found the difference of the absolute values of the addends. However, because |-13|>|4|, the sign of the sum should be the same as the sign on -13. The correct answer is -9.
D. Incorrect. You added the absolute values of the addends, and gave the sum the wrong sign. The temperature rose (added) 4 degrees from -13, so the high temperature is -13+4. Since the addends have different signs, you must find the difference of the absolute values. |-13|=13 and |4|=4. The difference is 13-4=9. The sign of the sum is the same as the addend with the greater absolute value. since |-13|>|4|, the sum is negative. The correct answer is -9.

Summary

As with integers, adding real numbers is done following two rules. When the signs are the same, you add the absolute values of the addends and use the same sign. When the signs are different, you subtract the absolute values and use the same sign as the addend with the greater absolute value. Adding 0, which is neither positive nor negative, is done using the additive identity of 0: $x+0=x$ and $0+x=x$, for any value of x.

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9.2.3: Subtracting Real Numbers

Learning Objectives

- Subtract two or more real numbers.
- Simplify combinations that require both addition and subtraction of real numbers.
- Solve application problems requiring subtraction of real numbers.

Introduction

Subtraction and addition are closely related. They are called **inverse operations**, because one "undoes" the other. So, just as with integers, you can rewrite subtraction as addition to subtract real numbers.

Additive Inverses

Inverse operations, such as addition and subtraction, are a key idea in algebra. Suppose you have $10 and you loan a friend $5. An hour later, she pays you back the $5 she borrowed. You are back to having $10. You could represent the transaction like this:

10-5+5=10.

This works because a number minus itself is 0.

$$3-3=0 \quad 63.5-63.5=0 \quad 39,283-39,283=0$$

So, adding a number and then subtracting the same number is like adding 0.

Thinking about this idea in terms of **opposite** numbers, you can also say that a number plus its opposite is also 0. Notice that each example below consists of a positive and a negative number pair added together.

$$3+(-3)=0 \quad -63.5+63.5=0 \quad 39,283+(-39,283)=0$$

Two numbers are **additive inverses** if their sum is 0. Since this means the numbers are opposites (same absolute value but different signs), "additive inverse" is another, more formal term for the opposite of a number. (Note that 0 is its own additive inverse.)

Subtracting Real Numbers

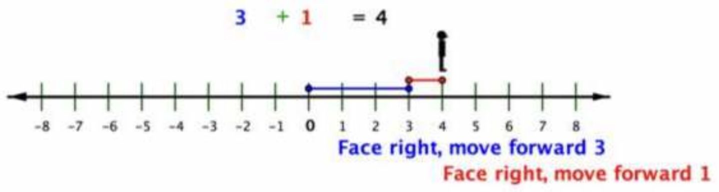
You can use the additive inverses or opposites to rewrite subtraction as addition. If you are adding two numbers with different signs, you find the difference between their absolute values and keep the sign of the number with the greater absolute value.

When the greater number is positive, it's easy to see the connection.

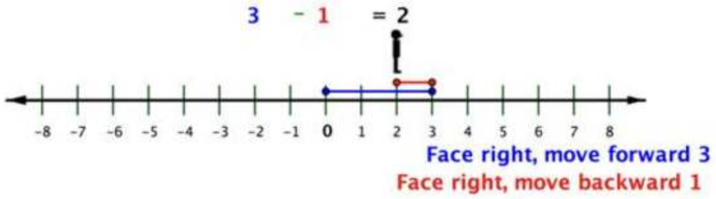
$$13+(-7)=13-7$$

Both equal 6.

Let’s see how this works. When you add positive numbers, you are moving forward, facing in a positive direction.

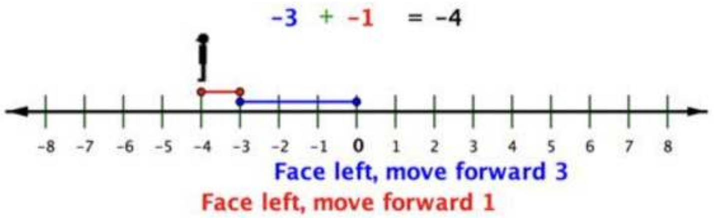


When you subtract positive numbers, you can imagine moving backward, but still facing in a positive direction.



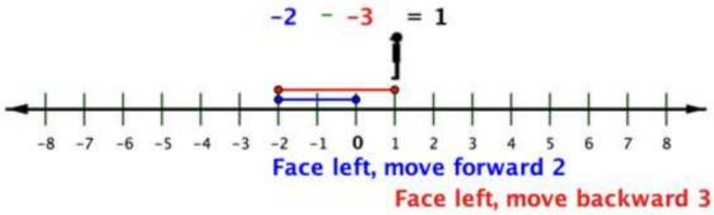
Now let's see what this means when one or more of the numbers is negative.

Recall that when you add a negative number, you move forward, but face in a negative direction (to the left).



-3 + -1 = -4
-8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
Face left, move forward 3
Face left, move forward 1

How do you subtract a negative number? First face and move forward in a negative direction to the first number, -2. Then continue facing in a negative direction (to the left), but move *backward* to subtract -3.



-2 - -3 = 1
-8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
Face left, move forward 2
Face left, move backward 3

But isn't this the same result as if you had added positive 3 to -2? -2+3=1.

Supplemental Interactive Activity

Use the interactive number line below to find the answers to the following pairs of sums and differences, and compare the answers. You will have to specify both numbers and whether you are adding or subtracting.

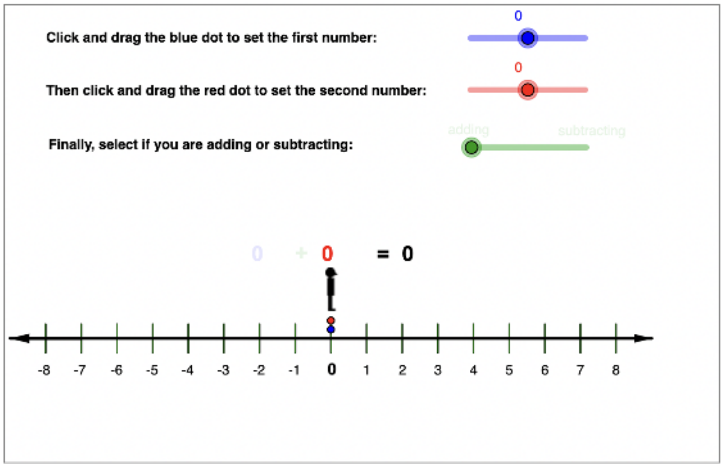
$3-4 \text{ and } 3+(-4)$

$2-(-3) \text{ and } 2+3$

$-1-5 \text{ and } -1+(-5)$

$-2-(-1) \text{ and } -2+1$

In each addition problem, you move in one direction some distance forward. In the paired subtraction problem, you move in the opposite direction the same distance backward. Notice how each gives you the same result!



Click and drag the blue dot to set the first number:
0
Then click and drag the red dot to set the second number:
0
Finally, select if you are adding or subtracting:
adding
subtracting
0 + 0 = 0
-8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8

> To subtract a real number, you can rewrite the problem as adding the opposite (additive inverse).

Note, that while this always works, whole number subtraction is still the same. You can subtract 38-23 just as you have always done. Or, you could also rewrite it as

38+(-23). Both ways you will get the same answer.

38-23=38+(-23)=15.

It’s your choice in these cases.

✓ Example

Find 23-73.

Solution

	You can't use your usual method of subtraction, because 73 is greater than 23.
$23+(-73)$	Rewrite the subtraction as adding the opposite.
$\lvert 23\rvert = 23$ and $\lvert -73\rvert = 73$ $73-23=50$	The addends have different signs, so find the difference of their absolute values.
$23-73=-50$	Since $\lvert -73\rvert > \lvert 23\rvert$, the final answer is negative.

✓ Example

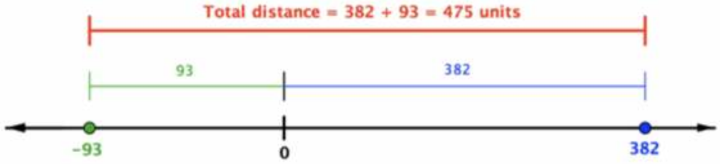
Find $382-(-93)$**.**

Solution

$382+93$ $382+93=475$	Rewrite the subtraction as adding the opposite. The opposite of -93 is 93. So, this becomes a simple addition problem.

$382-(-93)=475$

Another way to think about subtracting is to think about the distance between the two numbers on the number line. In the example above, 382 is to the *right* of 0 by 382 units, and -93 is to the *left* of 0 by 93 units. The distance between them is the sum of their distances to 0: 382+93.



✓ Example

Find $22\frac{1}{3}-x$**, when** $x=-\frac{3}{5}$**.**

Solution

$22\frac{1}{3}-(-\frac{3}{5})$	Substitute $-\frac{3}{5}$ for x in the expression.
$22\frac{1}{3}+\frac{3}{5}$	Rewrite the subtraction as adding the opposite. The opposite of $-\frac{3}{5}$ is $\frac{3}{5}$.
$22\frac{1\cdot 5}{3\cdot 5}+\frac{3\cdot 3}{5\cdot 3}=22\frac{5}{15}+\frac{9}{15}$ $22\frac{5}{15}+\frac{9}{15}=22\frac{14}{15}$	This is now just adding two rational numbers. Remember to find a common denominator when adding fractions. 3 and 5 have a common multiple of 15; change denominators of both fractions to 15 (and make the necessary changes in the numerator!) before adding.

$22\frac{14}{15}$

Exercise

Find $-32.3-(-16.3)$.

A. -48.6
B. -16
C. 16
D. 48.6

Answer

A. Incorrect. You added -32.3 and -16.3. To subtract, change the problem to adding the opposite of -16.3, which gives -32.3+16.3. Then use the rules for adding two numbers with different signs. Since the difference between 32.3 and 16.3 is 16, and |-32.3|>|16.3|, the correct answer is -16.
B. Correct. To subtract, change the problem to adding the opposite of -16.3, which gives -32.3+16.3. Then use the rules for adding two numbers with different signs. Since the difference between 32.3 and 16.3 is 16, and |-32.3|>|16.3|, the correct answer is -16.
C. Incorrect. You used the wrong sign. To subtract, change the problem to adding the opposite of -16.3, which gives -32.3+16.3. Then use the rules for adding two numbers with different signs. Since the difference between 32.3 and 16.3 is 16, and |-32.3|>|16.3|, the correct answer is -16.
D. Incorrect. You added the opposites of both numbers. To subtract, change the problem to adding the opposite of -16.3, which gives -32.3+16.3. Then use the rules for adding two numbers with different signs. Since the difference between 32.3 and 16.3 is 16, and |-32.3|>|16.3|, the correct answer is -16.

Adding and Subtracting More Than Two Real Numbers

When you have more than two **real numbers** to add or subtract, work from left to right as you would when adding more than two whole numbers. Be sure to change subtraction to addition of the opposite when needed.

Example

Find -23+16-(-32)-4+6.

Solution

$\mathbf{-23+16}-(-32)-4+6$ $\mathbf{-7}-(-32)-4+6$	Start with -23+16. The addends have different signs, so find the difference and use the sign of the addend with the greater absolute value. -23+16=-7.
$\mathbf{-7-(-32)}-4+6$ $\mathbf{-7+32}-4+6$	Now you have -7-(-32). Rewrite this subtraction as addition of the opposite. The opposite of -32 is 32, so this becomes -7+32, which equals 25.
$\mathbf{25-4}+6$	You now have 25-4. You *could* rewrite this as an addition problem, but you don't need to.
$\mathbf{21+6}$	Complete the final addition of 21+6.

-23+16-(-32)-4+6=27

Exercise

Find 32-(-14)-2+(-82).

A. -66
B. -38
C. 98
D. 126

Answer

A. Incorrect. You probably subtracted -14 incorrectly. To subtract 32-(-14), write the subtraction as addition of the opposite, giving 32+14=46. Then subtract 2 to get 44, and add -82 to get the correct answer of -38.
B. Correct. To subtract 32-(-14), write the subtraction as addition of the opposite, giving 32+14=46. Then subtract 2 to get 44, and add -82 to get -38.
C. Incorrect. You may have missed the negative signs on -14 and -82. To subtract 32-(-14), write the subtraction as addition of the opposite, giving 32+14=46. Then subtract 2 to get 44, and add -82 to get the correct answer of -38.
D. Incorrect. You probably subtracted -14 correctly, but you added 82 instead of -82 as the last step. To subtract 32-(-14), write the subtraction as addition of the opposite, giving 32+14=46. Then subtract 2 to get 44, and add -82 to get the correct answer of -38.

Applications of Subtraction

Situations that use negative numbers can require subtraction as well as addition. As you saw above, sometimes subtracting two positive numbers can give a negative result. You should be sure that a negative number makes sense in the problem.

✓ Example

Boston is, on average, 7 degrees warmer than Bangor, Maine. The low temperature on one cold winter day in Boston was 3^{o}F. About what low temperature would you expect Bangor to have on that day?

Solution

If the temperature in Boston is x, the temperature in Bangor is $x-7$.	The phrase "7 degrees warmer" means you can subtract 7 degrees from Boston's temperature to estimate Bangor's temperature. (Note that you can also add 7 degrees to Bangor's temperature to estimate Boston's temperature. Be careful about which should have the greater number!)
$x=3$	On that day, Boston's low was 3^{o}.
Bangor's temperature is $3-7$	Substitute 3 for x to get Bangor's temperature.
$3-7=3+(-7)$	Since 3<7, rewrite the subtraction problem as addition of the opposite.dd the numbers. Since one is positive and the other is negative, you find the difference of |-7| and |3|, which is 4. Since |-7|>|3|, the final sum is negative.

You would expect the low temperature in Bangor, Maine to be -4^{o}F.

✓ Example

Everett paid several bills without balancing his checkbook first! When the last check he wrote was still to be deducted from his balance, Everett's account was already overdrawn. The balance was -$201.35. The final check was for $72.66, and another $25 will be subtracted as an overdraft charge. What will Everett's account balance be after that last check and the overdraft charge are deducted?

Solution

$-201.35-72.66-25$	The new balance will be the existing balance of -$201.35, minus the check's amount and the overdraft charge.
$-201.35-72.66-25$ $-201.35+(-72.66)-25$	Start with the first subtraction, $-201.35-72.66$. Rewrite it as the addition of the opposite of 72.66.
$-274.01-25$	Since the addends have the same signs, the sum is the sum of their absolute values (201.35+72.66) with the same sign (negative).

$-274.01+(-25)$	Again, rewrite the subtraction as the addition of the opposite.
$-274.01+(-25)=-299.01$	Add, by adding the sum of their absolute values and use the same sign as both addends.

Everett’s account balance will be -$299.01.

Example

One winter, Phil flew from Syracuse, NY to Orlando, FL. The temperature in Syracuse was -20°F. The temperature in Orlando was 75°F. What was the difference in temperatures between Syracuse and Orlando?

Solution

$75-(-20)$	To find the difference between the temperatures, you need to subtract. We subtract the ending temperature from the beginning temperature to get the change in temperature.
$75+20$	Rewrite the subtraction as adding the opposite. The opposite of -20 is 20.
$75+20=95$	There is a 95 degree difference between 75° and -20°.

The difference in temperatures is 95 degrees.

Exercise

Louise noticed that her bank balance was -$33.72 before her paycheck was deposited. After the check had been deposited, the balance was $822.98. No other deductions or deposits were made. How much money was she paid?

Answer

$856.70. The amount she was paid is the difference between the two balances: $822.98-(-33.72)$. This is the same as $822.98+33.72$, or 856.70.

Summary

Subtracting a number is the same as adding its opposite (also called its additive inverse). To subtract, you can rewrite the subtraction as adding the opposite and then use the rules for the addition of real numbers.

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9.2.4: Multiplying and Dividing Real Numbers

Learning Objectives

- Multiply two or more real numbers.
- Simplify by using the identity property of 1.
- Divide real numbers.
- Solve application problems requiring multiplication or division of real numbers.

Introduction

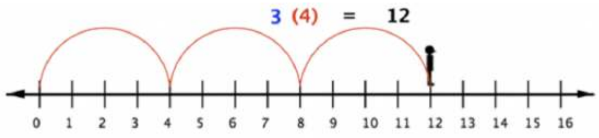
After addition and subtraction, the next operations you learned how to do were multiplication and division. You may recall that multiplication is a way of computing “repeated addition,” and this is true for negative numbers as well.

Multiplication and division are **inverse operations**, just as addition and subtraction are. You may recall that when you divide fractions, you multiply by the reciprocal.

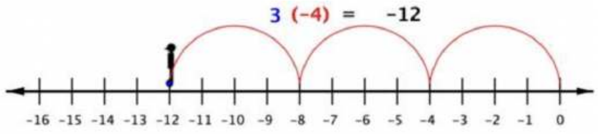
Multiplying Real Numbers

Multiplying real numbers is not that different from multiplying whole numbers and positive fractions. However, you haven't learned what effect a negative sign has on the product.

With whole numbers, you can think of multiplication as repeated addition. Using the number line, you can make multiple jumps of a given size. For example, the following picture shows the product $3 \cdot 4$ as 3 jumps of 4 units each.

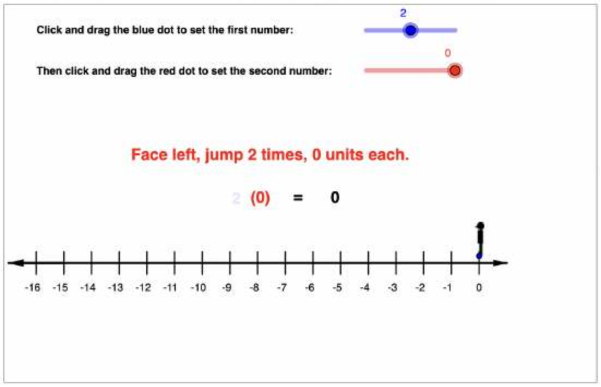


So to multiply 3(-4), you can face left (toward the negative side) and make three “jumps” forward (in a negative direction).



Supplemental Interactive Activity

Use the interactive number line to see how to multiply integers.



Click and drag the blue dot to set the first number: 2

Then click and drag the red dot to set the second number: 0

Face left, jump 2 times, 0 units each.

2 (0) = 0

The product of a positive number and a negative number (or a negative and a positive) is negative. You can also see this by using patterns. In the following list of products, the first number is always 3. The second number decreases by 1 with each row (3, 2, 1, 0, -1, -2). Look for a pattern in the products of these numbers. What numbers would fit the pattern for the last two products?

$3(3) = 9$
$3(2) = 6$
$3(1) = 3$
$3(0) = 0$
$3(-1) = ?$
$3(-2) = ?$

Notice that the pattern is the same if the order of the **factors** is switched:

$3(3) = 9$
$2(3) = 6$
$1(3) = 3$
$0(3) = 0$
$-1(3) = ?$
$-2(3) = ?$

Take a moment to think about that pattern before you read on.

As the factor decreases by 1, the product decreases by 3. So $3(-1) = -3$ and $3(-2) = -6$.

If you continue the pattern further, you see that multiplying 3 by a negative integer gives a negative number. This is true in general.

The Product of a Positive Number and a Negative Number.

To multiply a **positive number** and a **negative number**, multiply their absolute values. The product is **negative.**

You can use the pattern idea to see how to multiply two negative numbers. Think about how you would complete this list of products.

$-3(3) = -9$
$-3(2) = -6$
$-3(1) = -3$
$-3(0) = 0$
$-3(-1) = ?$
$-3(-2) = ?$

As the factor decreases by 1, the product increases by 3. So $-3(-1) = 3, -3(-2) = 6$.

Multiplying -3 by a negative integer gives a *positive* number. This is true in general.

The Product of Two Numbers with the Same Sign (both positive or both negative).

To multiply two **positive numbers**, multiply their absolute values. The product is **positive**.

To multiply two **negative numbers**, multiply their absolute values. The product is **positive**.

Example

Find $-3.8(0.6)$**.**

Solution

$\begin{array}{r} 3.8 \\ \times \quad 0.6 \\ \hline 2.28 \end{array}$	Multiply the absolute values as you normally would. 3.8 times 0.6.
	Place the decimal point by counting place values.
	3.8 has 1 place after the decimal point, and 0.6 has 1 place after the decimal point, so the product has 1+1 or 2 places after the decimal point.
$-3.8(0.6) = -2.28$	The product of a negative and a positive is negative.

✓ Example

Find $\left(-\frac{3}{4}\right)\left(-\frac{2}{5}\right)$**.**

Solution

$\left(\frac{3}{4}\right)\left(\frac{2}{5}\right)=\frac{6}{20}=\frac{3}{10}$	Multiply the absolute values of the numbers. First, multiply the numerators together to get the product's numerator. Then, multiply the denominators together to get the product's denominator. Rewrite in lowest terms, if needed.
$\left(-\frac{3}{4}\right)\left(-\frac{2}{5}\right)=\frac{3}{10}$	The product of two negative numbers is positive.

✓ Example

Find $43y$ **when** $y=-3$**.**

Solution

$43(-3)$	Substitute -3 for y in the expression.
$43(3)=129$	Multiply 43 and 3.
$43(-3)=-129$	The product of a positive number and a negative number is negative.

To summarize:

$\textbf{positive}\cdot\textbf{positive}:$ The product is $\textbf{positive.}$

$\textbf{negative}\cdot\textbf{negative}:$ The product is $\textbf{positive.}$

$\textbf{negative}\cdot\textbf{positive}:$ The product is $\textbf{negative.}$

$\textbf{positive}\cdot\textbf{negative}:$ The product is $\textbf{negative.}$

You can see that the product of two negative numbers is a positive number. So, if you are multiplying more than two numbers, you can count the number of negative factors.

Multiplying More Than Two Negative Numbers

If there are an **even** number (0, 2, 4, ...) If there are an **even** number (1, 3, 5, ...) of negative factors, the product is **negative**.

✓ Example

Find $3(-6)(2)(-3)(-1)$**.**

Solution

$3(6)(2)(3)(1)$ $18(2)(3)(1)$ $36(3)(1)$ $108(1)$ 108	Multiply the absolute values of the numbers.
$3(-6)(2)(-3)(-1)$	Count the number of negative factors. There are three (-6, -3, -1).
$3(-6)(2)(-3)(-1)=-108$	Since there are an odd number of negative factors, the product is negative.

Exercise

Find $(-30)(-0.5)$.

A. -150
B. -15
C. 15
D. 150

Answer

A. Incorrect. You multiplied 30 and 5 and forgot to adjust the decimal point. You also used the wrong sign. To multiply 30 and 0.5, multiply 30 and 5 to get 150, then place the decimal. Since 0.5 has one digit to the right of the decimal point, the decimal point in the product needs to be placed with one digit to the right of it to get 15.0, or 15. The two original factors are both negative. Since there is an even number of negative factors, the product is positive. The correct answer is 15.
B. Incorrect. You correctly multiplied 30 and 0.5 to get 15, but the two original factors are both negative. Since there is an even number of negative factors, the product is positive. The correct answer is 15.
C. Correct. First, multiply 30 and 0.5. Multiply 30 and 5 to get 150, then place the decimal point. Since 0.5 has one digit to the right of the decimal point, the decimal point in the product needs to be placed with one digit to the right of it to get 15.0, or 15. The two original factors are both negative. Since there is an even number of negative factors, the product is positive.
D. Incorrect. First, multiply 30 and 0.5. Multiply 30 and 5 to get 150, then place the decimal point. Since 0.5 has one digit to the right of the decimal point, the decimal point in the product needs to be placed with one digit to the right of it to get 15.0, or 15. The two original factors are both negative. Since there is an even number of negative factors, the product is positive. The correct answer is 15.

The Multiplicative Identity

There is a number that can be added, again and again, without ever changing the sum. That number, 0, is called the additive identity.

There is also a number that can be included as a *factor* as many times as you want, and it will never change the value of the product. That number, 1, is called the multiplicative identity.

$$\begin{array}{ll} 7(1)=7 & -7(1)=-7 \\ 1(3.6)=3.6 & -\frac{2}{23}(1)=-\frac{2}{23} \\ x(1)=x & (1)\ x=x \end{array}$$

The **identity property of 1** states that $x \cdot 1=x$ and $1 \cdot x=x$.

You can think of it in this way: Multiplying by 1 lets the other number keep its identity.

Exercise

What is $1(y)$, when $y=-3$?

A. -3
B. 1
C. 3

Answer

A. Correct. Substituting -3 for y gives $1(-3)$, and $1(-3)=-3$.
B. Incorrect. The product of a number and 1 is the other number, not 1. Substituting -3 for y gives $1(-3)$, and $1(-3)$ $=-3$.
C. Incorrect. The identity property says $1 \text{ . any number } = \text{ that number }$. Substituting -3 for y gives $1(-3)$, and $1(-3)$ $=-3$.

Multiplicative Inverses

You might recall that two numbers are additive inverses if their sum is 0, the additive identity.

3 and -3 are additive inverses because $3+(-3)=0$.

Two numbers are **multiplicative inverses** if their product is 1, the multiplicative identity.

$\frac{2}{3}$ and $\frac{3}{2}$ are multiplicative inverses because $\frac{2}{3}\left(\frac{3}{2}\right)=\frac{6}{6}=1$.

You may remember that when you divided fractions, you multiplied by the **reciprocal**. *Reciprocal* is another name for the multiplicative inverse (just as *opposite* is another name for additive inverse).

An easy way to find the multiplicative inverse is to just “flip” the numerator and denominator as you did to find the reciprocal. Here are some examples:

The reciprocal of $\frac{4}{9}$ is $\frac{9}{4}$ because $\frac{4}{9}\left(\frac{9}{4}\right)=\frac{36}{36}=1$.

The reciprocal of 3 is $\frac{1}{3}$ because $\frac{3}{1}\left(\frac{1}{3}\right)=\frac{3}{3}=1$.

The reciprocal of $-\frac{5}{6}$ is $-\frac{6}{5}$ because $-\frac{5}{6}\left(-\frac{6}{5}\right)=\frac{30}{30}=1$.

The reciprocal of 1 is 1 as $1(1)=1$.

? Exercise

What is the reciprocal, or multiplicative inverse, of -12?

A. -12
B. 1
C. $\frac{1}{12}$
D. 12
E. $-\frac{1}{12}$

Answer

A. Incorrect. $-12(-12)=144$, not 1. Since $-12=-\frac{12}{1}$ and $-\frac{12}{1}\left(-\frac{1}{12}\right)=1$, the correct answer is $-\frac{1}{12}$.
B. Incorrect. The reciprocal gives a product of 1; 1 is the reciprocal of 1. Since $-12=-\frac{12}{1}$ and $-\frac{12}{1}\left(-\frac{1}{12}\right)=1$, the correct answer is $-\frac{1}{12}$.
C. Incorrect. The numerator and denominator are properly "flipped," but the sign is incorrect. The product needs to be positive 1, so since -12 is negative, its reciprocal must also be negative. Since $-12=-\frac{12}{1}$ and $-\frac{12}{1}\left(-\frac{1}{12}\right)=1$, the correct answer is $-\frac{1}{12}$.
D. Incorrect. $12(-12)=-144$, not 1. Since $-12=-\frac{12}{1}$ and $-\frac{12}{1}\left(-\frac{1}{12}\right)=1$, the correct answer is $-\frac{1}{12}$.
E. Correct. $-12=-\frac{12}{1}$ and $-\frac{12}{1}\left(-\frac{1}{12}\right)=1$.

Dividing Real Numbers

When you divided by positive fractions, you learned to multiply by the reciprocal. You also do this to divide **real numbers**.

Think about dividing a bag of 26 marbles into two smaller bags with the same number of marbles in each. You can also say each smaller bag has *one half* of the marbles.

$26\div 2=26\left(\frac{1}{2}\right)=13$

Notice that 2 and $\frac{1}{2}$ are reciprocals.

Try again, dividing a bag of 36 marbles into smaller bags.

Number of bags	**Dividing by number of bags**	**Multiplying by reciprocal**
3	$\frac{36}{3}=12$	$36\left(\frac{1}{3}\right)=\frac{36}{3}=\frac{12(3)}{3}=12$
4	$\frac{36}{4}=9$	$36\left(\frac{1}{4}\right)=\frac{36}{4}=\frac{9(4)}{4}=9$
6	$\frac{36}{6}=6$	$36\left(\frac{1}{6}\right)=\frac{36}{6}=\frac{6(6)}{6}=6$

Dividing by a number is the same as multiplying by its reciprocal. (That is, you use the reciprocal of the **divisor**, the second number in the division problem.)

Example

Find $28 \div \frac{4}{3}$.

Solution

$28 \div \frac{4}{3} = 28\left(\frac{3}{4}\right)$	Rewrite the division as multiplication by the reciprocal. The reciprocal of $\frac{4}{3}$ is $\frac{3}{4}$.
$\frac{28}{1}\left(\frac{3}{4}\right) = \frac{28(3)}{4} = \frac{4(7)(3)}{4} = 7(3) = 21$	Multiply.

$28 \div \frac{4}{3} = 21$

Exercise

Find $\frac{6}{7} \div \frac{3}{10}$. Write the answer in lowest terms.

A. $\frac{9}{35}$
B. $\frac{7}{20}$
C. $\frac{27}{10}$
D. $\frac{20}{7}$

Answer

A. Incorrect. You multiplied without first rewriting the problem using the reciprocal. $\frac{6}{7} \div \frac{3}{10} = \frac{6}{7}\left(\frac{10}{3}\right) = \frac{60}{7(3)} = \frac{20(3)}{7(3)} = \frac{20}{7}$.
B. Incorrect. You used the reciprocal of the *first* number (the dividend) rather than the second number you're dividing by (the divisor). $\frac{6}{7} \div \frac{3}{10} = \frac{6}{7}\left(\frac{10}{3}\right) = \frac{60}{7(3)} = \frac{20(3)}{7(3)} = \frac{20}{7}$.
C. Incorrect. You performed the procedure incorrectly. Rewrite the problem as multiplication, using the reciprocal of the divisor. Then multiply and simplify. $\frac{6}{7} \div \frac{3}{10} = \frac{6}{7}\left(\frac{10}{3}\right) = \frac{60}{7(3)} = \frac{20(3)}{7(3)} = \frac{20}{7}$.
D. Correct. Rewrite the problem as multiplication, using the reciprocal of the divisor. Then multiply and simplify. $\frac{6}{7} \div \frac{3}{10} = \frac{6}{7}\left(\frac{10}{3}\right) = \frac{60}{7(3)} = \frac{20(3)}{7(3)} = \frac{20}{7}$.

Now let's see what this means when one or more of the numbers is negative. A number and its reciprocal have the same sign. Since division is rewritten as multiplication using the reciprocal of the divisor, and taking the reciprocal doesn’t change any of the signs, division follows the same rules as multiplication.

Rules of Division

When dividing, rewrite the problem as multiplication using the reciprocal of the divisor as the second factor.

When one number is **positive** and the other is **negative,** the **quotient** is **negative**.

When **both** numbers are **negative,** the quotient is **positive**.

When **both** numbers are **positive,** the quotient is **positive**.

Example

Find $24 \div \left(-\frac{5}{6}\right)$.

Solution

$24 \div \left(-\frac{5}{6}\right) = 24\left(-\frac{6}{5}\right)$	Rewrite the division as multiplication by the reciprocal.
$\frac{24}{1}\left(-\frac{6}{5}\right) = -\frac{144}{5}$	Multiply. Since one number is positive and one is negative, the product is negative.

$24 \div \left(-\frac{5}{6}\right) = -\frac{144}{5}$

✓ Example

Find $4x \div (-6)$ **when** $x = -\frac{2}{3}$**.**

Solution

$4\left(-\frac{2}{3}\right) \div (-6)$	Substitute $-\frac{2}{3}$ for x in the expression.
$\frac{4}{1}\left(-\frac{2}{3}\right)\left(-\frac{1}{6}\right)$	Rewrite the division as multiplication by the reciprocal.
$\frac{4(2)(1)}{3(6)} = \frac{8}{18}$	Multiply. There is an even number of negative numbers, so the product is positive.
$4x \div (-6) = \frac{4}{9}$ when $x = -\frac{2}{3}$	Write the fraction in lowest terms.

Remember that a fraction bar also indicates division. So, a negative sign in front of a fraction goes with the numerator, the denominator, or the whole fraction: $-\frac{3}{4} = \frac{-3}{4} = \frac{3}{-4}$. In each case, the overall fraction is negative because there's only one negative in the division.

Applications of Multiplication and Division

Situations that require multiplication or division may use negative numbers and rational numbers.

✓ Example

Carl didn't know his bank account was at exactly 0 when he wrote a series of $100 checks. As each check went through, $125 was charged against his account. (In addition to the $100 for the check, there was a $25 overdraft charge.) After 1 check, his account was -$125 dollars. After 6 of these checks, what was his account balance?

Solution

$-\$125(6)$	Each check reduces the account by $125; this is represented by -$125. To find the amount it reduces for multiple checks, multiply the number of checks by the amount charged.
$-\$125(6) = -\$750$	Multiply. Since there is one negative number, the product is negative.

Carl’s account balance is now -$750.

✓ Example

Brenda thought she was taking 3 chocolate bars to a picnic with 5 friends. When she brought out the chocolate, she discovered her brother had eaten half of one bar, so she only had $2\frac{1}{2}$ bars to divide among the 6 people (herself and her 5 friends). If each person gets the same amount, how much of a full bar does each person get?

Solution

$2\frac{1}{2} \div 6$	Since the candy is being shared among 6 people, divide the amount of chocolate by 6.
$2\frac{1}{2}\left(\frac{1}{6}\right)$	Rewrite the problem as multiplication, using the reciprocal of the divisor.
$\frac{5}{2}\left(\frac{1}{6}\right)$	Change the mixed number to an improper fraction. Multiply.

Each person gets $\frac{5}{12}$ of a full bar.

✓ Example

During a storm, the temperature dropped by $\frac{1}{2}$ degree every minute. At the beginning of the storm, the temperature was 83ºF. An expression giving the temperature t minutes after the storm began is $-\frac{1}{2}t+83$. What was the temperature after 8 minutes?

Solution

$-\frac{1}{2}(8)+83$	Substitute 8 for t in the expression.
$-\frac{1}{2}\left(\frac{8}{1}\right)+83=$ $-\frac{8}{2}+83=-4+83$	First, multiply. You may find it helpful to rewrite 8 as $\frac{8}{1}$ first. Notice that since you are multiplying one negative and one positive number, the product is negative. Finally, add. Use the rules for adding numbers with different signs.

The temperature was 79ºF after 8 minutes.

? Exercise

Over the course of an 18-year research project, the height of an oceanside cliff actually dropped due to soil erosion. At the end of this period, its height was measured as -3 inches compared to what it had been at the beginning of the research project. What was the average amount that the cliff’s height changed each year?

A. -6 inches
B. $-\frac{1}{6}$ inches
C. $\frac{1}{6}$ inches
D. 6 inches

Answer

A. Incorrect. You divided the number of years by the change in height. The correct answer is $\frac{-3}{18}=\frac{-1}{6}=-\frac{1}{6}$ or $-\frac{1}{6}$ in.
B. Correct. $\frac{-3}{18}=\frac{-1}{6}=-\frac{1}{6}$ or $-\frac{1}{6}$ in.
C. Incorrect. You divided the numerical part correctly, but since you are dividing a negative height by a positive number of years, the answer is negative. The correct answer is $\frac{-3}{18}=\frac{-1}{6}=-\frac{1}{6}$ or $-\frac{1}{6}$ in.
D. Incorrect. You divided the number of years by the change in height and used the wrong sign. The correct answer is $\frac{-3}{18}=\frac{-1}{6}=-\frac{1}{6}$ or $-\frac{1}{6}$ in.

Summary

With multiplication and division, you can find the sign of the final answer by counting how many negative numbers are used in the product or quotient. If there are an even number of negatives, the result is positive. If there are an odd number of negatives, the result is negative. Division can be rewritten as multiplication by using the reciprocal or multiplicative inverse of the divisor.

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SECTION OVERVIEW

9.3: Properties of Real Numbers

Topic hierarchy

9.3.1: Associative, Commutative, and Distributive Properties

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9.3.1: Associative, Commutative, and Distributive Properties

Learning Objectives

- Identify and use the commutative properties for addition and multiplication.
- Identify and use the associative properties for addition and multiplication.
- Identify and use the distributive property.

Introduction

There are many times in algebra when you need to simplify an expression. The properties of real numbers provide tools to help you take a complicated expression and simplify it.

The associative, commutative, and distributive properties of algebra are the properties most often used to simplify algebraic expressions. You will want to have a good understanding of these properties to make the problems in algebra easier to solve.

The Commutative Properties of Addition and Multiplication

You may encounter daily routines in which the order of tasks can be switched without changing the outcome. For example, think of pouring a cup of coffee in the morning. You would end up with the same tasty cup of coffee whether you added the ingredients in either of the following ways:

- Pour 12 ounces of coffee into mug, then add splash of milk.
- Add a splash of milk to mug, then add 12 ounces of coffee.

The order that you add ingredients does not matter. In the same way, it does not matter whether you put on your left shoe or right shoe first before heading out to work. As long as you are wearing both shoes when you leave your house, you are on the right track!

In mathematics, we say that these situations are commutative—the outcome will be the same (the coffee is prepared to your liking; you leave the house with both shoes on) no matter the order in which the tasks are done.

Likewise, the **commutative property of addition** states that when two numbers are being added, their order can be changed without affecting the sum. For example, $30+25$ has the same sum as $25+30$.

$30+25=55$

$25+30=55$

Multiplication behaves in a similar way. The **commutative property of multiplication** states that when two numbers are being multiplied, their order can be changed without affecting the product. For example, $7\cdot 12$ has the same product as $12\cdot 7$.

$7\cdot 12=84$

$12\cdot 7=84$

These properties apply to all real numbers. Let’s take a look at a few addition examples.

Original Equation	**Rewritten Equation**
$1.2+3.8=5$	$3.8+1.2=5$
$\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$	$\frac{1}{8}+\frac{1}{2}=\frac{5}{8}$
$14+(-10)=4$	$(-10)+14=4$
$\frac{1}{3}+\left(-1\frac{2}{3}\right)=-1\frac{1}{3}$	$\left(-1\frac{2}{3}\right)+\frac{1}{3}=-1\frac{1}{3}$
$(-5.2)+(-3.6)=-8.8$	$(-3.6)+(-5.2)=-8.8$

Commutative Property of Addition

For any real numbers a and b, $a+b=b+a$.

Subtraction is not commutative. For example, $4-7$ does not have the same difference as $7-4$. The $-$ sign here means subtraction.

However, recall that $4-7$ can be rewritten as $4+(-7)$, since subtracting a number is the same as adding its opposite. Applying the commutative property for addition here, you can say that $4+(-7)$ is the same as $(-7)+4$. Notice how this expression is very different than $7-4$.

Now look at some multiplication examples.

Original Equation	**Rewritten Equation**
$4.5 \cdot 2=9$	$2 \cdot 4.5=9$
$(-5) \cdot 3=-15$	$3 \cdot(-5)=-15$
$\frac{1}{5} \cdot 5=1$	$5 \cdot \frac{1}{5}=1$
$\left(-\frac{1}{4}\right) \cdot\left(-\frac{8}{10}\right)=\frac{1}{5}$	$\left(-\frac{8}{10}\right) \cdot\left(-\frac{1}{4}\right)=\frac{1}{5}$

Commutative Property of Multiplication

For any real numbers a and b, $a \cdot b=b \cdot a$.

Order does not matter as long as the two quantities are being multiplied together. This property works for real numbers and for variables that represent real numbers.

Just as subtraction is not commutative, neither is division commutative. $4 \div 2$ does not have the same quotient as $2 \div 4$.

✓ Example

Write the expression $(-15.5)+35.5$ in a different way, using the commutative property of addition, and show that both expressions result in the same answer.

Solution

$(-15.5)+35.5=20$	Adding.
$35.5+(-15.5)$	Using the commutative property, you can switch the -15.5 and the 35.5 so that they are in a different order.
$35.5+(-15.5)$ $35.5-15.5=20$	Adding 35.5 and -15.5 is the same as subtracting 15.5 from 35.5. The sum is 20.

$(-15.5)+35.5=20$ and $35.5+(-15.5)=20$

? Exercise

Rewrite $52 \cdot y$ in a different way, using the commutative property of multiplication. Note that y represents a real number.

A. $5 y \cdot 2$
B. $52 y$
C. $26 \cdot 2 \cdot y$
D. $y \cdot 52$

Answer

A. Incorrect. You cannot switch one digit from 52 and attach it to the variable y. The correct answer is $y \cdot 52$.

B. Incorrect. This is another way to rewrite $52 \cdot y$, but the commutative property has not been used. The correct answer is $y \cdot 52$.
C. Incorrect. You do not need to factor 52 into $26 \cdot 2$. The correct answer is $y \cdot 52$.
D. Correct. The order of factors is reversed.

The Associative Properties of Addition and Multiplication

The **associative property of addition** states that numbers in an addition expression can be grouped in different ways without changing the sum. You can remember the meaning of the associative property by remembering that when you *associate* with family members, friends, and co-workers, you end up forming groups with them.

Below are two ways of simplifying the same addition problem. In the first example, 4 is grouped with 5, and $4+5=9$.

$4+5+6=9+6=15$

Here, the same problem is worked by grouping 5 and 6 first, $5+6=11$.

$4+5+6=4+11=15$

In both cases, the sum is the same. This illustrates that changing the grouping of numbers when adding yields the same sum.

Mathematicians often use parentheses to indicate which operation should be done first in an algebraic equation. The addition problems from above are rewritten here, this time using parentheses to indicate the associative grouping.

$(4+5)+6=9+6=15$

$4+(5+6)=4+11=15$

It is clear that the parentheses do not affect the sum; the sum is the same regardless of where the parentheses are placed.

Associative Property of Addition

For any real numbers a, b, and c,

$(a+b)+c=a+(b+c)$.

The example below shows how the associative property can be used to simplify expressions with real numbers.

✓ Example

Rewrite $7+2+8.5-3.5$ in two different ways using the associative property of addition. Show that the expressions yield the same answer.

Solution

$7+2+8.5-3.5$ $7+2+8.5+(-3.5)$	The associative property does not apply to expressions involving subtraction. So, re-write the expression as addition of a negative number.
$(7+2)+8.5+(-3.5)$ $9+8.5+(-3.5)$ $17.5+(-3.5)$ $17.5-3.5=14$	Group 7 and 2, and add them together. Then, add 8.5 to that sum. Finally, add -3.5, which is the same as subtracting 3.5. Subtract 3.5. The sum is 14.
$7+2+(8.5+(-3.5))$ $7+2+5$ $9+5$	Group 8.5 and -3.5, and add them together to get 5. Then add 7 and 2, and add that sum to the 5. The sum is 14.

$(7+2)+8.5-3.5=14$ and $7+2+(8.5+(-3.5))=14$

Multiplication has an associative property that works exactly the same as the one for addition. The **associative property of multiplication** states that numbers in a multiplication expression can be regrouped using parentheses. For example, the expression below can be rewritten in two different ways using the associative property.

Original expression: $-\frac{5}{2}\cdot 6\cdot 4$

Expression 1: $(-\frac{5}{2}\cdot 6)\cdot 4=(-\frac{30}{2})\cdot 4=-15\cdot 4=-60$

Expression 2: $-\frac{5}{2}\cdot(6\cdot 4)=-\frac{5}{2}\cdot 24=-\frac{120}{2}=-60$

The parentheses do not affect the product. The product is the same regardless of where the parentheses are.

Associative Property of Multiplication

For any real numbers a, b, and c, $(a\cdot b)\cdot c=a\cdot(b\cdot c)$.

Exercise

Rewrite $\frac{1}{2}\cdot(\frac{5}{6}\cdot 6)$ using only the associative property.

A. $(\frac{1}{2}\cdot\frac{5}{6})\cdot 6$
B. $(\frac{5}{6}\cdot 6)\cdot\frac{1}{2}$
C. $6\cdot(\frac{5}{6}\cdot\frac{1}{2})$
D. $\frac{1}{2}\cdot 5$

Answer

A. Correct. Here, the numbers are regrouped. Now $\frac{1}{2}$ and $\frac{5}{6}$ are grouped in parentheses instead of $\frac{5}{6}$ and 6.
B. Incorrect. The order of numbers is not changed when you are rewriting the expression using the associative property of multiplication. How they are *grouped* should change. The correct answer is $(\frac{1}{2}\cdot\frac{5}{6})\cdot 6$.
C. Incorrect. The order of numbers is not changed when you are rewriting the expression using the associative property of multiplication. **Only** how they are *grouped* should change. The correct answer is $(\frac{1}{2}\cdot\frac{5}{6})\cdot 6$.
D. Incorrect. Multiplying within the parentheses is not an application of the property. The correct answer is $(\frac{1}{2}\cdot\frac{5}{6})\cdot 6$.

Using the Associative and Commutative Properties

You will find that the associative and commutative properties are helpful tools in algebra, especially when you evaluate expressions. Using the commutative and associative properties, you can reorder terms in an expression so that compatible numbers are next to each other and grouped together. Compatible numbers are numbers that are easy for you to compute, such as $5+5$, or $3\cdot 10$, or $12-2$, or $100\div 20$. (The main criteria for compatible numbers is that they “work well” together.) The two examples below show how this is done.

Example

Evaluate the expression $4\cdot(x\cdot 27)$ when $x=-\frac{3}{4}$.

Solution

$4\cdot(x\cdot 27)$	Original expression.
$4\cdot((-\frac{3}{4})\cdot 27)$	Substitute $-\frac{3}{4}$ for x.
$(4\cdot(-\frac{3}{4}))\cdot 27$ $(-\frac{12}{4})\cdot 27$	Use the associative property of multiplication to regroup the factors so that 4 and $-\frac{3}{4}$ are next to each other. Multiplying 4 by $-\frac{3}{4}$ first makes the expression a bit easier to evaluate than multiplying $-\frac{3}{4}$ by 27.
$-3\cdot 27=-81$	Multiply. 4 times $-\frac{3}{4}=-3$, and -3 times 27 is -81.

$4\cdot(x\cdot 27)=-81$ when $x=(-\frac{3}{4})$

Example

Simplify: $4+12+3+4-8$

Solution

$4+12+3+4-8$	Original expression.
$12+3+4+4+(-8)$	Identify compatible numbers. $4+4$ is 8, and there is a $-8.$ present. Recall that you can think of -8 as $+(-8)$. Use the commutative property of addition to group them together.
$12+3+(4+4+(-8))$	Use the associative property to group $4+4+(-8)$.
$12+3+0$	Add $4+4+(-8)$.
$12+3+0=15$	Add the rest of the terms.

$4+12+3+4-8=15$

Exercise

Simplify the expression: $-5+25-15+2+8$

A. 5
B. 15
C. 30
D. 55

Answer

A. Incorrect. When you use the commutative property to rearrange the addends, make sure that negative addends carry their negative signs. The correct answer is 15.
B. Correct. Use the commutative property to rearrange the expression so that compatible numbers are next to each other, and then use the associative property to group them.
C. Incorrect. Check your addition and subtraction, and think about the order in which you are adding these numbers. Use the commutative property to rearrange the addends so that compatible numbers are next to each other. The correct answer is 15.
D. Incorrect. It looks like you ignored the negative signs here. When you use the commutative property to rearrange the addends, make sure that negative addends carry their negative signs. The correct answer is 15.

The Distributive Property

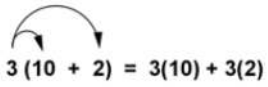
The **distributive property of multiplication** is a very useful property that lets you rewrite expressions in which you are multiplying a number by a sum or difference. The property states that the product of a sum or difference, such as $6(5-2)$, is equal to the sum or difference of products, in this case, $6(5)-6(2)$.

$$6(5-2)=6(3)=18$$
$$6(5)-6(2)=30-12=18$$

The distributive property of multiplication can be used when you multiply a number by a sum. For example, suppose you want to multiply 3 by the sum of $10+2$.

$$3(10+2)=?$$

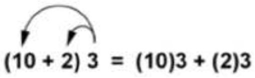
According to this property, you can add the numbers 10 and 2 first and then multiply by 3, as shown here: $3(10+2)=3(12)=36$. Alternatively, you can first multiply *each* addend by the 3 (this is called **distributing** the 3), and then you can add the products. This process is shown here.



$$3(10+2)=3(12)=36$$
$$3(10)+3(2)=30+6=36$$

The products are the same.

Since multiplication is commutative, you can use the distributive property regardless of the order of the factors.



The Distributive Properties

For any real numbers a, b, and c:

Multiplication distributes over addition:

$a(b+c)=ab+ac$

Multiplication distributes over subtraction:

$a(b-c)=ab-ac$

Exercise

Rewrite the expression $10(9-6)$ using the distributive property.

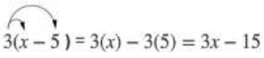
A. $10(6)-10(9)$
B. $10(3)$
C. $10(6-9)$
D. $10(9)-10(6)$

Answer

A. Incorrect. Since subtraction isn't commutative, you can't change the order. The correct answer is $10(9)-10(6)$.
B. Incorrect. This is a correct way to find the answer. But the question asked you to rewrite the problem using the distributive property. The correct answer is $10(9)-10(6)$
C. Incorrect. You changed the order of the 6 and the 9. Note that subtraction is not commutative and you did not use the distributive property. The correct answer is $10(9)-10(6)$.
D. Correct. The 10 is correctly distributed so that it is used to multiply the 9 and the 6 separately.

Distributing with Variables

As long as variables represent real numbers, the distributive property can be used with variables. The distributive property is important in algebra, and you will often see expressions like this: $3(x-5)$. If you are asked to expand this expression, you can apply the distributive property just as you would if you were working with integers.



Remember, when you multiply a number and a variable, you can just write them side by side to express the multiplied quantity. So, the expression "three times the variable x" can be written in a number of ways: $3x$, $3(x)$, or $3\cdot x$.

Example

Use the distributive property to expand the expression $9(4+x)$**.**

Solution

$9(4+x)$	Original expression.

$9(4)+9(x)$	Distribute the 9 and multiply.
$36+9x$	Multiply.

$9(4+x)=36+9x$

✓ Example

Use the distributive property to evaluate the expression $5(2x-3)$ **when** $x=2$**.**

Solution

$5(2x-3)$	Original expression.
$5(2x)-5(3)$	Distribute the 5.
$10x-15$	Multiply.
$10(2)-15$ $20-15=5$	Substitute 2 for x, and evaluate.

When $x=2, 5(2x-3)=5$.

In the example above, what do you think would happen if you substituted $x=2$ before distributing the 5? Would you get the same answer of 5? The example below shows what would happen.

✓ Example

Use the distributive property to evaluate the expression $5(2x-3)$ **when** $x=2$**.**

Solution

$5(2x-3)$	Original expression.
$5(2(2)-3)$	Substitute 2 for x.
$5(4-3)$ $5(4)-5(3)$	Multiply.
$20-15=5$	Subtract and evaluate.

When $x=2, 5(2x-3)=5$.

Combining Like Terms

The distributive property can also help you understand a fundamental idea in algebra: that quantities such as $3x$ and $12x$ can be added and subtracted in the same way as the numbers 3 and 12. Let's look at one example and see how it can be done.

✓ Example

Add: $3x+12x$

Solution

$3(x)+12(x)$	$3x$ is 3 times x, and $12x$ is 12 times x
$x(3+12)$	From studying the distributive property (and also using the commutative property), you know that $x(3+12)$ is the same as $3(x)+12(x)$.
$x(15)$ or $15x$	Combine the terms within the parentheses: $3+12=15$.

$3x + 12x = 15x$

Do you see what happened? By thinking of the x as a distributed quantity, you can see that $3x + 12x = 15x$. (If you're not sure about this, try substituting any number for in this expression…you will find that it holds true!)

Groups of terms that consist of a coefficient multiplied by the same variable are called "like terms". The table below shows some different groups of like terms:

Groups of Like Terms
$3x, 7x, -8x, -0.5x$
$-1.1y, -4y, -8y$
$12t, 25t, 100t, 1t$
$4ab, -8ab, 2ab$

Whenever you see like terms in an algebraic expression or equation, you can add or subtract them just like you would add or subtract real numbers. So, for example,

$10y + 12y = 22y$, and $8x - 3x - 2x = 3x$.

Be careful not to combine terms that do not have the same variable: $4x + 2y$ is not $6xy$!

✓ Example

Simplify: $10y + 5y + 9x - 6x - x$.

Solution

$10y + 5y + 9x - 6x - x$	There are like terms in this expression, since they all consist of a coefficient multiplied by the variable x or y. Note that $-x$ is the same as $(-1)x$.
$15y + 2x$	Add like terms. $10y + 5y = 15y$, and $9x - 6x - x = 2x$.

$10y + 5y + 9x - 6x - x = 15y + 2x$

? Exercise

Simplify: $12x - x + 2x - 8x$.

A. $23x$
B. 5
C. $5x$
D. x

Answer

A. Incorrect. It looks like you added all of the terms. Notice that $-x$ and $-8x$ are negative. The correct answer is $5x$.
B. Incorrect. You combined the integers correctly, but remember to include the variable too! The correct answer is $5x$.
C. Correct. When you combine these like terms, you end up with a sum of $5x$
D. Incorrect. It looks like you subtracted all of the terms from $12x$. Notice that $-x$ and $-8x$ are negative, but that $2x$ is positive. The correct answer is $5x$.

Summary

The commutative, associative, and distributive properties help you rewrite a complicated algebraic expression into one that is easier to deal with. When you rewrite an expression by a commutative property, you change the order of the numbers being added or multiplied. When you rewrite an expression using an associative property, you group a different pair of numbers together using parentheses. You can use the commutative and associative properties to regroup and reorder any number in an expression as long as the expression is made up entirely of addends or factors (and not a combination of them). The distributive property can be used to rewrite expressions for a variety of purposes. When you are multiplying a number by a sum, you can add and then multiply. You can also multiply each addend first and then add the products together. The same principle applies if you are multiplying a number by a difference.

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SECTION OVERVIEW

9.4: Simplifying Expressions

Topic hierarchy

9.4.1: Order of Operations

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9.4.1: Order of Operations

Learning Objectives

- Use the order of operations to simplify expressions.
- Simplify expressions containing absolute values.

Introduction

People need a common set of rules for performing basic calculations. What does $3+5\cdot 2$ equal? Is it 16 or 13? Your answer depends on how you understand the **order of operations** — a set of rules that tell you the order in which addition, subtraction, multiplication, and division are performed in any calculation.

Mathematicians have developed a standard order of operations that tells you which calculations to make first in an expression with more than one operation. Without a standard procedure for making calculations, two people could get two different answers to the same problem.

The Four Basic Operations

The building blocks of the order of operations are the **arithmetic operations**: addition, subtraction, multiplication, and division. The order of operations states:

- multiply or divide first, going from left to right
- then add or subtract in order from left to right

What is the correct answer for the expression $3+5\cdot 2$? Use the order of operations listed above.

Multiply first. $3+5\cdot 2=3+10$

Then add. $3+10=13$

This order of operations is true for all real numbers.

Example

Simplify $7-5+3\cdot 8$**.**

Solution

$7-5+3\cdot 8$	According to the order of operations, multiplication comes before addition and subtraction. Multiply $3\cdot 8$.
$7-5+24$	Now, add and subtract from left to right. $7-5$ comes first.
$2+24=26$	Finally, add $2+24$.

$7-5+3\cdot 8=26$

When you are applying the order of operations to expressions that contain fractions, decimals, and negative numbers, you will need to recall how to do these computations as well.

Example

Simplify: $3\cdot\frac{1}{3}-8\div\frac{1}{4}$

Solution

$3\cdot\frac{1}{3}-8\div\frac{1}{4}$	According to the order of operations, multiplication comes before addition and subtraction. Multiply $3\cdot\frac{1}{3}$ first.
$1-8\div\frac{1}{4}$	Now, divide $8\div\frac{1}{4}$.
$8\div\frac{1}{4}=\frac{8}{1}\cdot\frac{4}{1}=32$	

$1-32=-31$	Subtract.

$3\cdot\frac{1}{3}-8\div\frac{1}{4}=-31$

Exponents

When you are evaluating expressions, you will sometimes see exponents used to represent repeated multiplication. Recall that an expression such as 7^2 is **exponential notation** for $7\cdot 7$. (Exponential notation has two parts: the **base** and the **exponent** or the **power**. In 7^2, 7 is the base and 2 is the exponent; the exponent determines how many times the base is multiplied by itself.)

Exponents are a way to represent repeated multiplication; the order of operations places it *before* any other multiplication, division, subtraction, and addition is performed.

✓ Example

Simplify: $3^2\cdot 2^3$

Solution

$3^2\cdot 2^3$	This problem has exponents and multiplication in it. According to the order of operations, simplifying 3^2 and 2^3 comes before multiplication.
$9\cdot 2^3$	3^2 is $3\cdot 3$, which equals 9.
$9\cdot 8$	2^3 is $2\cdot 2\cdot 2$, which equals 8.
$9\cdot 8=72$	Multiply.

$3^2\cdot 2^3=72$

✓ Example

Simplify: $\left(\frac{1}{2}\right)^2+\left(\frac{1}{4}\right)^3\cdot 32$

Solution

$\left(\frac{1}{2}\right)^2+\left(\frac{1}{4}\right)^3\cdot 32$	This problem has exponents, multiplication, and addition in it. According to the order of operations, simplify the terms with the exponents first, then multiply, then add.
$\frac{1}{4}+\left(\frac{1}{4}\right)^3\cdot 32$	Evaluate: $\left(\frac{1}{2}\right)^2=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$
$\frac{1}{4}+\frac{1}{64}\cdot 32$	Evaluate: $\left(\frac{1}{4}\right)^3=\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{64}$
$\frac{1}{4}+\frac{32}{64}$	Multiply.
$\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$	Simplify. $\frac{32\div 32}{64\div 32}=\frac{1}{2}$, so you can add $\frac{1}{4}+\frac{1}{2}$.

$\left(\frac{1}{2}\right)^2+\left(\frac{1}{4}\right)^3\cdot 32=\frac{3}{4}$

? Exercise

Simplify: $100-5^2\cdot 4$

A. -300
B. 0
C. 100
D. 300

Answer

A. Incorrect. You may have found $4 \cdot 5 = 20$, squared 20, and then subtracted 400 from 100. The order of operations states that you should simplify the term with the exponent first, then multiply, then subtract. $5^2 = 25$, and $25 \cdot 4 = 100$, and $100 - 100 = 0$. The correct answer is 0.
B. Correct. To simplify this expression, simplify the term with the exponent first, then multiply, then subtract. $5^2 = 25$, and $25 \cdot 4 = 100$, and $100 - 100 = 0$.
C. Incorrect. The order of operations states that you should simplify the term with the exponent first, then multiply, then subtract. $5^2 = 25$, and $25 \cdot 4 = 100$, and $100 - 100 = 0$. The correct answer is 0.
D. Incorrect. You may have found that $5^2 = 25$, subtracted that from 100, and multiplied by 4. The order of operations states that you should simplify the term with the exponent first, then multiply, then subtract. $5^2 = 25$, and $25 \cdot 4 = 100$, and $100 - 100 = 0$. The correct answer is 0.

Grouping Symbols

The final piece that you need to consider in the order of operations is **grouping symbols**. These include parentheses $(\quad)$, brackets $[\quad]$, braces $\{\quad\}$, and even fraction bars. These symbols are often used to help organize mathematical expressions (you will see them a lot in algebra).

Grouping symbols are used to clarify which operations to do first, especially if a specific order is desired. If there is an expression to be simplified within the grouping symbols, follow the order of operations.

The Order of Operations

- Perform all operations within grouping symbols first. Grouping symbols include parentheses $(\quad)$, brackets $[\quad]$, braces $\{\quad\}$, and fraction bars.
- Evaluate exponents or square roots.
- Multiply or divide, from left to right.
- Add or subtract, from left to right.

When there are grouping symbols within grouping symbols, calculate from the inside to the outside. That is, begin simplifying within the innermost grouping symbols first.

Remember that parentheses can also be used to show multiplication. In the example that follows, both uses of parentheses—as a way to represent a group, as well as a way to express multiplication—are shown.

Example

Simplify: $(3+4)^2 + (8)(4)$

Solution

$(3+4)^2 + (8)(4)$	This problem has parentheses, exponents, multiplication, and addition in it. The first set of parentheses is a grouping symbol. The second set indicates multiplication.
$(3+4)^2 + (8)(4)$	Grouping symbols are handled first. Add numbers in parentheses.
$7^2 + (8)(4)$	Simplify 7^2.
$49 + \mathbf{(8)(4)}$	Perform multiplication.
$49 + 32 = 81$	Perform addition.

$(3+4)^2 + (8)(4) = 81$

✓ Example

Simplify: $(1.5+3.5)-2(0.5\cdot 6)^2$

Solution

$(1.5+3.5)-2(0.5\cdot 6)^2$	This problem has parentheses, exponents, multiplication, subtraction, and addition in it.
$5-2(0.5\cdot 6)^2$	Grouping symbols are handled first. Add numbers in the first set of parentheses. Multiply numbers in the second set of parentheses.
$5-2(3)^2$	Evaluate exponents.
$5-2\cdot 9$	Multiply.
$5-18=-13$	Subtract.

$(1.5+3.5)-2(0.5\cdot 6)^2=-13$

✓ Example

Simplify: $\frac{5-[3+(2\cdot(-6))]}{3^2+2}$

Solution

$\frac{5-[3+(2\cdot(-6))]}{3^2+2}$	This problem has brackets, parentheses, fractions, exponents, multiplication, subtraction, and addition in it. Grouping symbols are handled first. The parentheses around the -6 aren't a grouping symbol, they are simply making it clear that the negative sign belongs to the 6. Start with the innermost set of parentheses that are a grouping symbol, here it is in the numerator of the fraction, ($2\cdot -6$), and begin working out. (The fraction line acts as a type of grouping symbol, too; you simplify the numerator and denominator independently, and then divide the numerator by the denominator at the end.)
$\frac{5-[3+(-12)]}{3^2+2}$	Add the values in the brackets.
$\frac{5-[-9]}{3^2+2}$	Subtract $5-[-9]=5+9=14$.
$\frac{14}{3^2+2}$	The top of the fraction is all set, but the bottom (denominator) has remained untouched. Apply the order of operations to that as well. Begin by evaluating $3^2=9$.
$\frac{14}{9+2}$ $\frac{14}{11}$	Now add. $9+2=11$.

$\frac{5-[3+(2\cdot(-6))]}{3^2+2}=\frac{14}{11}$

? Exercise

Simplify $\left[\frac{3^3+3}{(-2)(-3)}\right]^2+1$.

A. 25
B. 26
C. 151
D. $\frac{11}{6}$

Answer

A. Incorrect. You may have forgotten to add the 1! Simplify the expressions within the grouping symbols first (5), then square that expression (25), then add 1. The correct answer is 26.
B. Correct. The entire quantity within the brackets is $5\cdot 5^2$ is 25, and $25+1=26$.
C. Incorrect. You may have squared the numerator of the fraction without simplifying the entire fraction first! Simplify the entire expression within the grouping symbols first (5), then square that expression (25), then add 1. The correct answer is 26.
D. Incorrect. You may have squared the denominator of the fraction without simplifying the entire fraction first! Simplify the entire expression within the grouping symbols first (5), then square that expression (25), then add 1. The correct answer is 26.

Remembering the Order of Operations

The Order of Operations

- Perform all operations within grouping symbols first. Grouping symbols include parentheses $(\quad)$, brackets $[\quad]$, braces $\{\quad\}$, and fraction bars.
- Evaluate exponents or square roots.
- Multiply or divide, from left to right.
- Add or subtract, from left to right.

The order of operations is important to know, but it is sometimes hard to remember. Some people use a saying to help them remember the order of operations. This saying is “Please Excuse My Dear Aunt Sally,” or PEMDAS for short. The first letter of each word begins with the same letter of an arithmetic operation.

The P in Please stands for Parentheses (and other grouping symbols).

The E in Excuse stands for Exponents.

The M and D in My Dear stand for Multiplication and Division (from left to right).

The A and S in Aunt Sally stand for Addition and Subtraction (from left to right).

Note: Even though multiplication comes before division in the saying, division could be performed first. Whether multiplication or division is performed first is determined by which comes first when reading from left to right. The same is true of addition and subtraction. Don’t let the saying confuse you about this!

Absolute Value Expressions

Absolute value expressions are one final method of grouping that you may see. Recall that the absolute value of a quantity is always positive or 0.

When you see an absolute value expression included within a larger expression, follow the regular order of operations and evaluate the expression within the absolute value sign. Then take the absolute value of that expression. The example below shows how this is done.

✓ Example

Simplify: $\frac{3+|2-6|}{2|3\cdot 1.5|-(-3)}$

Solution

$\frac{3+\vert 2-6\vert}{2\vert 3\cdot 1.5\vert-(-3)}$	This problem has absolute values, decimals, multiplication, subtraction, and addition in it. Grouping symbols, including absolute value, are handled first. Simplify the numerator, then the denominator. Evaluate $\vert 2-6\vert$.
$\frac{3+\vert -4\vert}{2\vert 3\cdot 1.5\vert-(-3)}$	Take the absolute value of $\vert -4\vert$.
$\frac{3+4}{2\vert 3\cdot 1.5\vert-(-3)}$	Add the numbers in the numerator.

$\frac{7}{2\|3\cdot 1.5\|-(-3)}$	Now that the numerator is simplified, turn to the denominator. Evaluate the absolute value expression first.
$\frac{7}{2\|4.5\|-(-3)}$ $\frac{7}{2\cdot 4.5-(-3)}$	The expression " $2\|4.5\|$" reads "2 times the absolute value of 4.5." Multiply 2 times 4.5.
$\frac{7}{9-(-3)}$ $\frac{7}{12}$	Subtract.

$$\frac{3+|2-6|}{2|3\cdot 1.5|-(-3)}=\frac{7}{12}$$

? Exercise

Simplify: $(5|3-4|)^3$.

A. -125
B. 1331
C. -49
D. 125

Answer

A. Incorrect. You may have forgotten to take the absolute value of the expression $|3-4|$. Remember that the absolute value will be either 0 or a positive number. $|3-4|=|-1|=1$ and 5 times 1 is 5. The correct answer is 125.
B. Incorrect. You may have ignored the absolute value expression and found 11^3. Evaluate $|3-4|$, multiply that by 5, and then cube that number: $|3-4|=|-1|=1$ and 5 times 1 is 5; $5^3=125$. The correct answer is 125.
C. Incorrect. You may have subtracted 4^3 from $5(3)$. Evaluate $|3-4|$, multiply that by 5, and then cube that number: $|3-4|=|-1|=1$ and 5 times 1 is 5; $5^3=125$ The correct answer is 125.
D. Correct. $|3-4|=|-1|=1$ and 5 times 1 is 5. 5 cubed is 125.

Summary

The order of operations gives us a standard, consistent method to use when simplifying strings of numbers and algebraic expressions. Without the order of operations, different people could come up with different answers to the same computation problem. Some people remember the order of operations by using the phrase “Please Excuse My Dear Aunt Sally” or, more simply, PEMDAS.

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CHAPTER OVERVIEW

10: Solving Equations and Inequalities

10.1: Solving Equations
  10.1.1: Solving One-Step Equations Using Properties of Equality
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  10.3.1: Compound Inequalities
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SECTION OVERVIEW

10.1: Solving Equations

Topic hierarchy

10.1.1: Solving One-Step Equations Using Properties of Equality

10.1.2: Solving Multi-Step Equations

10.1.3: Special Cases and Applications

10.1.4: Formulas

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10.1.1: Solving One-Step Equations Using Properties of Equality

Learning Objectives

- Solve algebraic equations using the addition property of equality.
- Solve algebraic equations using the multiplication property of equality.

Introduction

Writing and solving equations is an important part of mathematics. Algebraic equations can help you model situations and solve problems in which quantities are unknown. The simplest type of algebraic equation is a linear equation that has just one variable.

Expressions and Equations

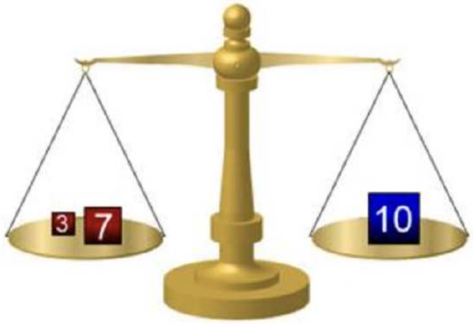
An **equation** is a mathematical statement that two expressions are equal. An equation will always contain an equal sign with an expression on each side. **Expressions** are made up of **terms**, and the number of terms in each expression in an equation may vary.

Algebraic equations contain **variables**, symbols that stand for an unknown quantity. Variables are often represented with letters, like x, y, or z. Sometimes a variable is multiplied by a number. This number is called the **coefficient** of the variable. For example, the coefficient of $3x$ is 3.

Using the Addition Property of Equality

An important property of equations is one that states that you can add the same quantity to both sides of an equation and still maintain an equivalent equation. Sometimes people refer to this as keeping the equation "balanced." If you think of an equation as being like a balance scale, the quantities on each side of the equation are equal, or balanced.

Let's look at a simple numeric equation, $3+7=10$, to explore the idea of an equation as being balanced.



The expressions on each side of the equal sign are equal, so you can add the same value to each side and maintain the equality. Let's see what happens when 5 is added to each side.

$$3+7+5=10+5$$

Since each expression is equal to 15, you can see that adding 5 to each side of the original equation resulted in a true equation. The equation is still "balanced."

On the other hand, let's look at what would happen if you added 5 to only one side of the equation.

$$\begin{aligned} 3+7&=10 \\ 3+7+5&=10 \\ 15&\neq 10 \end{aligned}$$

Adding 5 to only one side of the equation resulted in an equation that is false. The equation is no longer "balanced", and it is no longer a true equation!

Addition Property of Equality

For all real numbers a, b, and c: If $a=b$, then $a+c=b+c$.

If two expressions are equal to each other, and you add the same value to both sides of the equation, the equation will remain equal.

When you solve an equation, you find the value of the variable that makes the equation true. In order to solve the equation, you **isolate the variable**. Isolating the variable means rewriting an equivalent equation in which the variable is on one side of the equation and everything else is on the other side of the equation.

When the equation involves addition or subtraction, use the inverse operation to “undo” the operation in order to isolate the variable. For addition and subtraction, your goal is to change any value being added or subtracted to 0, the additive identity.

Example

Solve $x-6=8$**.**

Solution

$x-6=8$	This equation means that if you begin with some unknown number, x, and subtract 6, you will end up with 8. You are trying to figure out the value of the variable x.
$\begin{array}{rr} x-6 & =\ \ 8 \\ +6 & +6 \\ \hline x+0 & =14 \end{array}$	Using the Addition Property of Equality, add 6 to both sides of the equation to isolate the variable. You choose to add 6, as 6 is being subtracted from the variable. Subtracting 6 from both sides leaves you with $x+0=14$.

$x=14$

Since subtraction can be written as addition (adding the opposite), the **addition property of equality** can be used for subtraction as well. So just as you can add the same value to each side of an equation without changing the meaning of the equation, you can subtract the same value from each side of an equation.

Example

Solve $x+7=42$**.**

Solution

$x+7=42$	Since 7 is being added to the variable, subtract 7 to isolate the variable.
$\begin{array}{rr} x+7= & 42 \\ -7 & -7 \\ \hline x+0= & 35 \end{array}$	To keep the equation balanced, subtract 7 from both sides of the equation. This gives you $x+0=35$.

$x=35$

Advanced Example

Solve $12.5+x=-7.5$**.**

Solution

$12.5+x=-7.5$	Since 12.5 is being added to the variable, subtract 12.5 to isolate the variable.

$\begin{array}{rr} 12.5+x= & -7.5 \\ -12.5 & -12.5 \\ \hline 0+x= & -20 \end{array}$	To keep the equation balanced, subtract 12.5 from both sides of the equation. This gives you $0+x=-20$.

$x=-20$

The examples above are sometimes called **one-step equations** because they require only one step to solve. In these examples, you either added or subtracted a **constant** from both sides of the equation to isolate the variable and solve the equation.

? Exercise

What would you do to isolate the variable in the equation below, using only one step?

$x+10=65$

A. Add 10 to both sides of the equation.
B. Subtract 10 from the left side of the equation only.
C. Add 65 to both sides of the equation.
D. Subtract 10 from both sides of the equation.

Answer

A. Incorrect. Adding 10 to both sides of the equation gives an equivalent equation, $x+20=65+10$, but this step does not get the variable alone on one side of the equation. The correct answer is: Subtract 10 from both sides of the equation.
B. Incorrect. Subtracting 10 from the left side will isolate the variable, but subtracting 10 from only one side of the equation does not keep the equation balanced. According to the properties of equality, you must perform the same exact operation to each side of the equation, so you must also subtract 10 from 65 to keep the equation balanced. The correct answer is: Subtract 10 from both sides of the equation.
C. Incorrect. This step will not isolate the variable. It will only give an equivalent equation. $x+10+65=65+65$. The correct answer is: Subtract 10 from both sides of the equation.
D. Correct. Subtracting 10 from each side of the equation yields an equivalent equation with the variable isolated to give the solution: $x+10-10=65-10$, so $x=55$.

? Advanced Question

What would you do to isolate the variable in the equation below, using only one step? $x-\frac{1}{4}=\frac{7}{2}$

A. Subtract $\frac{1}{4}$ from both sides of the equation.
B. Add $\frac{1}{4}$ to both sides of the equation.
C. Subtract $\frac{7}{2}$ from both sides of the equation.
D. Add $\frac{7}{2}$ to both sides of the equation.

Answer

A. Incorrect. Subtracting \from$bothsidesoftheequationgivestheequation, frac14$ from both sides of the equation gives the equation, $x-\frac{1}{4}-\frac{1}{4}=\frac{7}{2}-\frac{1}{4}$, which is the same as $x-\frac{1}{4}-\frac{1}{4}=\frac{14}{4}-\frac{1}{4}$ or $x-\frac{1}{2}=\frac{13}{4}$. However, this step does not get the variable alone on one side of the equation. The correct answer is: Add $\frac{1}{4}$ to both sides of the equation.
B. Correct. Adding $\frac{1}{4}$ to each side of the equation yields an equivalent equation and isolates the variable: $x-\frac{1}{4}+\frac{1}{4}=\frac{7}{2}+\frac{1}{4}$ and $x=\frac{14}{4}+\frac{1}{4}$, so $x=\frac{15}{4}$.
C. Incorrect. Subtracting $\frac{7}{2}$ from both sides will result in the equivalent expression $x-\frac{1}{4}-\frac{7}{2}=\frac{7}{2}-\frac{7}{2}$, which can be rewritten $x-\frac{1}{4}-\frac{14}{4}=0$, or $x-\frac{15}{4}=0$, but this step does not get the variable alone on one side of the equation. The correct answer is: Add $\frac{1}{4}$ to both sides of the equation.
D. Incorrect. Adding $\frac{7}{2}$ to both sides will result in the equivalent expression $x-\frac{1}{4}+\frac{7}{2}=\frac{7}{2}+\frac{7}{2}$, which can be rewritten $x-\frac{1}{4}+\frac{14}{4}=\frac{14}{2}$, or $x+\frac{13}{4}=7$, but this step does not get the variable alone on one side of the equation. The correct answer is: Add $\frac{1}{4}$ to both sides of the equation.

With any equation, you can check your solution by substituting the value for the variable in the original equation. In other words, you evaluate the original equation using your solution. If you get a true statement, then your solution is correct.

✓ Example

Solve $x+10=-65$. Check your solution.

Solution

	$x+10=-65$	Since 10 is being added to the variable, subtract 10 from both sides.
	$\begin{array}{r} x+10=-65 \\ -10 \qquad -10 \\ \hline x=-75 \end{array}$	Note that subtracting 10 is the same as adding -10. You get $x=-75$.
Check:	$\begin{aligned} x+10&=-65 \\ -75+10&=-65 \\ -65&=-65 \end{aligned}$	To check, substitute the solution, -75 for x in the original equation. Simplify. This equation is true, so the solution is correct.

$x=-75$ is the solution to the equation

$x+10=-65$.

It is always a good idea to check your answer whether it is requested or not.

Using the Multiplication Property of Equality

Just as you can add or subtract the same exact quantity on both sides of an equation, you can also multiply both sides of an equation by the same quantity to write an equivalent equation. Let's look at a numeric equation, $5\cdot 3=15$, to start. If you multiply both sides of this equation by 2, you will still have a true equation.

$$\begin{aligned} 5\cdot 3&=15 \\ 5\cdot 3\cdot 2&=15\cdot 2 \\ 30&=30 \end{aligned}$$

This characteristic of equations is generalized in the **multiplication property of equality**.

Multiplication Property of Equality

For all real numbers a, b, and c: If $a=b$, then $a\cdot c=b\cdot c$ (or $ab=ac$).

If two expressions are equal to each other and you multiply both sides by the same number, the resulting expressions will also be equivalent.

When the equation involves multiplication or division, you can "undo" these operations by using the inverse operation to isolate the variable. When the operation is multiplication or division, your goal is to change the coefficient to 1, the multiplicative identity.

✓ Example

Solve $3x=24$. Check your solution.

Solution

$3x=24$	Divide both sides of the equation by 3 to isolate the variable (have a coefficient of 1).
$\begin{aligned} \frac{3x}{3}&=\frac{24}{3} \\ x&=8 \end{aligned}$	Dividing by 3 is the same as having multiplied by $\frac{1}{3}$.
$\begin{aligned} 3x&=24 \\ 3\cdot 8&=24 \\ 24&=24 \end{aligned}$	Check by substituting your solution, 8, for the variable in the original equation.he solution is correct!

$x=8$

You can also multiply the coefficient by the multiplicative inverse (reciprocal) in order to change the coefficient to 1!

✓ Example

Solve $\frac{1}{2}x=8$. Check your solution.

Solution

$\frac{1}{2}x=8$ $2\left(\frac{1}{2}x\right)=2(8)$	The coefficient of $\frac{1}{2}x$ is $\frac{1}{2}$. Since the multiplicative inverse of $\frac{1}{2}$ is 2, you can multiply both sides of the equation by 2 to get a coefficient of 1 for the variable.
$\frac{2}{2}x=16$ $x=16$	Multiply.
$\frac{1}{2}(16)=8$	Check by substituting your solution into the original equation.
$\frac{16}{2}=8$ $8=8$	The solution is correct!

$x=16$

✓ Example

Solve $\left(-\frac{1}{4}\right)x=2$. Check your solution.

Solution

$\left(-\frac{1}{4}\right)x=2$	The coefficient of the variable is $-\frac{1}{4}$. Multiply both sides by the multiplicative inverse of $-\frac{1}{4}$, which is -4.
$(-4)\left(-\frac{1}{4}\right)x=(-4)2$	Multiply.
$(1)\ x=-8$ $x=-8$	Any number multiplied by its multiplicative inverse is equal to 1, so $x=-8$.
$\left(-\frac{1}{4}\right)(-8)=2$ $\frac{8}{4}=2$ $2=2$	Check by substituting your solution into the original equation. The solution is correct.

$x=-8$

✓ Advanced Example

Solve $-\frac{7}{2}=\frac{x}{10}$. Check your solution.

Solution

	$10\left(-\frac{7}{2}\right)=10\left(\frac{x}{10}\right)$ $-\frac{70}{2}=\frac{10x}{10}$ $-\frac{70}{2}=x$ $-35=x$	This problem contains two fractions. Multiply both sides by 10 in order to isolate the variable x. Then simplify the fractions.

Check	$-\frac{7}{2}=\frac{x}{10}$ $-\frac{7}{2}=\frac{-35}{10}$ $-\frac{7}{2}\cdot\frac{5}{5}=\frac{-35}{10}$ $-\frac{35}{10}=\frac{-35}{10}$	Check your answer by substituting -35 in for x. The solution is correct.

$x=-35$

? Exercise

Solve for x: $5x=-100$

A. $x=20$
B. $x=-20$
C. $x=500$
D. $x=-500$

Answer

A. Incorrect. To isolate the variable, you can divide both sides by 5, because any number divided by itself is 1. $\frac{-100}{5}=-20$. The correct answer is -20.
B. Correct. Dividing both sides by 5, you find that $x=-20$. To check this answer, you can substitute -20 in for x in the original equation to get a true statement: $5(-20)=-100$.
C. Incorrect. You probably multiplied both sides by -5. To isolate the variable, remember to divide, not multiply, both sides by 5 because any number divided by itself is 1. The correct answer is -20.
D. Incorrect. You probably multiplied both sides by 5. To isolate the variable, remember to divide, not multiply, both sides by 5 because any number divided by itself is 1. The correct answer is -20.

? Advanced Question

Solve for y: $4.2=7y$

A. $y=0.6$
B. $y=29.4$
C. $y=1.67$
D. $y=-2.8$

Answer

A. Correct. To isolate the variable y on the right side of the equation, you have to divide both sides of the equation by 7. $\frac{4.2}{7}=0.6$, so $y=0.6$.
B. Incorrect. It looks like you multiplied $4.2\cdot 7$ to get an answer of 29.4. Remember that if you multiply the left side by 7, you also have to multiply the right by 7; this does not isolate the variable! The correct answer is $y=0.6$.
C. Incorrect. It looks like you divided 7 by 4.2 to get an answer of 1.67. However, dividing both sides by 4.2 does not isolate the variable! The correct answer is $y=0.6$.
D. Incorrect. It looks like you subtracted 7 from 4.2 to get an answer of -2.8. Remember that the expression $7y$ means “7 times y.” To isolate the variable, you need to use the inverse of multiplication, not subtraction. The correct answer is $y=0.6$.

Summary

Equations are mathematical statements that combine two expressions of equal value. An algebraic equation can be solved by isolating the variable on one side of the equation using the properties of equality. To check the solution of an algebraic equation, substitute the value of the variable into the original equation.

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10.1.2: Solving Multi-Step Equations

Learning Objectives

- Use properties of equality together to isolate variables and solve algebraic equations.
- Use the properties of equality and the distributive property to solve equations containing parentheses, fractions, and/or decimals.

Introduction

There are some **equations** that you can solve in your head quickly. For example, what is the value of y in the equation $2y = 6$? Chances are you didn't need to get out a pencil and paper to calculate that $y = 3$. You only needed to do one thing to get the answer, divide 6 by 2.

Other equations are more complicated. Solving $4\left(\frac{1}{3}t+\frac{1}{2}\right)=6$ without writing anything down is difficult! That's because this equation contains not just a **variable** but also fractions and **terms** inside parentheses. This is a **multi-step equation**, one that takes several steps to solve. Although multi-step equations take more time and more operations, they can still be simplified and solved by applying basic algebraic rules.

Using Properties of Equalities

Remember that you can think of an equation as a balance scale, with the goal being to rewrite the equation so that it is easier to solve but still balanced. The **addition property of equality** and the **multiplication property of equality** explain how you can keep the scale, or the equation, balanced. Whenever you perform an operation to one side of the equation, if you perform the same exact operation to the other side, you'll keep both sides of the equation equal.

If the equation is in the form, $ax+b=c$, where x is the variable, you can solve the equation as before. First "undo" the addition and subtraction, and then "undo" the multiplication and division.

Example

Solve $3y+2=11$.

Solution

$\begin{array}{rr} 3y+2= & 11 \\ -2 & -2 \\ \hline 3y \qquad = & 9 \end{array}$	Subtract 2 from both sides of the equation to get the term with the variable by itself, so $3y=9$.
$\frac{3y}{3}=\frac{9}{3}$ $y=3$	Divide both sides of the equation by 3 to get a coefficient of 1 for the variable.

$y=3$

Example

Solve $\frac{1}{4}x-2=3$.

Solution

$\begin{array}{rr} \frac{1}{4}x-2= & 3 \\ +2 & +2 \\ \hline \frac{1}{4}x \qquad = & 5 \end{array}$	Add 2 to both sides of the equation to get the term with the variable by itself, so $\frac{1}{4}x=5$.
$\frac{4x}{4}=5(4)$ $x=20$	Multiply both sides of the equation by 4 to get a coefficient of 1 for the variable.

$x=20$

If the equation is not in the form, $ax+b=c$, you will need to perform some additional steps to get the equation in that form.

In the example below, there are several sets of **like terms**. You must first combine all like terms.

Example

Solve $3x+5x+4-x+7=88$**.**

Solution

$3x+5x+4-x+7=88$	There are three like terms $3x,\ 5x$ and $-x$ involving a variable.
$7x+4+7=88$ $7x+11=88$	Combine these like terms. 4 and 7 are also like terms and can be added.
$\begin{array}{rr} 7x+11= & 88 \\ -11 & -11 \\ \hline \frac{7x}{7}= & \frac{77}{7} \end{array}$	The equation is now in the form $ax+b=c$. So, we can solve $7x+11=88$ as before. Subtract 11 from both sides. Divide both sides by 7.

$x=11$

Some equations may have the variable on both sides of the equal sign. We need to “move” one of the variable terms in order to solve the equation.

Example

Solve $6x+5=10+5x$**. Check your solution.**

Solution

	$6x+5=10+5x$	This equation has x terms on both the left and the right. To solve an equation like this, you must first get the variables on the same side of the equal sign.
	$\begin{array}{rr} 6x+5= & 10+5x \\ -5x & -5x \\ \hline x+5= & 10 \end{array}$	You can subtract $5x$ on each side of the equal sign, which gives a new equation: $x+5=10$. This is now a one-step equation!
	$\begin{array}{rr} x+5= & 10 \\ -5 & -5 \\ \hline x= & 5 \end{array}$	Subtract 5 from both sides and you get $x=5$.
Check	$6x+5=10+5x$ $6(5)+5=10+5(5)$ $30+5=10+25$ $35=35$	Check your solution by substituting 5 for x in the original equation. This is a true statement, so the solution is correct.

$x=5$

Here are some steps to follow when you solve multi-step equations.

Solving multi-step equations

1. If necessary, simplify the expressions on each side of the equation, including combining like terms.
2. Get all variable terms on one side and all numbers on the other side using the addition property of equality. ($ax+b=c$ or $c=ax+b$)
3. Isolate the variable term using the inverse operation or additive inverse (opposite) using the addition property of equality.
4. Isolate the variable using the inverse operation or multiplicative inverse (reciprocal) using the multiplication property of equality to write the variable with a coefficient of 1.

5. Check your solution by substituting the value of the variable in the original equation.

The examples below illustrate this sequence of steps.

✓ Example

Solve for y.

$-20y+15=2-16y+11$

Solution

	$-20y+15=2-16y+11$ $-20y+15=-16y+13$	Step 1. On the right side, combine like terms: $2+11=13$.
	$\begin{array}{rr} -20y+15= & -16y+13 \\ +20y & +20y \\ \hline 15= & 4y+13 \\ -13 & -13 \\ \hline 2= & 4y \\ \frac{2}{4}= & \frac{4y}{4} \\ \frac{1}{2}= & y \end{array}$	Step 2. Add $20y$ to both sides to remove the variable term from the left side of the equation. Step 3. Subtract 13 from both sides. Step 4. Divide $4y$ by 4 to solve for y.
Check	$-20y+15=2-16y+11$ $-20\left(\frac{1}{2}\right)+15=2-16\left(\frac{1}{2}\right)+11$ $-10+15=2-8+11$ $5=5$	Step 5. To check your answer, substitute $\frac{1}{2}$ for y in the original equation. The statement $5=5$ is true, so $y=\frac{1}{2}$ is the solution.

$y=\frac{1}{2}$

✓ Advanced Example

Solve $3y+10.5=6.5+2.5y$. Check your solution.

Solution

	$3y+10.5=6.5+2.5y$	This equation has y terms on both the left and the right. To solve an equation like this, you must first get the variables on the same side of the equal sign.
	$\begin{array}{rr} 3y+10.5= & 6.5+2.5y \\ -2.5y & -2.5y \\ \hline 0.5y+10.5= & 6.5 \end{array}$	Add $-2.5y$ to both sides so that the variable remains on one side only.
	$\begin{array}{rr} 0.5y+10.5= & 6.5 \\ -10.5 & -10.5 \\ \hline 0.5y\ = & -4 \end{array}$	Now isolate the variable by subtracting 10.5 from both sides.
	$10(0.5y)=10(-4)$ $5y=-40$ $\frac{5y}{5}=\frac{-40}{5}$ $y=-8$	Multiply both sides by 10 so that $0.5y$ becomes $5y$, then divide by 5.
Check	$3y+10.5=6.5+2.5y$ $3(-8)+10.5=6.5+(-20)$ $-24+10.5=6.5+(-20)$ $-13.5=-13.5$	Check your solution by substituting -8 in for y in the original equation.his is a true statement, so the solution is correct.

$y=-8$

Advanced Question

Identify the step that will *not* lead to a correct solution to the problem.

$3a-\frac{11}{2}=-\frac{3a}{2}+\frac{25}{2}$

A. Multiply both sides of the equation by 2.
B. Add $\frac{11}{2}$ to both sides of the equation.
C. Add $\frac{3a}{2}$ to the left side, and add $-3a$ to the right side.
D. Rewrite $3a$ as $\frac{6a}{2}$.

Answer

A. Incorrect. Multiplying both sides by 2 keeps both sides equal; the new equation will be $6a-11=-3a+25$. This will lead to the correct solution. The step that will not lead to a correct solution is: Add $\frac{3a}{2}$ to the left side, and add $-3a$ to the right side.
B. Incorrect. Adding $\frac{11}{2}$ to both sides of the equation will keep both sides equal; the new equation will be $3a=-\frac{3a}{2}+\frac{36}{2}$. This will lead to the correct solution. The step that will not lead to a correct solution is: Add $\frac{3a}{2}$ to the left side, and add $-3a$ to the right side.
C. Correct. Adding unequal amounts to the left and to the right will unbalance the equation, and you will no longer be able to solve accurately for a.
D. Incorrect. Rewriting $3a$ as $\frac{6a}{2}$ does not change the value of the fraction at all, so this will keep both sides equal. The step that will not lead to a correct solution is: Add $\frac{3a}{2}$ to the left side, and add $-3a$ to the right side.

Solving Equations Involving Parentheses, Fractions, and Decimals

More complex multi-step equations may involve additional symbols such as parentheses. The steps above can still be used. If there are parentheses, you use the distributive property of multiplication as part of Step 1 to simplify the expression. Then you solve as before.

The Distributive Property of Multiplication

For all real numbers a, b, and c, $a(b+c)=ab+ac$.

What this means is that when a number multiplies an expression inside parentheses, you can distribute the multiplication to each term of the expression individually. Then, you can follow the routine steps described above to **isolate the variable** to solve the equation.

Example

Solve for a.

$4(2a+3)=-3(a-1)+31$

Solution

$4(2a+3)=-3(a-1)+31$ $8a+12=-3a+3+31$	Apply the distributive property to expand $4(2a+3)$ to $8a+12$ and $-3(a-1)$ to $-3a+3$.
$\begin{array}{rcr} 8a+12 & = & -3a+34 \\ +3a & & +3a \\ \hline 11a+12 & = & +34 \\ -12 & & -12 \\ \hline \frac{11a}{11} & = & \frac{22}{11} \\ a & = & 2 \end{array}$	Combine like terms. Add $3a$ to both sides to move the variable terms to one side. Subtract 12 to isolate the variable term. Divide both terms by 11 to get a coefficient of 1.

$a=2$

✓ Example

In which of the following equations is the distributive property properly applied to the equation $2(y+3)=7$?

A. $y+6=7$
B. $2y+6=14$
C. $2y+6=7$
D. $2y+3=7$

Solution

A. Incorrect. All of the terms inside the parentheses must be multiplied by the value outside. The correct answer is $2y+6=7$.
B. Incorrect. When applying the distributive property, multiplication is spread only to the terms inside the parentheses, not to the other parts of the equation. The correct answer is $2y+6=7$.
C. Correct. Since the distributive property allows us to distribute the multiplication of an entire expression to each of the terms of the expression separately, $2y+6=7$ is correct.
D. Incorrect. All of the terms inside the parentheses must be multiplied by the value outside. The correct answer is $2y+6=7$.

If you prefer not working with fractions, you can use the multiplication property of equality to multiply both sides of the equation by a common denominator of all of the fractions in the equation. See the example below.

✓ Example

Solve $\frac{1}{2}x-3=2-\frac{3}{4}x$ by clearing the fractions in the equation first.

Solution

$\frac{1}{2}x-3=2-\frac{3}{4}x$ $4\left(\frac{1}{2}x-3\right)=4\left(2-\frac{3}{4}x\right)$ $4\left(\frac{1}{2}x\right)-4(3)=4(2)-4\left(\frac{3}{4}x\right)$ $\frac{4}{2}x-12=8-\frac{12}{4}x$	Multiply both sides of the equation by 4, the common denominator of the fractional coefficients.se the distributive property to expand the expressions on both sides. Multiply.
$\begin{array}{rr} 2x-12= & 8-3x \\ +3x & +3x \\ \hline 5x-12= & 8 \\ +12 & +12 \\ \hline \frac{5x}{5} \quad = & \frac{20}{5} \\ x= & 4 \end{array}$	Add $3x$ to both sides to move the variable terms to only one side. Add 12 to both sides to move the **constant** terms to the other side. Divide to isolate the variable.

$x=4$

Of course, if you like to work with fractions, you can just apply your knowledge of operations with fractions and solve.

✓ Example

Solve $\frac{1}{2}x-3=2-\frac{3}{4}x$.

Solution

$\begin{array}{rr} \frac{1}{2}x-3= & 2-\frac{3}{4}x \\ +\frac{3}{4}x \quad = & +\frac{3}{4}x \\ \hline \frac{5}{4}x-3= & 2 \\ +3 & +3 \\ \hline \frac{5}{4}x \quad = & 5 \end{array}$	Add $\frac{3}{4}x$ to both sides to get the variable terms on one side. $\frac{1}{2}+\frac{3}{4}=\frac{2}{4}+\frac{3}{4}=\frac{5}{4}-\frac{3}{4}+\frac{3}{4}=0$ Add 3 to both sides to get the constant terms on the other side.

$\frac{4}{5}\cdot\frac{5}{4}x=\frac{4}{5}\cdot 5$ $x=\frac{20}{5}$ $x=4$	To get a coefficient of 1, multiply the variable term by its multiplicative inverse.

$x=4$

Advanced Example

Solve $\frac{1}{2}(2+a)=\frac{3a+4}{3^2}$. Check your solution.

Solution

	$\frac{1}{2}(2+a)=\frac{3a+4}{3^2}$	Solving this equation will require multiple steps. Begin by evaluating $3^2=9$.
	$\frac{1}{2}\cdot 2+\frac{1}{2}\cdot a=\frac{3a+4}{9}$ $1+\frac{1}{2}a=\frac{3a+4}{9}$ $1+\frac{a}{2}=\frac{3a+4}{9}$	Now distribute the $\frac{1}{2}$ on the left side of the equation.
	$18\left(1+\frac{a}{2}\right)=18\left(\frac{3a+4}{9}\right)$ $18\cdot 1+18\cdot\frac{a}{2}=\frac{18(3a+4)}{9}$ $18+\frac{18a}{2}=\frac{18(3a+4)}{9}$ $18+\frac{18a}{2}=\frac{9\cdot 2(3a+4)}{9}$ $18+9a\cdot\frac{2}{2}=2(3a+4)\cdot\frac{9}{9}$ $18+9a=2(3a+4)$ $18+9a=2\cdot 3a+2\cdot 4$ $18+9a=6a+8$	Multiply both sides of the equation by 18, the common denominator of the fractions in the problem. Use the distributive property to expand the expression on the left side. Then remove a factor of 1 from both sides. On the left, you can think of $\frac{18a}{2}$ as $\frac{2}{2}$. On the right, you can think of $\frac{18(3a+4)}{9}$ as $\frac{9}{9}\cdot\frac{2(3a+4)}{1}$. Continue solving for a using the distributive property.
	$\begin{array}{rr} 18+9a= & 6a+8 \\ -6a & -6a \\ \hline 18+3a= & 8 \\ -18 & -18 \\ \hline 3a= & -10 \\ \frac{3a}{3}= & \frac{-10}{3} \\ a= & -\frac{10}{3} \end{array}$	Then isolate the variable, and solve the remaining one-step problem.
Check	$\frac{1}{2}\left[2+\left(-\frac{10}{3}\right)\right]=\frac{3\left(-\frac{10}{3}\right)+4}{3^2}$ $\frac{1}{2}\left[2+\left(-\frac{10}{3}\right)\right]=\frac{-10+4}{9}$ $\frac{1}{2}\left[\frac{6}{3}+\left(-\frac{10}{3}\right)\right]=\frac{-10+4}{9}$ $\frac{1}{2}\left(-\frac{4}{3}\right)=\frac{-10+4}{9}$ $\frac{-4}{6}=\frac{-10+4}{9}$ $\frac{-4}{6}=\frac{-6}{9}$ $-\frac{2}{3}\cdot\frac{2}{2}=-\frac{2}{3}\cdot\frac{3}{3}$ $-\frac{2}{3}=-\frac{2}{3}$	Check your solution by substituting $-\frac{10}{3}$ in for a in the original equation. This is a true statement, so the solution is correct.

$a=-\frac{10}{3}$

Exercise

To clear the fractions from $\frac{1}{3} - \frac{2y}{9} = 19$, we can multiply both sides of the equation by which of the following numbers?

$3 \quad 6 \quad 9 \quad 27$

A. 9
B. 9 and 27
C. 6
D. 3 or 9

Answer

A. Incorrect. While 9 is a common denominator of $\frac{1}{3}$ and $\frac{2}{9}$, so is 27. Any denominator will work, not just the least one. The correct answer is 9 and 27.
B. Correct. Both 9 and 27 are common denominators of $\frac{1}{3}$ and $\frac{2}{9}$.
C. Incorrect. You clear fractions by multiplying them by a common denominator. 6 is not a common denominator of $\frac{1}{3}$ and $\frac{2}{9}$. The correct answer is 9 and 27.
D. Incorrect. While 9 is a common denominator of $\frac{1}{3}$ and $\frac{2}{9}$, 3 is not. The correct answer is 9 and 27.

Regardless of which method you use to solve equations containing variables, you will get the same answer. You can choose the method you find easier! Remember to check your answer by substituting your solution into the original equation.

Just as you can clear fractions from an equation, you can clear decimals from the equation in the same way. Find a common denominator and use the multiplication property of equality to multiply both sides of the equation.

Example

Solve $0.4x - 0.25 = 1.75$ **by clearing the decimals first.**

Solution

	$0.4x - 0.25 = 1.75$ $100(0.4x - 0.25) = 100(1.75)$	$0.4\left(\frac{4}{10}\right)$ and $0.25\left(\frac{25}{100}\right)$ and $1.75\left(\frac{175}{100}\right)$ have a common denominator of 100. Multiply both sides by 100.
	$\begin{array}{rcr} 40x - 25 & = & 175 \\ +25 & & +25 \\ \hline \frac{40x}{40} & = & \frac{200}{40} \\ x & = & 5 \end{array}$	Apply the distributive property to clear the parentheses. Solve as before. Add 25 to both sides. Divide both sides by 40.
Check:	$0.4x - 0.25 = 1.75$ $0.4(5) - 0.25 = 1.75$ $2 - 1.25 = 1.75$ $1.75 = 1.75$	Substitute $x = 5$ into the original equation. Evaluate. The solution checks.

$x = 5$

Advanced Question

Solve for a: $\frac{1}{4}(a+3) = 2 - a$

A. $a = 2$
B. $a = 1$
C. $a = 0$
D. $a = -2$

Answer

A. Incorrect. Try multiplying both sides of the equation by 4, as $4 \cdot \frac{1}{4} = 1$. The new equation will be $4\left[\frac{1}{4}(a+3)\right] = 4(2 - a)$, which reduces to $a + 3 = 8 - 4a$. The correct answer is: $a = 1$.
B. Correct. You can solve this equation by multiplying both sides by 4, since $4 \cdot \frac{1}{4} = 1$. The resulting equation, $a + 3 = 4(2 - a)$ can be rewritten as $a + 3 = 8 - 4a$, and then $5a = 5$. You find that $a = 1$.
C. Incorrect. Substituting $a = 0$ into the equation, you find $\frac{1}{4}(0+3) = 2 - 0$, so $\frac{3}{4} = 2$. This is not accurate. The correct answer is: $a = 1$.
D. Incorrect. Try multiplying both sides of the equation by 4, as $4 \cdot \frac{1}{4} = 1$. The new equation will be $4\left[\frac{1}{4}(a+3)\right] = 4(2 - a)$, which reduces to $a + 3 = 8 - 4a$. The correct answer is: $a = 1$.

Summary

Complex, multi-step equations often require multi-step solutions. Before you can begin to isolate a variable, you may need to simplify the equation first. This may mean using the distributive property to remove parentheses, or multiplying both sides of an equation by a common denominator to get rid of fractions. Sometimes it requires both techniques.

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10.1.3: Special Cases and Applications

Learning Objectives

- Solve equations that have one solution, no solution, or an infinite number of solutions.
- Solve application problems by using an equation in one variable.

Introduction

When you follow the steps to solve an **equation**, you try to isolate the variable. You have a solution when you get the equation $x =$ some value. There are equations, however, that have no solution, and other equations that have an infinite number of solutions. How does this work?

Example

Solve for x: $12+2x-8=7x+5-5x$

Solution

$12+2x-8=7x+5-5x$	Combine **like terms** on both sides of the equation.
$\begin{array}{rr} 2x+4= & 2x+5 \\ -2x & -2x \\ \hline 4= & 5 \end{array}$	Isolate the x term by subtracting $2x$ from both sides.

This is *not* a solution! You did *not* find a value for x. Solving for x the way you know how, you arrive at the false statement $4 = 5$. Surely 4 cannot be equal to 5!

This may make sense when you consider the second line in the solution where like terms were combined. If you multiply a number by 2 and add 4 you would never get the same answer as when you multiply that same number by 2 and add 5. Since there is no value of x that will ever make this a true statement, the solution to the equation above is “no solution.”

Be careful that you do not confuse the solution $x=0$ with “no solution.” The solution $x=0$ means that the value 0 satisfies the equation, so there is a solution. “No solution” means that there is no value, not even 0, which would satisfy the equation.

Also, be careful not to make the mistake of thinking that the equation $4=5$ means that 4 and 5 are values for x that are solutions. If you substitute these values into the original equation, you’ll see that they do not satisfy the equation. This is because there is truly *no solution*—there are no values for x that will make the equation $12+2x-8=7x+5-5x$ true.

Example

Solve for x.

$3x+8=3(x+2)$

Solution

$3x+8=3(x+2)$	Apply the distributive property to simplify.
$\begin{array}{rr} 3x+8= & 3x+6 \\ -\,3x & -\,3x \\ \hline 8= & 6 \end{array}$	Isolate the variable term. Since you know that $8=6$ is false, there is no solution.

There is no solution.

Advanced Example

Solve for y.

$8y=2[3(y+4)+y]$

Solution

$\begin{aligned} 8y &= 2[3(y+4)+y] \\ 8y &= 2[3y+12+y] \\ 8y &= 2[4y+12] \\ 8y &= 8y+24 \end{aligned}$	Apply the distributive property to simplify. When two sets of grouping symbols are used, evaluate the inner set and then evaluate the outer set.
$\begin{array}{rr} 8y = & 8y+24 \\ -8y & -8y \\ \hline 0 = & 0+24 \end{array}$	Isolate the variable term by subtracting $8y$ from both sides of the equation. Since you know that $0=24$ is false, there is no solution.

There is no solution.

Algebraic Equations with an Infinite Number of Solutions

You have seen that if an equation has no solution, you end up with a false statement instead of a value for x. You can probably guess that there might be a way you could end up with a true statement instead of a value for x.

✓ Example

Solve for x.

$5x+3-4x=3+x$

Solution

$5x+3-4x=3+x$	Combine like terms on both sides of the equation.
$\begin{array}{rr} x+3= & 3+x \\ -x & -x \\ \hline 3= & 3 \end{array}$	Isolate the x term by subtracting x from both sides.

You arrive at the true statement "$3=3$." When you end up with a true statement like this, it means that the solution to the equation is "all real numbers." Try substituting $x=0$ into the original equation—you will get a true statement! Try $x=-\frac{3}{4}$, and it also will check!

This equation happens to have an infinite number of solutions. Any value for x that you can think of will make this equation true. When you think about the context of the problem, this makes sense—the equation $x+3=3+x$ means "some number plus 3 is equal to 3 plus that same number." We know that this is always true—it's the commutative property of addition!

✓ Example

Solve for x.

$5(x-7)+42=3x+7+2x$

Solution

$\begin{aligned} 5(x-7)+42 &= 3x+7+2x \\ 5x-35+42 &= 5x+7 \end{aligned}$	Apply the distributive property and combine like terms to simplify.
$\begin{array}{rr} 5x+7= & 5x+7 \\ -5x & -5x \\ \hline 7= & 7 \end{array}$	Isolate the x term by subtracting $5x$ from both sides. You get the true statement $7=7$, so you know that x can be all real numbers.

$x=$ all real numbers

When solving an equation, multiplying both sides of the equation by zero is not a good choice. Multiplying both side of an equation by 0 will always result in an equation of $0=0$, but an equation of $0=0$ does not help you know what the solution to the original equation is.

Example

Solve for x.

$x=x+2$

Solution

	Multiply both sides by zero.
$\begin{aligned} x&=x+2 \\ 0(x)&=0(x+2) \\ 0&=0 \end{aligned}$	While it is true that $0=0$, and you may be tempted to conclude that x is true of all real numbers, that is not the case.
Check: $x=x+2$ $3=3+2$ $3=5$	For example, check and see if $x=3$ will solve the equation. Clearly 3 never equals 5, so $x=3$ is not a solution. The equation has no solutions. It was not helpful to have multiplied both sides of the equation by zero.
Better Method: $\begin{array}{rr} x= & x+2 \\ -x & -x \\ \hline 0= & 2 \end{array}$	It would have been better to have started by subtracting x from both sides, resulting in $0=2$, resulting in a false statement telling us that there are no solutions.

There is no solution.

Exercise

In solving the algebraic equation $2(x-5)=2x+10$, you end up with $-10=10$. What does this mean?

A. $x=-10$ and 10
B. There is no solution to the equation.
C. You must have made a mistake in solving the equation.
D. $x=$ all real numbers

Answer

A. Incorrect. Any solution to an equation must satisfy the equation. If you substitute -10 into the original equation, you get $-30=-10$. If you substitute 10 for x in the original equation, you get $10=30$. The correct answer is: There is no solution to the equation.
B. Correct. Whenever you end up with a false statement like $-10=10$, it means there is no solution to the equation.
C. Incorrect. A false statement like this looks like a mistake and it's always good to check the answer. In this case, though, there is not a mistake in the algebra. The correct answer is: There is no solution to the equation.
D. Incorrect. If you substitute some real numbers into the equation, you will see that they do not satisfy the equation. The correct answer is: There is no solution to the equation.

Advanced Question

How many solutions are there for the equation:

$$2\left[\frac{1}{4}(4y-8)+3y\right]+7=3(y+1)+5y$$

A. There is one solution.
B. There are two solutions.
C. There are an infinite number of solutions.
D. There are no solutions.

Answer

A. Incorrect. Try substituting *any* value in for y in this equation and think about what you find. The correct answer is: There are an infinite number of solutions to the equation.
B. Incorrect. Try substituting *any* two values in for y in this equation and think about what you find. When dealing with sets of parentheses, make sure to evaluate the inner parentheses first, and then move to the outer set. The correct answer is: There are an infinite number of solutions to the equation.
C. Correct. When you evaluate the expressions on either side of the equals sign, you get $8y+3=8y+3$. If you were to move the variables to the left side and the constants to the right, you would end up with $0=0$. Since you have a true statement, the equation is true for all values of y
D. Incorrect. Recall that statements such as $3=5$ are indicative of an equation having no solutions. The correct answer is: There are an infinite number of solutions to the equation.

Application Problems

The power of algebra is how it can help you model real situations in order to answer questions about them. This requires you to be able to translate real-world problems into the language of algebra, and then be able to interpret the results correctly. Let's start by exploring a simple word problem that uses algebra for its solution.

Amanda's dad is twice as old as she is today. The sum of their ages is 66. Use an algebraic equation to find the ages of Amanda and her dad.

One way to solve this problem is to use trial and error—you can pick some numbers for Amanda's age, calculate her father's age (which is twice Amanda's age), and then combine them to see if they work in the equation. For example, if Amanda is 20, then her father would be 40 because he is twice as old as she is, but then their combined age is 60, not 66. What if she is 12? 15? 20? As you can see, picking random numbers is a very inefficient strategy!

You can represent this situation algebraically, which provides another way to find the answer.

✓ Example

Amanda's dad is twice as old as she is today. The sum of their ages is 66. Find the ages of Amanda and her dad.

Solution

We need to find Amanda's age and her father's age.	*What is the problem asking?*
Let $x =$ Amanda's age $2x =$ Father's age	Assign a variable to the unknown. The father's age is two times Amanda's age.
$x+2x=66$	Amanda's age added to her father's age is equal to 66.
$x+2x=66$ $\frac{3x}{3}=\frac{66}{3}$ $x=22$	Solve the equation for the variable.
Amanda's age $=22$ Father's age $=2x=2(22)=44$	Use Amanda's age to find her father's age.
Amanda's father's age is double Amanda's age and their sum is 66. The solutions make sense.	*Do the answers make sense?*

Amanda is 22 years old, and her father is 44 years old.

Let's try a new problem. Consider that the rental fee for a landscaping machine includes a one-time fee plus an hourly fee. You could use algebra to create an **expression** that helps you determine the total cost for a variety of rental situations. An equation containing this expression would be useful for trying to stay within a fixed expense budget.

Example

A landscaper wants to rent a tree stump grinder to prepare an area for a garden. The rental company charges a $26 one-time rental fee plus $48 for each hour the machine is rented.

Write an expression for the rental cost for any number of hours.

Solution

The problem asks for an algebraic expression for the rental cost of the stump grinder for any number of hours. An expression will have terms, one of which will contain a variable, but it will not contain an equal sign.	*What is the problem asking?*
Look at the values in the problem: $\$26 =$ one-time fee $\$48 =$ per-hour fee	*What information is important to finding an answer?*
Think about what this means, and try to identify a pattern. 1 hour rental: $\$26+\$48$ 2 hour rental: $\$26+\$48+\$48$ 3 hour rental: $\$26+\$48+\$48+\$48$ Notice that the number of "+48" in the problem is the same as the number of hours the machine is being rented. Since multiplication is repeated addition, you could also represent it like this: 1 hour rental: $\$26+\$48\ (1)$ 2 hour rental: $\$26+\$48\ (2)$ 3 hour rental: $\$26+\$48\ (3)$	
Now let's use a variable, h, to represent the number of hours the machine is rented.	What is the variable?
Rental for h hours: $26+48h$	*What expression models this situation?*
	The total rental fee is determined by multiplying the number of hours by $\$48$ and adding $\$26$

The rental cost for h hours is $26+48h$.

Using the information provided in the problem, you were able to create a general expression for this relationship. This means that you can find the rental cost of the machine for any number of hours!

Let's use this new expression to solve another problem.

Example

A landscaper wants to rent a tree stump grinder to prepare an area for a garden. The rental company charges a $26 one-time rental fee plus $48 for each hour the machine is rented.

What is the maximum number of hours the landscaper can rent the tree stump grinder, if he can spend no more than $290?

(The machine cannot be rented for part of an hour.)

Solution

$26+48h$, where $h =$ the number of hours.	*What expression models this situation?*
$26+48h$, where $h =$ the number of hours. $26+48h=290$	Write an equation to help you find out when the expense equals $290.

$\begin{array}{rr} 26+48h= & 290 \\ -26\qquad\quad & -26 \\ \hline 48h= & 264 \\ \frac{48h}{48}= & \frac{264}{48} \end{array}$ $h=5.5$ hours	Solve the equation.
Does $26+48(5.5)=290?$ $26+264=290$ $290=290$	Check the solution.
You found that $h=5.5$. Since the machine cannot be rented for part of an hour, the landscaper can only rent the machine for 5 hours and will have some money left over.	Interpret the answer.

The landscaper can rent the machine for 5 hours.

It is often helpful to follow a list of steps to organize and solve application problems.

Solving Application Problems

Follow these steps to translate problem situations into algebraic equations you can solve.

1. Read and understand the problem.
2. Determine the **constants** and variables in the problem.
3. Write an equation to represent the problem.
4. Solve the equation.
5. Check your answer.
6. Write a sentence that answers the question in the application problem.

Let's try applying the problem-solving steps with some new examples.

✓ Example

Gina has found a great price on paper towels. She wants to stock up on these for her cleaning business. Paper towels cost $1.25 per package. If she has $60 to spend, how many packages of paper towels can she purchase? Write an equation that Gina could use to solve this problem and show the solution.

Solution

The problem asks for how many packages of paper towels Gina can purchase.	*What is the problem asking you?*
The paper towels cost $1.25 per package. Gina has $60 to spend on paper towels.	*What are the constants?*
Let $p=$ the number of packages of paper towels.	*What is the variable?*
$1.25p=60$	*What equation represents this situation?*
$\frac{1.25p}{1.25}=\frac{60}{1.25}$ $p=48$	*Solve for* p. *Divide both sides of the equation by 1.25.*

	$60 \div 1.25 = 6,000 \div 125$ 48 125\longdiv$6,000$ 500 $\overline{1,000}$ $\underline{1,000}$ 0 48 125\longdiv$6,000$
$\text{Does } 1.25p = 60?$ $1.25(48) = 60$ $60 = 60$	*Check your solution.* Substitute 48 in for p in your equation.

Gina can purchase 48 packages of paper towels.

✓ Example

Levon and Maria were shopping for candles to decorate tables at a restaurant. Levon bought 5 packages of candles plus 3 single candles. Maria bought 11 single candles plus 4 packages of candles. Each package of candles contains the same number of candles. After finishing shopping, Maria and Levon realized that they had each purchased the same exact number of candles. How many candles are in a package?

Solution

The problem asks how many candles are contained in one package.	*What is the problem asking you?*
Levon bought 5 packages and 3 single candles.	*What are the constants?*
Maria bought 4 packages and 11 single candles.	
Let $c =$ the number of candles in one package.	*What is the variable?*
$5c + 3$ $4c + 11$ $5c + 3 = 4c + 11$	*What expression represents the number of candles Levon purchased?* *What expression represents the number of candles Maria purchased?* *What equation represents the situation?* Maria and Levon bought the same number of candles.
$\begin{array}{rcr} 5c+3 & = & 4c+11 \\ -4c & & -4c \\ \hline 1c+3 & = & 11 \\ -3 & & -3 \\ \hline 1c & & 8 \\ c & = & 8 \end{array}$	*Solve for* c. Subtract $4c$ from both sides. Subtract 3 from both sides.
$5c + 3 = 4c + 11$ $5(8) + 3 = 4(8) + 11$ $40 + 3 = 32 + 11$ $43 = 43$	*Check your solution.* Substitute 8 for c in the original equation.

There are 8 candles in one package of candles.

✓ Advanced Example

The money from two vending machines is being collected. One machine contains 30 dollar bills and a bunch of dimes. The other machine contains 38 dollar bills and a bunch of nickels. The number of coins in both machines is equal, and the amount of money that the machines collected is also equal. How many coins are in each machine?

Solution

The problem asks how many coins are in each machine.	*What is the problem asking you?*
One machine has 30 dollar bills and a bunch of dimes.	*What are the constants and what are the unknowns?*
Another machine has 38 dollar bills and a bunch of nickels—the same number of coins as the first machine.	
Let $c =$ the number of coins in each machine.	*What is the variable?*
$30+0.10c$	*What expression represents the amount of money in the first machine?*
$38+0.05c$	*What expression represents the amount of money in the second machine?*
$30+0.10c=38+0.05c$	*What equation represents the situation?* The amount of money in both machines is the same.
$\begin{array}{rr} 30+0.10c= & 38+0.05c \\ -0.05c & -0.05c \\ \hline 30+0.05c= & 38 \\ -30 & -30 \\ \hline 0.05c= & 8 \\ c= & 160 \end{array}$	*Solve for* c.
$\begin{aligned} 30+0.10c &= 38+0.05c \\ 30+0.10(160) &= 38+0.05(160) \\ 30+16 &= 38+8 \\ 46 &= 46 \end{aligned}$	*Check your solution.* Substitute 160 for c in the original equation.

There are 160 coins in each machine.

Advanced Question

Albert and Bryn are buying candy at the corner store. Albert buys 5 bags and 3 individual pieces; Bryn buys 3 bags and then eats 2 pieces of candy from one of the bags. Each bag has the same number of pieces of candy.

After Bryn eats the 2 pieces, she has exactly half the number of pieces of candy as Albert. How many pieces of candy are in each bag?

Pick the equation that could be used to solve the problem above. Use the variable b to represent the number of pieces of candy in one bag.

A. $5b+3=\frac{1}{2}(3b+2)$
B. $5b+3=3b+2$
C. $\frac{1}{2}(5b+3)=3b-2$
D. $\frac{1}{2}(3b+3)=5b-2$

Answer

A. Incorrect. For the equation $5b+3=\frac{1}{2}(3b+2)$ to be correct, the situation would have to be “Albert had 5 bags and 3 pieces, and Bryn had 3 bags and 2 pieces. Albert had half as much as Bryn.” The correct answer is: $\frac{1}{2}(5b+3)=3b-2$.
B. Incorrect. For the equation $5b+3=3b+2$ to be correct, the situation would have to be “Albert had 5 bags and 3 pieces, and Bryn had 3 bags and 2 pieces. They had the same amount.” The correct answer is: $\frac{1}{2}(5b+3)=3b-2$.
C. Correct. The amount of candy that Albert has can be represented by $5b+3$, and the amount of candy Bryn has can be represented by $3b-2$. Since Bryn has half as much as Albert, the final equation is $\frac{1}{2}(5b+3)=3b-2$.
D. Incorrect. For the equation $\frac{1}{2}(3b+3)=5b-2$ to be correct, the situation would have to be “Albert had 3 bags and 3 pieces, and Bryn had 5 bags and then ate 2 pieces. Bryn had half as much as Albert.” The correct answer is $\frac{1}{2}(5b+3)=3b-2$.

Summary

Some equations are considered special cases. These are equations that have no solution and equations whose solution is the set of all real numbers. When you use the steps for solving an equation, and you get a false statement rather than a value for the variable, there is no solution. When you use the steps for solving an equation, have avoided multiplying both sides of the equation by zero, and you get a true statement rather than a value for the variable, the solution is all real numbers. Algebra is a powerful tool for modeling and solving real-world problems.

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10.1.4: Formulas

Learning Objectives

- Evaluate a formula using substitution.
- Rearrange formulas to isolate specific variables.

Introduction

Many real-world problems have well-known **equations** that describe relationships among different quantities. These equations that state a rule for a relationship are called **formulas**.

You may have used formulas to figure out things like the area of a rectangle:

$$area = length \cdot width$$

or maybe the speed of a moving object

$$speed = distance \div time$$

or to convert from one system of measurement to another. Most formulas include more than one **variable**. Although they have a special name, formulas are written and solved like any other equations.

Evaluating Formulas Using Substitution

There are many formulas that we know that relate to the activities of our daily lives. For example, the formula $distance = rate \cdot time$. expresses a frequently used relationship in algebra. Formulas are also commonly used for measurement in geometry. For example, the formula for the area of a parallelogram, a four-sided figure with two pairs of parallel sides, is base times height, or $A = bh$.

Formulas are written so that one variable is already isolated. You need to only evaluate the expression on the other side for the given value(s) for the variable.

Example

Juan drove his car for 7 hours at 50 miles per hour. Use the formula $d = r \cdot t$ where $d = \text{ distance }$, $r = \text{ rate }$, and $t = \text{ time }$ to find the distance Juan drove.

Solution

$d = r \cdot t$ $d = 50 \cdot 7$ $d = 350$	Substitute in the values given for the rate (or speed) and the time traveled. Then compute.

Juan drove his car a distance of 350 miles.

The next example illustrates a geometry formula.

Example

Using the formula $A = bh$, find the area (A) of a parallelogram with a base (b) of 3.5 inches and a height (h) of 7.2 inches.

Solution

$A = b \cdot h$ $A = 3.5 \cdot 7.2$ $A = 25.20$	Substitute the given lengths into the formula for area. Multiply.

The area of the parallelogram is 25.2 square inches.

You can also solve for any of the variables in a formula using the algebra techniques you have learned.

Example

Find the base (b) of a triangle with an area (A) of 20 square feet and a height (h) of 8 feet. Use the formula for the area of a triangle, $A=\frac{1}{2}bh$.

Solution

$A=\frac{1}{2}bh$	Substitute the given lengths into the formula.
$20=\frac{1}{2}b\cdot 8$ $20=\frac{8}{2}b$ $20=4b$ $\frac{20}{4}=\frac{4b}{4}$ $5=b$	Solve for b.

The base of the triangle measures 5 feet.

Advanced Example

In order to entice customers to invest their money, many banks will offer interest-bearing accounts. They work like this: a customer deposits a certain amount of money (called the Principal, or p), which then grows slowly according to the interest rate (R, measured in percent) and the length of time (T, usually measured in months) that the money stays in the account. The amount earned over time is called the interest (I), which is then given to the customer.

The simplest way to calculate interest earned on an account is through the formula $I=P\cdot R\cdot T$.

If a customer deposits a principal of \$2000 at a monthly rate of 0.7%, what is the total amount that she has after 24 months?

Solution

$I=P\cdot R\cdot T$	Substitute in the values given for the Principal, Rate, and Time.
$I=\$2000\cdot 0.7\%\cdot 24$ $I=\$2000\cdot 0.007\cdot 24$ $I=336$	Rewrite 0.7% as the decimal 0.007, then multiply.
$2000+336=2336$	Add the interest and the original principal amount to get the total amount in her account.

She has \$2336 after 24 months.

Solving a Formula for a Specified Variable

If you were going to use the formula $A=\frac{1}{2}bh$ to find numerous values of b, given A and h, it would be more efficient to solve the formula for b first. Since formulas are equations, you can solve for a different variable just as you solve an equation. This is called solving the formula for b.

In the example below, we solve the formula $d=r\cdot t$ for t. This would be useful if we wanted to calculate the amount of time a number of car trips will take to complete.

Example

Solve the equation $d=r\cdot t$ for t.

Solution

$d=r\cdot t$	To solve this formula for t, isolate this variable on one side of the equation.
$\frac{1}{r}\cdot d=\frac{1}{r}\cdot r\cdot t$	You can do this by using the multiplication property of equality to multiply both sides by $\frac{1}{r}$.

$\frac{d}{r}=\frac{r\cdot t}{r}$	Multiplying by $\frac{1}{r}$ is the equivalent of dividing by r as long as $r \neq 0$.
$\frac{d}{r}=t\cdot\frac{r}{r}$	The quantity $\frac{r}{r}=1$, so $t\cdot\frac{r}{r}=t$.
$\frac{d}{r}=t$	

$t=\frac{d}{r}$

Note the answer $\frac{d}{r}=t$ is a correct answer. However, we traditionally write the variable on the left. So with equality, the expressions can be switched to $t=\frac{d}{r}$.

Exercise

Solve the formula for the volume of a rectangular solid, $V=lwh$, for width (w).

A. $w=\frac{lh}{V}$
B. $w=\frac{V}{hl}$
C. $w=Vhl$
D. $\frac{V}{w}=lh$

Answer

A. Incorrect. To isolate the variable w, you should have divided both sides of the formula by lh (or hl). The correct answer is $w=\frac{V}{hl}$.
B. Correct. To isolate the variable w, you just need to divide both sides of the formula by l and h. The order in which these variables are written doesn't affect their product. So, $w=\frac{V}{hl}$.
C. Incorrect. To isolate the variable w, you should have divided both sides of the formula by lh (or hl), not just exchanged V and w in the formula. The correct answer is $w=\frac{V}{hl}$.
D. Incorrect. This equation is true, but it does not solve for w. To isolate the w, you must divide both sides of the formula by lh (or hl), not by w. The correct answer is $w=\frac{V}{hl}$.

Advanced Question

Solve the formula for the area of a trapezoid, $A=\frac{1}{2}h\left(b_1+b_2\right)$ for b_1.

A. $b_1=\frac{1}{2}\left(Ab_2\right)-h$
B. $b_1=\frac{h}{2A}-b_2$
C. $b_1=\frac{2A}{h}-b_2$
D. $b_1=\frac{1}{2}h\left(A+b_2\right)$

Answer

A. Incorrect. To isolate the variable b_1, begin by multiplying both sides of the equation by 2. Then divide by h. The new equation looks like this: $\frac{2A}{h}=b_1+b_2$. The correct solution is: $b_1=\frac{2A}{h}-b_2$.
B. Incorrect. It looks like you multiplied and divided incorrectly. To remove the fraction from the right side, multiply both sides of the equation by 2. This gives you the new equation: $2A=h\left(b_1+b_2\right)$. Now divide by h in order to isolate b_1+b_2. The correct solution is: $b_1=\frac{2A}{h}-b_2$.
C. Correct. To isolate the variable b_1, multiply both sides of the equation by 2, divide by h, and then subtract b_2. Written in terms of b_1, the formula is $b_1=\frac{2A}{h}-b_2$.
D. Incorrect. To isolate the variable b_1, begin by multiplying both sides of the equation by 2. Then divide by h. The new equation looks like this: $\frac{2A}{h}=b_1+b_2$. The correct solution is: $b_1=\frac{2A}{h}-b_2$.

Solving Complex Formulas for a Specific Variable

Some formulas can be solved for a different variable with just a single step. Many others are multi-step formulas, and these will require a series of steps. It may sound like a big task, but it can be done, one step at a time.

For example, let’s look at the formula for the perimeter of a rectangle, $p=2l+2w$. This formula is useful if we are given the length and width. But what if you are given the perimeter and lengths of the sides instead? If you are asked to calculate the width of several rectangles, it might be more efficient to solve the formula for w, the width, first.

The steps for rearranging a more complex formula are shown in the example.

✓ Example

Express the formula for the perimeter of a rectangle, $p=2l+2w$, in terms of the width, w.

Solution

$\begin{array}{rcr} p & = & 2l+2w \\ -2l & & -2l \\ \hline p-2l= & & 2w \end{array}$	Isolate the term containing the variable, w, by subtracting the other term in its expression, $2l$, from both sides.
$\frac{p-2l}{2}=\frac{2w}{2}$ $\frac{p-2l}{2}=w$	Next, clear the coefficient of w by dividing both sides of the equation by 2.
$w=\frac{p-2l}{2}$	You can rewrite the equation so the isolated variable is on the left side.

$w=\frac{p-2l}{2}$

Let’s look at a more complex formula that includes parentheses and fractions, the formula for converting from the Fahrenheit temperature scale to the Celsius scale.

You may or may not recall the formula for converting from the Celsius scale to the Fahrenheit scale. However, you can solve the formula that you do know (or are given) for the other to find that formula.

✓ Example

Solve the formula shown below for converting from the Fahrenheit scale to the Celsius scale for F.

$C=(F-32)\cdot\frac{5}{9}$

Solution

$C=(F-32)\cdot\frac{5}{9}$	To isolate the variable F, it would be best to clear the fraction involving F first.
$\left(\frac{9}{5}\right)C=(F-32)\left(\frac{5}{9}\right)\left(\frac{9}{5}\right)$ $\frac{9}{5}C=F-32$	Multiply both sides of the equation by $\frac{9}{5}$.
$\frac{9}{5}C+32=F-32+32$ $\frac{9}{5}C+32=F$	Add 32 to both sides.

$F=\frac{9}{5}C+32$

✓ Advanced Example

Express the formula for the surface area of a cylinder, $S=2\pi rh+2\pi r^2$, in terms of the height, h.

Solution

$S=2\pi rh+2\pi r^2$ $\begin{array}{rcr} S & = & 2\pi rh+2\pi r^2 \\ -2\pi r^2 & & -2\pi r^2 \\ \hline S-2\pi r^2= & & 2\pi rh \end{array}$	Isolate the term containing the variable, h, by subtracting $2\pi r^2$ from both sides.
$\frac{S-2\pi r^2}{2\pi r}=\frac{2\pi rh}{2\pi r}$	Next, isolate the variable h by dividing both sides of the equation by $2\pi r$.

$\frac{S-2\pi r^2}{2\pi r}=h$	You can rewrite the equation so the isolated variable is on the left side.

$h=\frac{S-2\pi r^2}{2\pi r}$

Summary

Formulas are a type of equation. They usually contain multiple variables, describe important relationships, and provide a quick way to calculate frequently needed quantities. Although they are written to find the value for a particular variable, formulas can be solved for other variables in the formula following standard algebraic rules.

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SECTION OVERVIEW

10.2: Solving Inequalities

Topic hierarchy

10.2.1: Solving One-Step Inequalities

10.2.2: Multi-Step Inequalities

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10.2.1: Solving One-Step Inequalities

Learning Objectives

- Represent inequalities on a number line.
- Use the addition property of inequality to isolate variables and solve algebraic inequalities, and express their solutions graphically.
- Use the multiplication property of inequality to isolate variables and solve algebraic inequalities, and express their solutions graphically.

Introduction

Sometimes there is a range of possible values to describe a situation. When you see a sign that says "Speed Limit 25," you know that it doesn't mean that you have to drive exactly at a speed of 25 miles per hour (mph). This sign means that you are not supposed to go faster than 25 mph, but there are many legal speeds you could drive, such as 22 mph, 24.5 mph or 19 mph. In a situation like this, which has more than one acceptable value, **inequalities** are used to represent the situation rather than **equations**.

What is an Inequality?

An inequality is a mathematical statement that compares two expressions using an inequality sign. In an inequality, one expression of the inequality can be greater or less than the other expression. Special symbols are used in these statements. The box below shows the symbol, meaning, and an example for each inequality sign.

Inequality Signs

$x \neq y \quad x \text{ is not equal to } y.$

Example: The number of days in a week is **not equal** to 9.

$x > y \quad x \text{ is greater than } y. \text{ Example: } 6 > 3$

Example: The number of days in a month is **greater than** the number of days in a week.

$x < y \quad x \text{ is less than } y$

Example: The number of days in a week is **less than** the number of days in a year.

$x \geq y \quad x \text{ is greater than or equal to } y$

Example: 31 is **greater than or equal** to the number of days in a month.

$x \leq y \quad x \text{ is less than or equal to } y$

Example: The speed of a car driving legally in a 25 mph zone is **less than or equal** to 25 mph.

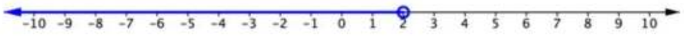
The important thing about inequalities is that there can be multiple solutions. For example, the inequality "31 ≥ the number of days in a month" is a true statement for every month of the year—no month has more than 31 days. It holds true for January, which has 31 days ($31 \geq 31$); September, which has 30 days (≥ 30); and February, which has either 28 or 29 days depending upon the year ($31 \geq 28 \text{ and } 31 \geq 29$).

The inequality $x > y$ can also be written as $y < x$. The sides of any inequality can be switched as long as the inequality symbol between them is also reversed.

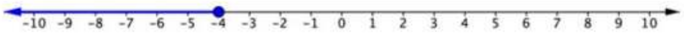
Representing Inequalities on a Number Line

Inequalities can be graphed on a number line. Below are three examples of inequalities and their graphs.

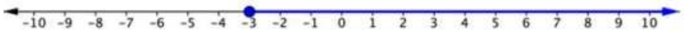
$x < 2$



$x \leq -4$



$x \geq -3$



Each of these graphs begins with a circle—either an open or closed (shaded) circle. This point is often called the *end point* of the solution. A closed, or shaded, circle is used to represent the inequalities *greater than or equal to* (≥) or *less than or equal to* (≤). The point is part of the solution. An open circle is used for *greater than* (>) or *less than* (<). The point is *not* part of the solution.

The graph then extends endlessly in one direction. This is shown by a line with an arrow at the end. For example, notice that for the graph of $x \geq -3$ shown above, the end point is -3, represented with a closed circle since the inequality is *greater than or equal to -3*. The blue line is drawn to the right on the number line because the values in this area are greater than -3. The arrow at the end indicates that the solutions continue infinitely.

Solving Inequalities Using Addition & Subtraction Properties

You can solve most inequalities using the same methods as those for solving equations. Inverse operations can be used to solve inequalities. This is because when you add or subtract the same value from both sides of an inequality, you have maintained the inequality. These properties are outlined in the blue box below.

Addition and Subtraction Properties of Inequality

If $a > b$, then $a + c > b + c$

If $a > b$, then $a - c > b - c$

Because inequalities have multiple possible solutions, representing the solutions graphically provides a helpful visual of the situation. The example below shows the steps to solve and graph an inequality.

✓ Example

Solve for x.

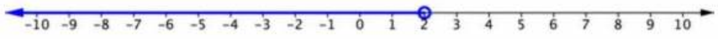
$x + 3 < 5$

Solution

$$\begin{array}{lcr} x+3< & & 5 \\ -3 & & -3 \\ \hline x & < & 2 \end{array}$$	Isolate the variable by subtracting 3 from both sides of the inequality.

$x < 2$

The graph of the inequality $x < 2$ is shown below.



Just as you can check the solution to an equation, you can check a solution to an inequality. First, you check the end point by substituting it in the related equation. Then you check to see if the inequality is correct by substituting any other solution to see if it is one of the solutions. Because there are multiple solutions, it is a good practice to check more than one of the possible solutions. This can also help you check that your graph is correct.

The example below shows how you could check that $x<2$ is the solution to $x+3<5$.

Example

Check that $x<2$ is the solution to $x+3<5$.

Solution

$x+3=5$ Does $2+3=5$? $5=5$	Substitute the end point 2 into the related equation, $x+3=5$.
$x+3<5$ Is $0+3<5$? $3<5$ It checks!	Pick a value less than 2, such as 0, to check into the inequality. (This value will be on the shaded part of the graph.)

$x<2$ is the solution to $x+3<5$

The following examples show additional inequality problems. The graph of the solution to the inequality is also shown. Remember to check the solution. This is a good habit to build!

Advanced Example

Solve for x.

$\frac{15}{2}+x>-\frac{37}{4}$

Solution

$\frac{15}{2}-\frac{15}{2}+x-\frac{37}{4}-\frac{15}{2}$ $x>-\frac{37}{4}-\frac{15}{2}$ $x>-\frac{37}{4}-\frac{30}{4}$ $x>-\frac{67}{4}$	Subtract $\frac{15}{2}$ from both sides to isolate the variable.

$x>-\frac{67}{4}$

Example

Solve for x.

$x-10\leq -12$

Solution

$\begin{array}{rcr} x-10\leq & & -12 \\ +10 & & +10 \\ \hline x & \leq & -2 \end{array}$	Isolate the variable by adding 10 to both sides of the inequality.

$x\leq -2$

The graph of this solution in shown below. Notice that a closed circle is used because the inequality is "less than or equal to" (≤). The blue arrow is drawn to the left of the point -2 because these are the values that are less than -2.

Example

Check that $x\leq -2$ is the solution to $x-10\leq -12$.

Solution

$x-10=-12$ Does $-2-10=-12$? $-12=-12$	Substitute the end point -2 into the related equation $x-10=-12$.

$x - 10 \leq 12$ Is $-5 - 10 \leq 12$? $-15 \leq 12$ It checks!	Pick a value less than -2, such as -5, to check in the inequality. (This value will be on the shaded part of the graph.)

$x \leq -2$ is the solution to $x - 10 \leq 12$.

✓ Example

Solve for a.

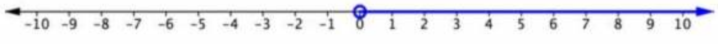
$a - 17 > -17$

Solution

$$\begin{array}{lr} a - 17 > & -17 \\ \;\;+17 & +17 \\ \hline a \quad > & 0 \end{array}$$	Isolate the variable by adding 17 to both sides of the inequality.

$a > 0$

The graph of this solution in shown below. Notice that an open circle is used because the inequality is “greater than” (>). The arrow is drawn to the right of 0 because these are the values that are greater than 0.



✓ Example

Check that $a > 0$ is the solution to

$a - 17 > -17$.

Solution

$a - 17 = -17$ Does $0 - 17 = -17$? $-17 = -17$	Substitute the end point, 0, into the related equation.
$a - 17 > -17$ Is $20 - 17 > -17$? $3 > -17$ It checks!	Pick a value greater than 0, such as 20, to check in the inequality. (This value will be on the shaded part of the graph.)

$a > 0$ is the solution to $a - 17 > -17$.

? Advanced Question

Solve for x: $0.5x \leq 7 - 0.5x$.

A. $x \leq 0$
B. $x > 35$
C. $x \leq 7$
D. $x \geq 5$

Answer

A. Incorrect. To find the value of x, try adding $0.5x$ to both sides. The correct answer is $x \leq 7$.
B. Incorrect. To find the value of x, try adding $0.5x$ to both sides. The correct answer is $x \leq 7$.
C. Correct. When you add $0.5x$ to both sides it creates $1x$, so $x \leq 7$.
D. Incorrect. To find the value of x, try adding $0.5x$ to both sides. The correct answer is $x \leq 7$.

Solving Inequalities Involving Multiplication

Solving an inequality with a variable that has a coefficient other than 1 usually involves multiplication or division. The steps are like solving one-step equations involving multiplication or division EXCEPT for the inequality sign. Let's look at what happens to the inequality when you multiply or divide each side by the same number.

Let's start with the true statement:	$10>5$	Let's try again by starting with the same true statement:	$10>5$
Next, multiply both sides by the same positive number:	$10\cdot 2>5\cdot 2$	This time, multiply both sides by the same negative number:	$10\cdot -2>5\cdot -2$
20 is greater than 10, so you still have a true inequality:	$20>10$	Wait a minute! -20 is *not* greater than -10, so you have an untrue statement.	$-20>-10$
When you multiply by a positive number, leave the inequality sign as it is!		You must "reverse" the inequality sign to make the statement true:	$-20<-10$

When you multiply by a negative number, "reverse" the inequality sign.

Whenever you multiply or divide both sides of an inequality by a negative number, the inequality sign must be reversed in order to keep a true statement.

These rules are summarized in the box below.

Multiplication and Division Properties of Inequality

If $a>b$, then $ac>bc$, if $c>0$
If $a>b$, then $ac<bc$, if $c<0$

If $a>b$, then $\frac{a}{c}>\frac{b}{c}$, if $c>0$

If $a>b$, then $\frac{a}{c}<\frac{b}{c}$, if $c<0$

Keep in mind that you only change the sign when you are multiplying and dividing by a *negative* number. If you add or subtract a negative number, the inequality stays the same.

Example

Solve for x.

$-\frac{1}{3}>-12x$

Solution

$-\frac{1}{3}\div -12<-12x\div -12$ $-\frac{1}{3}\cdot -\frac{1}{12}<\frac{-12x}{-12}$ $\frac{1}{36}<x$	Divide both sides by -12 to isolate the variable. Since you are dividing by a negative number, you need to change the direction of the inequality sign.
Check Does $-\frac{1}{3}=-12\left(\frac{1}{36}\right)$? $-\frac{1}{3}=-\frac{12}{36}$ $-\frac{1}{3}=-\frac{1}{3}$	Check your solution by first checking the end point, $\frac{1}{36}$, in the related equation.

Is $-\frac{1}{3} > -12(2)$ $-\frac{1}{3} > -24$ It checks!	Pick a value greater than $\frac{1}{36}$, such as 2, to check in the inequality.

$x > \frac{1}{36}$

Example

Solve for x.

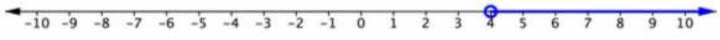
$3x > 12$

Solution

$\frac{3x}{3} > \frac{12}{3}$ $x > 4$	Divide both sides by 3 to isolate the variable.
Check Does $3 \cdot 4 = 12?$ $12 = 12$ Is $3 \cdot 10 > 12?$ $30 > 12$	Check your solution by first checking the end point, 4, and then checking another solution for the inequality.

$x > 4$

The graph of this solution is shown below.



There was no need to make any changes to the inequality sign because both sides of the inequality were divided by *positive 3*. In the next example, there is division by a negative number, so there is an additional step in the solution!

Example

Solve for x.

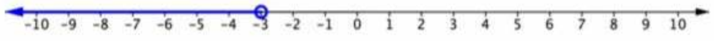
$-2x > 6$

Solution

$\frac{-2x}{-2} < \frac{6}{-2}$ $x < -3$	Divide each side of the inequality by -2 to isolate the variable, and change the direction of the inequality sign because of the division by a negative number.
Check: Does $-2(-3) = 6?$ $6 = 6$ Is $-2(-6) > 6?$ $12 > 6$ It checks!	Check your solution by first checking the end point, -3, and then checking another solution for the inequality.

$x < -3$

Because both sides of the inequality were divided by a negative number, -2, the inequality symbol was switched from > to <. The graph of this solution is shown below.



Exercise

Solve for y: $-10y \geq 150$

A. $y=-15$
B. $y \geq -15$
C. $y \leq -15$
D. $y \geq 15$

Answer

A. Incorrect. While -15 is a solution to the inequality, it is not the only solution. The solution must include an inequality sign. The correct answer is $y \leq -15$.
B. Incorrect. This solution does not satisfy the inequality. For example $y=0$, which is a value greater than -15, results in an untrue statement. 0 is not greater than When dividing by a negative number, you must change the inequality symbol. The correct answer is $y \leq -15$.
C. Correct. Dividing both sides by -10 leaves y isolated on the left side of the inequality and -15 on the right. Since you divided by a negative number, the ≥ must be switched to ≤.
D. Incorrect. Divide by -10, not 10, to isolate the variable. The correct answer is $y \leq -15$.

Advanced Question

Solve for a: $-\frac{a}{5}<\frac{35}{8}$

A. $a>-\frac{175}{8}$
B. $a<-\frac{175}{8}$
C. $a>-\frac{7}{8}$
D. $a<-\frac{7}{8}$

Answer

A. Correct. By multiplying both sides by -5 and flipping the inequality sign from < to >, you found that $a>-\frac{175}{8}$.
B. Incorrect. You correctly multiplied by -5, but remember that the inequality sign flips when you multiply by a negative number. The correct response is: $a>-\frac{175}{8}$.
C. Incorrect. It looks like you divided both sides by -5. While you remembered to flip the inequality sign correctly, division is not the correct operation here. The correct response is: $a>-\frac{175}{8}$.
D. Incorrect. It looks like you divided both sides by -5. Division is not the correct operation here, and remember to flip the inequality sign when you multiply or divide by a negative number. The correct response is: $a>-\frac{175}{8}$.

Summary

Solving inequalities is very similar to solving equations, except you have to reverse the inequality symbols when you multiply or divide both sides of an inequality by a negative number. Since inequalities can have multiple solutions, it is customary to represent the solution to an inequality graphically as well as algebraically. Because there is usually more than one solution to an inequality, when you check your answer you should check the end point and one other value to check the direction of the inequality.

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10.2.2: Multi-Step Inequalities

Learning Objectives

- Use the properties of inequality together to isolate variables and solve algebraic inequalities, and express their solutions graphically.
- Simplify and solve algebraic inequalities using the distributive property to clear parentheses and fractions.

Introduction

Solving multi-step inequalities is very similar to solving equations—what you do to one side, you need to do to the other side in order to maintain the “balance” of the **inequality**. The properties of inequality can help you understand how to add, subtract, multiply, or divide within an inequality.

Using Properties Together to Solve Inequalities

A popular strategy for solving equations, isolating the variable, also applies to solving inequalities. By adding, subtracting, multiplying, and/or dividing, you can rewrite the inequality so that the variable is on one side and everything else is on the other. As with one-step inequalities, the solutions to multi-step inequalities can be graphed on a number line.

Example

Solve for p.

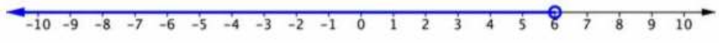
$4p+5<29$**.**

Solution

$\begin{array}{rcr} 4p+5< & & 29 \\ -5 & & -5 \\ \hline \frac{4p}{4} & < & \frac{24}{4} \end{array}$	Begin to isolate the variable by subtracting 5 from both sides of the inequality.
$p<6$	Divide both sides of the inequality by 4 to express the variable with a coefficient of 1.

$p<6$

To graph this inequality, you draw an open circle at the end point, 6, on the number line. The circle is open because the inequality is less than 6 and *not equal* to 6. The values where p is less than 6 are found all along the number line to the left of 6. Draw a blue line with an arrow on the number line pointing in that direction.



To check the solution, substitute the end point, 6, into the original inequality written as an equation, which is called the related equation, to see if you get a true statement. Then check another solution, such as 0, to see if the inequality is correct.

Example

Check that $p<6$ **is the solution to the inequality** $4p+5<29$**.**

Solution

$\begin{array}{r} 4p+5=29 \\ \text{Does } 4(6)+5=29? \\ 24+5=29 \\ 29=29 \text{ Yes!} \end{array}$	Check the end point, 6, in the related equation.
$\begin{array}{r} 4p+5<29 \\ \text{Is } 4(0)+5<29? \\ 0+5<29 \\ 5<29 \text{ Yes!} \end{array}$	Try another value to check the inequality. Let’s use $p=0$.

$p<6$ is the solution to the inequality $4p+5<29$.

Example

Solve for x.

$3x-7\geq 41$

Solution

$\begin{array}{rr} 3x-7\geq & 41 \\ +7 & +7 \\ \hline \frac{3x}{3}\quad\geq & \frac{48}{3} \\ x\geq & 16 \end{array}$	Begin to isolate the variable by adding 7 to both sides of the inequality. Divide both sides of the inequality by 3 to express the variable with a coefficient of 1.
Check $3x-7=41$ Does $3(16)-7=41?$ $48-7=41$ $41=41$ Yes! $3x-7\geq 41$ Is $3(20)-7\geq 41?$ $60-7\geq 41$ $53\geq 41$ Yes!	First, check the end point, 16, in the related equation. Then, try another value to check the inequality. Let's use $x=20$.

$x\geq 16$

When solving multi-step equations, pay attention to situations in which you multiply or divide by a negative number. In these cases, you must reverse the inequality sign.

Example

Solve for p.

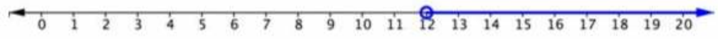
$-6p+14<-58$

Solution

$\begin{array}{rr} -6p+14< & -58 \\ -14 & -14 \\ \hline \frac{-6p}{-6}\quad> & \frac{-72}{-6} \\ p> & 12 \end{array}$	Begin to isolate the variable by subtracting 14 from both sides of the inequality. Divide both sides of the inequality by -6 to express the variable with a coefficient of 1. Dividing by a negative number results in reversing the inequality sign.
Check $-6p+14=-58$ Does $-6(12)+14=-58?$ $-72+14=-58$ $-58=-58$ Yes!	Check the solution. First, check the end point, 12, in the related equation.
$-6p+14<-58$ Is $-6(100)+14<-58?$ $-600+14<-58$ $-586<-58$ Yes!	Then, try another value to check the inequality. Try 100.

$p>12$

The graph of the inequality $p>12$ has an open circle at 12 with an arrow stretching to the right.



✓ Advanced Example

Solve for x.

$\frac{1}{3}(x+3)+\frac{1}{2}>-\frac{9}{2}$.

Solution

$\frac{1}{3}(x+3)+\frac{1}{2}-\frac{1}{2}>-\frac{9}{2}-\frac{1}{2}$ $\frac{1}{3}(x+3)>-\frac{10}{2}$ $\frac{1}{3}(x+3)>-5$ $3\cdot\frac{1}{3}(x+3)>3\cdot(-5)$ $x+3>-15$ $x+3-3>-15-3$ $x>-18$	To isolate the variable, subtract $\frac{1}{2}$ from both sides of the inequality. Then multiply by 3 so that the coefficient in front of the parentheses is 1. Then subtract 3 from both sides.
Check $\frac{1}{3}(-18+3)+\frac{1}{2}=-\frac{9}{2}$ $\frac{1}{3}(-15)+\frac{1}{2}=-\frac{9}{2}$ $-5+\frac{1}{2}=-\frac{9}{2}$ $-\frac{10}{2}+\frac{1}{2}=-\frac{9}{2}$ $-\frac{9}{2}=-\frac{9}{2}$	Check the solution. First, check the end point, -18, in the related equation.
$\frac{1}{3}(0+3)+\frac{1}{2}>-\frac{9}{2}$ $\frac{1}{3}(3)+\frac{1}{2}>-\frac{9}{2}$ $1+\frac{1}{2}>-\frac{9}{2}$ $\frac{2}{2}+\frac{1}{2}>-\frac{9}{2}$ $\frac{3}{2}>-\frac{9}{2}$	Now check any value for x that is within the region $x>-18$. We will use $x=0$.
	The statement is true.

$x>-18$

? Exercise

A student is solving the inequality $\frac{5}{2}+7x\leq 4x-\frac{7}{2}$. If she combines like terms, which of the following inequalities could she see?

A. $-6\leq 3x$
B. $3x\leq -6$
C. $\frac{19x}{2}\leq\frac{x}{2}$
D. $x\leq\frac{-6x}{7}$

Answer

A. Incorrect. It looks like you tried to combine like terms but made sign mistakes while adding and subtracting. To isolate the variable term on the right, you need to subtract $7x$ from both sides; $4x-7x=-3x$. The correct answer is: $3x\leq -6$.
B. Correct. You correctly combined like terms. $\frac{5}{2}+7x\leq 4x-\frac{7}{2}$ becomes $7x-4x\leq -\frac{7}{2}-\frac{5}{2}$. which is the same as $3x\leq -6$.
C. Incorrect. Remember that you cannot add variable and non-variable terms like you add integers. The correct answer is: $3x\leq -6$.
D. Incorrect. Try subtracting $\frac{5}{2}$ and $4x$ from each side before collecting like terms. The correct answer is: $3x\leq -6$.

Using the Distributive Property to Clear Parentheses and Fractions

As with equations, the distributive property can be applied to simplify expressions that are part of an inequality. Once the parentheses have been cleared, solving the inequality will be straightforward.

✓ Example

Solve for x.

$2(3x-5) \leq 4x+6$

Solution

$\begin{array}{rcr} 2(3x-5) & \leq & 4x+6 \\ 6x-10 & \leq & 4x+6 \\ -4x & & -4x \\ \hline 2x-10 & \leq & 6 \\ +10 & & +10 \\ \hline \frac{2x}{2} & \leq & \frac{16}{2} \\ x & \leq & 8 \end{array}$	Distribute to clear the parentheses. Subtract $4x$ from both sides to get the variable term on one side only. Add 10 to both sides to isolate the variable. Divide both sides by 2 to express the variable with a coefficient of 1.
Check $2(3x-5)=4x+6$ Does $2(3\cdot 8-5)=4\cdot 8+6?$ $2(24-5)=32+6$ $2(19)=38$ $38=38$ Yes! Is $2(3\cdot 0-5)\leq 4\cdot 0+6?$ $2(-5)\leq 6$ $-10\leq 6$	Check the solution. First, check the end point, 8, in the related equation. Then, choose another solution and evaluate the inequality for that value to make sure it is a true statement. Try 0.

$x \leq 8$

✓ Example

Solve for a.

$\frac{2a-4}{6} < 2$

Solution

$6\cdot\frac{2a-4}{6} < 2\cdot 6$	Clear the fraction by multiplying both sides of the equation by 6.
$\begin{array}{rcr} 2a-4 & < & 12 \\ +4 & & +4 \\ \hline \frac{2a}{2} & < & \frac{16}{2} \end{array}$	Add 4 to both sides to isolate the variable.
$a<8$	Divide both sides by 2 to express the variable with a coefficient of 1.
Check $\frac{2a-4}{6}=2$ Does $\frac{2(8)-4}{6}=2?$ $\frac{16-4}{6}=2$ $\frac{12}{6}=2$ $2=2$ Yes!	Check the solution. First, check the end point, 8, in the related equation.

Is $\frac{2(5)-4}{6}<2?$ $\frac{10-4}{6}<2$ $\frac{6}{6}<2$ $1<2$ Yes!	Then, choose another solution and evaluate the inequality for that value to make sure it is a true statement. Try 5.

$a<8$

✓ Advanced Example

Solve for d.

$\frac{3}{5}(2d-5)\le 4\left(7-\frac{1}{5}d\right)$

Solution

$\frac{3}{5}(2d-5)\le 4\left(7-\frac{1}{5}d\right)$ $\frac{3}{5}(2d)+\frac{3}{5}(-5)\le 4(7)+4\left(-\frac{1}{5}d\right)$ $\frac{6d}{5}+\left(\frac{-15}{5}\right)\le 28+\left(\frac{-4d}{5}\right)$ $\frac{6d}{5}-3\le 28-\frac{4d}{5}$	This inequality contains two parentheses. Use the distributive property to expand both sides of the inequality.
$\frac{6d}{5}-3+3\le 28-\frac{4d}{5}+3$ $\frac{6d}{5}\le 31-\frac{4d}{5}$ $\frac{6d}{5}+\frac{4d}{5}\le 31-\frac{4d}{5}+\frac{4d}{5}$ $\frac{10d}{5}\le 31$ $2d\le 31$ $\frac{2d}{2}\le\frac{31}{2}$ $d\le\frac{31}{2}$	Now that both sides have been expanded, combine like terms and find the range of values for d.
$\frac{3}{5}\left(2\cdot\frac{31}{2}-5\right)=4\left(7-\frac{1}{5}\cdot\frac{31}{2}\right)$ $\frac{3}{5}\left(\frac{62}{2}-5\right)=4\left(7-\frac{31}{10}\right)$ $\frac{3}{5}(31-5)=4\left(\frac{70}{10}-\frac{31}{10}\right)$ $\frac{3}{5}(26)=4\left(\frac{39}{10}\right)$ $\frac{78}{5}=4\left(\frac{39}{10}\right)$ $\frac{78}{5}=\frac{156}{10}$ $\frac{78}{5}=\frac{78}{5}\cdot\frac{2}{2}$ $\frac{78}{5}=\frac{78}{5}$	Check the solution. First, check the end point, $\frac{31}{2}$, in the related equation.
	It results in a true statement.
$\frac{3}{5}(2\cdot 0-5)\le 4\left(7-\frac{1}{5}\cdot 0\right)$ $\frac{3}{5}(0-5)\le 4(7-0)$ $\frac{3}{5}(-5)\le 4(7)$ $\frac{-15}{5}\le 28$ $-3\le 28$	Now try any value for d that is within the region $d\le\frac{31}{2}$. We will try $d=0$.

	This is also a true statement.

$d \leq \frac{31}{2}$

Exercise

Which is the most logical first step for solving for the variable in the inequality: $8x+7<3(2x+1)$

A. Reverse the inequality sign.
B. Use the distributive property to clear the parentheses by multiplying each of the terms in the parentheses by 3.
C. Subtract $2x$ from both sides of the inequality.
D. Divide both sides of the inequality by 3.

Answer

A. Incorrect. The inequality sign is reversed as a result of dividing or multiplying both sides of an inequality by a negative number. This usually occurs as a later step in a multi-step inequality and, in the case of this example, will not happen at all. The correct answer is to use the distributive property to clear the parentheses by multiplying each of the terms in the parentheses by 3.
B. Correct. The distributive property clears the parentheses so that you can isolate the variable.
C. Incorrect. It is necessary to distribute the 3 before subtracting any terms with x in it. The correct answer is to use the distributive property to clear the parentheses by multiplying each of the terms in the parentheses by 3.
D. Incorrect. Dividing both sides by 3, although legal to do, will result in a complicated fraction on the left side of the inequality and is not a logical first step. The correct answer is to use the distributive property to clear the parentheses by multiplying each of the terms in the parentheses by 3.

Exercise

Solve for x: $5[2(3-x)-1] \leq 27$

A. $x \leq -5$
B. $x \leq 27$
C. $x \geq \frac{1}{8}$
D. $x \geq -\frac{1}{5}$

Answer

A. Incorrect. Try substituting any value that is less than or equal to -5 for x into the inequality. You will not get a true statement, so x is not less than or equal to -5. The correct answer is: $x \geq -\frac{1}{5}$.
B. Incorrect. You need to evaluate the left side of the inequality and then solve for x. Remember that $5[2(3-x)-1]=5[6-2x-1]$. The correct answer is: $x \geq -\frac{1}{5}$.
C. Incorrect. Make sure you are evaluating the left side of the inequality correctly: $5[2(3-x)-1]=5[6-2x-1]$. The correct answer is: $x \geq -\frac{1}{5}$.
D. Correct. Evaluating the interior parentheses first, you find that $5[2(3-x)-1]=5[6-2x-1]=5[5-2x]=25-10x$. Solving for x, you subtract 25 from both sides and then divide by -10, which requires you to flip the inequality sign from ≤ to ≥.

Summary

Inequalities can have a range of answers. The solutions are often graphed on a number line in order to visualize all of the solutions. Multi-step inequalities are solved using the same processes that work for solving equations with one exception. When you multiply or divide both sides of an inequality by a negative number, you must reverse the inequality symbol. The inequality symbols stay the same whenever you add or subtract *either positive or negative* numbers to both sides of the inequality.

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SECTION OVERVIEW

10.3: Compound Inequalities and Absolute Value

Topic hierarchy

10.3.1: Compound Inequalities

10.3.2: Equations and Inequalities and Absolute Value

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10.3.1: Compound Inequalities

Learning Objectives

- Solve compound inequalities in the form of or and express the solution graphically.
- Solve compound inequalities in the form of and and express the solution graphically.
- Solve compound inequalities in the form $a < x < b$.
- Identify cases with no solution.

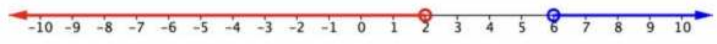
Introduction

Many times, solutions lie between two quantities, rather than continuing endlessly in one direction. For example, systolic (top number) blood pressure that is between 120 and 139mm Hg is called borderline high blood pressure. This can be described using a compound inequality, $b < 139$ *and* $b > 120$. Other compound inequalities are joined by the word "or."

When two inequalities are joined by the word *and*, the solution of the compound inequality occurs when *both* inequalities are true at the same time. It is the overlap, or intersection, of the solutions for each inequality. When the two inequalities are joined by the word *or*, the solution of the compound inequality occurs when *either* of the inequalities is true. The solution is the combination, or union, of the two individual solutions.

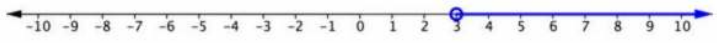
Solving and Graphing Compound Inequalities in the Form of "or"

Let's take a closer look at a **compound inequality** that uses or to combine two inequalities. For example, $x > 6$ or $x < 2$. The solution to this compound inequality is all the values of x in which x is *either* greater than 6 ***or*** x is less than 2. You can show this graphically by putting the graphs of each **inequality** together on the same number line.

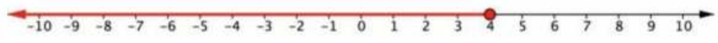


The graph has an open circle on 6 and a blue arrow to the right and another open circle at 2 and a red arrow to the left. In fact, the only parts that are not a solution to this compound inequality are the points 2 and 6 and all the points in between these values on the number line. Everything else on the graph is a solution to this compound inequality.

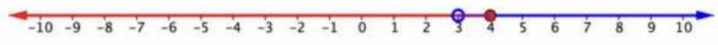
Let's look at another example of an or compound inequality, $x > 3$ ***or*** $x \leq 4$. The graph of $x > 3$ has an open circle on 3 and a blue arrow drawn to the right to contain all the numbers greater than 3.



The graph of $x \leq 4$ has a closed circle at 4 and a red arrow to the left to contain all the numbers less than 4.



What do you notice about the graph that combines these two inequalities?



Since this compound inequality is an or statement, it includes all of the numbers in each of the solutions, which in this case is all the numbers on the number line. (The region of the line greater than 3 and less than or equal to 4 is shown in purple because it lies on both of the original graphs.) The solution to the compound inequality $x > 3$ ***or*** $x \leq 4$ is the set of all real numbers!

You may need to solve one or more of the inequalities before determining the solution to the compound inequality, as in the example below.

Example

Solve for x.

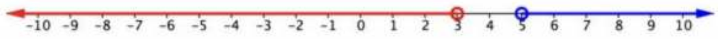
$3x - 1 < 8$ **or** $x - 5 > 0$

Solution

$\begin{array}{rr} x-5> & 0 \\ +5 & +5 \\ \hline x \quad > & 5 \end{array}$ OR $\begin{array}{rr} 3x-1< & 8 \\ +1 & +1 \\ \hline \frac{3x}{3} \quad < & \frac{9}{3} \\ x< & 3 \end{array}$	Solve each inequality by isolating the variable.
$x>5$ or $x<3$	Write both inequality solutions as a compound using *or*.

$x>5$ or $x<3$

The solution to this compound inequality can be shown graphically.



Remember to apply the properties of inequality when you are solving compound inequalities. The next example involves dividing by a negative to isolate a variable.

✓ Example

Solve for y.

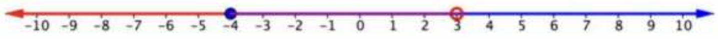
$2y+7<13$ or $-3y-2\leq 10$

Solution

$\begin{array}{rr} 2y+7< & 13 \\ -7 & -7 \\ \hline \frac{2y}{2} \quad < & \frac{6}{2} \\ y \quad < & 3 \end{array}$ OR	Solve each inequality separately.
$\begin{array}{rr} -3y-2\leq & 10 \\ +2 & +2 \\ \hline \frac{-3y}{-3} \quad \geq & \frac{12}{-3} \\ y \quad \geq & -4 \end{array}$	The inequality sign is reversed with division by a negative number.
$y<3$ or $y\geq -4$	Since y could be less than 3 or greater than or equal to -4, could be any number.

The solution is all real numbers.

This number line shows the solution set of $y<3$ or $y\geq -4$.



✓ Example

Solve for z.

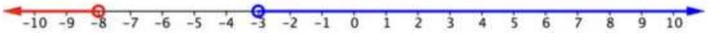
$5z-3>-18$ or $-2z-1>15$

Solution

$\begin{aligned} 5z-3> & \quad -18 \\ +3 & \quad +3 \\ \hline \frac{5z}{5} \quad > & \quad \frac{-15}{5} \\ z \quad > & \quad -3 \end{aligned}$ OR $\begin{aligned} -2z-1> & \quad 15 \\ +1 & \quad +1 \\ \hline \frac{-2z}{-2} \quad < & \quad \frac{16}{-2} \\ z \quad < & \quad -8 \end{aligned}$	Solve each inequality separately.
$z>-3$ or $z<-8$	Combine the solutions.

$z>-3$ or $z<-8$

This number line shows the solution set of $z>-3$ or $z<-8$.



? Exercise

Solve for h.

$h+3>12$ or $3-2h>9$

A. $h<3$ or $h>-3$
B. $h>9$ or $h>-3$
C. $h>-9$ or $h<3$
D. $h>9$ or $h<-3$

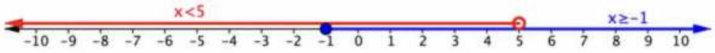
Answer

A. Incorrect. To solve the inequality $h+3>12$, subtract 3 from both sides to get $h>9$. When you divide both sides of an inequality by a negative number, reverse the inequality sign to get $h<-3$ for the solution to the second inequality. The correct answer is $h>9$ or $h<-3$.
B. Incorrect. To solve the inequality $3-2h>9$, subtract 3 from both sides and then divide by -2. When you divide both sides of an inequality by a negative number, reverse the inequality sign to get $h<-3$. The correct answer is $h>9$ or $h<-3$.
C. Incorrect. Check a few values for h that are greater than -9 but less than 3, and see if they make the inequality true. For example, if you substitute $h=2$ into each inequality, you get false statements: $2+3>9; 3-2(2)>9$. The correct answer is $h>9$ or $h<-3$.
D. Correct. Solving each inequality for h, you find that $h>9$ or $h<-3$.

Solving and Graphing Compound Inequalities in the Form of “and”

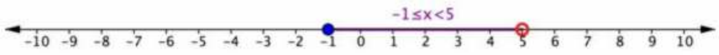
The solution of a compound inequality that consists of two inequalities joined with the word *and* is the intersection of the solutions of each inequality. In other words, both statements must be true at the same time. The solution to an *and* compound inequality are all the solutions that the two inequalities have in common. Graphically, you can think about it as where the two graphs overlap.

Think about the example of the compound inequality: $x<5$ **and** $x\geq-1$. The graph of each individual inequality is shown in color.



Since the word *and* joins the two inequalities, the solution is the overlap of the two solutions. This is where both of these statements are true at the same time.

The solution to this compound inequality is shown below.



Notice that in this case, you can rewrite $x \geq -1$ and $x < 5$ as $-1 \leq x < 5$ since the solution is between -1 and 5, including -1. You read $-1 \leq x < 5$ as "x is greater than or equal to -1 *and* less than 5." You can rewrite an *and* statement this way only if the answer is *between* two numbers.

Let's look at two more examples.

✓ Example

Solve for x.

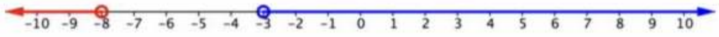
$1 - 4x \leq 21 \text{ and } 5x + 2 \geq 22$

Solution

$\begin{array}{rr} 1-4x \leq & 21 \\ -1 \qquad & -1 \\ \hline \frac{-4x}{-4} \geq & \frac{20}{-4} \\ x \geq & -5 \end{array}$ OR $\begin{array}{rr} -2z-1 > & 15 \\ +1 \qquad & +1 \\ \hline \frac{-2z}{-2} \quad < & \frac{16}{-2} \\ z \quad < & -8 \end{array}$	
$z > -3 \text{ or } z < -8$	Combine the solutions.

$z > -3 \text{ or } z < -8$

This number line shows the solution set of $z > -3 \text{ or } z < -8$.



? Exercise

Solve for h.

$h + 3 > 12 \text{ or } 3 - 2h > 9$

A. $h < 3 \text{ or } h > -3$
B. $h > 9 \text{ or } h > -3$
C. $h > -9 \text{ or } h < 3$
D. $h > 9 \text{ or } h < -3$

Answer

A. Incorrect. To solve the inequality $h + 3 > 12$, subtract 3 from both sides to get $h > 9$. When you divide both sides of an inequality by a negative number, reverse the inequality sign to get $h < -3$ for the solution to the second inequality. The correct answer is $h > 9 \text{ or } h < -3$.

B. Incorrect. To solve the inequality $3 - 2h > 9$, subtract 3 from both sides and then divide by -2. When you divide both sides of an inequality by a negative number, reverse the inequality sign to get $h < -3$. The correct answer is $h > 9 \text{ or } h < -3$.

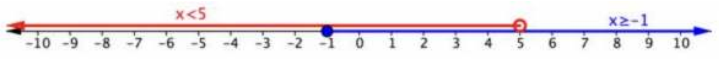
C. Incorrect. Check a few values for h that are greater than -9 but less than 3, and see if they make the inequality true. For example, if you substitute $h = 2$ into each inequality, you get false statements: $2 + 3 > 9; 3 - 2(2) > 9$ The correct answer is $h > 9 \text{ or } h < -3$.

D. Correct. Solving each inequality for h, you find that $h > 9 \text{ or } h < -3$.

Solving and Graphing Compound Inequalities in the Form of “and”

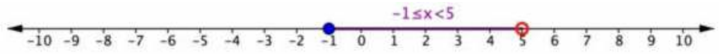
The solution of a compound inequality that consists of two inequalities joined with the word *and* is the intersection of the solutions of each inequality. In other words, both statements must be true at the same time. The solution to an *and* compound inequality are all the solutions that the two inequalities have in common. Graphically, you can think about it as where the two graphs overlap.

Think about the example of the compound inequality: $x<5$ **and** $x\geq -1$. The graph of each individual inequality is shown in color.



Since the word *and* joins the two inequalities, the solution is the overlap of the two solutions. This is where both of these statements are true at the same time.

The solution to this compound inequality is shown below.



Notice that in this case, you can rewrite $x\geq -1$ and $x<5$ as $-1\leq x<5$ since the solution is between -1 and 5, including -1. You read $-1\leq x<5$ as “ x is greater than or equal to -1 *and* less than 5.” You can rewrite an *and* statement this way only if the answer is *between* two numbers.

Let’s look at two more examples.

Example

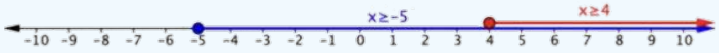
Solve for x.

$1-4x\leq 21$ and $5x+2\geq 22$

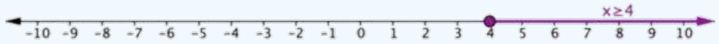
Solution

$\begin{array}{rr} 1-4x\leq & 21 \\ -1 & -1 \\ \hline \frac{-4x}{-4}\geq & \frac{20}{-4} \\ x\geq & -5 \end{array}$ AND $\begin{array}{rcr} 5x+2\geq & & 22 \\ -2 & & -2 \\ \frac{5x}{5} & \geq & \frac{20}{5} \\ x & \geq & 4 \end{array}$	Solve each inequality for x.
$x\geq -5$ and $x\geq 4$	Determine the intersection of the solutions.

The number line below shows the graphs of the two inequalities in the problem. The solution to the compound inequality is $x\geq 4$, as this is where the two graphs overlap.



And the solution can be represented as:



$x\geq 4$

Example

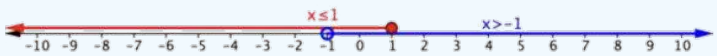
Solve for x.

$5x-2\leq 3$ and $4x+7>3$

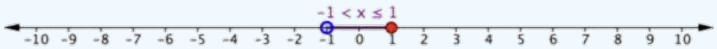
Solution

$\begin{array}{rr} 5x-2\leq & 3 \\ +2 & +2 \\ \hline \frac{5x}{5} \quad \leq & \frac{5}{5} \\ x \quad \leq & 1 \end{array}$ AND $\begin{array}{rr} 4x+7> & 3 \\ -7 & -7 \\ \hline \frac{4x}{4} \quad > & \frac{-4}{4} \\ x \quad > & -1 \end{array}$	Solve each inequality separately.
$x\leq 1$ and $x>-1$	Find the overlap between the solutions.

The two inequalities can be represented graphically as:



And the solution can be represented as:



$x\leq 1$ and $x>-1$

Rather than splitting a compound inequality in the form of $a<x<b$ into two inequalities $x<b$ and $x>a$, you can more quickly solve the inequality by applying the properties of inequality to all three segments of the compound inequality. Two examples are provided below.

Example

Solve for x.

$3<2x+3\leq 7$

Solution

$\begin{array}{rrr} 3< & 2x+3\leq & 7 \\ -3 & -3 & -3 \\ \hline \frac{0}{2}< & \frac{2x}{2} \quad \leq & \frac{4}{2} \end{array}$	Isolate the variable by subtracting 3 from all 3 parts of the inequality, and then dividing each part by 2.
$0<x\leq 2$	

$0<x\leq 2$

Example

Solve for x.

$-3\leq 2x+7<11$

Solution

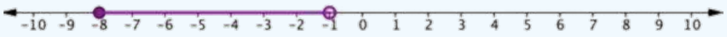
$\begin{array}{rrr} -3\leq & 2x+7< & 11 \\ -7 & -7 & -7 \\ \hline \frac{-10}{2}\leq & \frac{2x}{2} \quad < & \frac{4}{2} \end{array}$	Isolate the variable by subtracting 7 from all 3 parts of the inequality, and then dividing each part by 2.
$-5\leq x<2$	

$-5\leq x<2$

To solve inequalities like $a < x < b$, use the addition and multiplication properties of inequality to solve the inequality for x. Whatever operation you perform on the middle portion of the inequality, you must also perform to each of the outside sections as well. Pay particular attention to division or multiplication by a negative.

? Exercise

Which of the following compound inequalities represents the graph on the number line below?



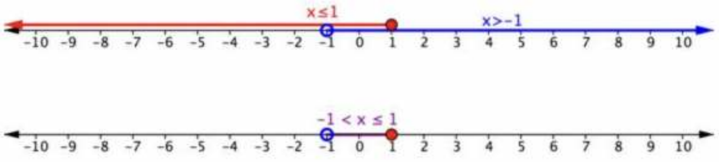
A. $-8 \geq x > -1$
B. $-8 \leq x < -1$
C. $-8 \leq x > -1$
D. $-8 \geq x < -1$

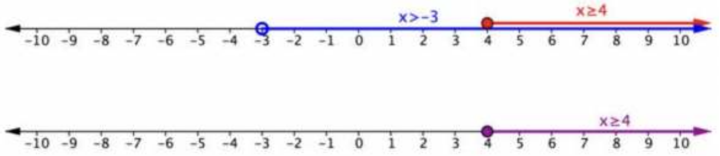
Answer

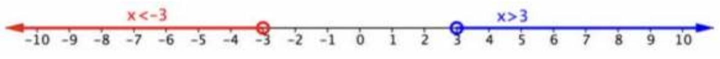
A. Incorrect. This compound inequality reads, “ x is less than or equal to -8 and greater than -1.” The shaded part of the graph includes values that are greater than or equal to -8 and less than -1. The correct answer is $-8 \leq x < -1$.
B. Correct. The selected region on the number line lies between -8 and -1 and includes -8, so x must be greater than or equal to -8 and less than -1.
C. Incorrect. This compound inequality reads, “ x is greater than or equal to -8 and greater than -1.” The values that are shaded are less -1, not greater. The correct answer is $-8 \leq x < -1$.
D. Incorrect. This compound inequality reads, “ x is less than or equal to -8 and less than -1.” The graph does not include values that are less than or equal to -8. It includes values that are greater than or equal to -8 and less than -1. The correct answer is $-8 \leq x < -1$.

Special Cases of Compound Inequalities

The solution to a compound inequality with *and* is always the overlap between the solution to each inequality. There are three possible outcomes for compound inequalities joined by the word *and*:

1. 

2. 

3. 

An example is shown below.

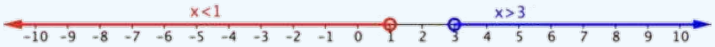
✓ Example

Solve for x.

$x+2>5$ and $x+4<5$

Solution

$\begin{array}{lr} x+2> & 5 \\ \quad -2 & -2 \\ \hline x \quad > & 3 \end{array}$ AND $\begin{array}{lr} x+4< & 5 \\ \quad -4 & -4 \\ \hline x \quad < & 1 \end{array}$	
$x > 3 \text{ and } x < 1$	Find the overlap between the solutions.



There is no overlap between $x > 3$ and $x < 1$, so there is no solution.

Summary

A compound inequality is a statement of two inequality statements linked together either by the word *or* or by the word *and*. Sometimes, an *and* compound inequality is shown symbolically, like $a < x < b$, and does not even need the word *and*. Because compound inequalities represent either a union or intersection of the individual inequalities, graphing them on a number line can be a helpful way to see or check a solution. Compound inequalities can be manipulated and solved much the same way any inequality is solved, paying attention to the properties of inequalities and the rules for solving them.

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10.3.2: Equations and Inequalities and Absolute Value

Learning Objectives

- Solve equations containing absolute values.
- Solve inequalities containing absolute values.
- Identify cases of equations and inequalities containing absolute values that have no solutions.

Introduction

The **absolute value** of a number or expression describes its distance from 0 on a number line. Since the absolute value expresses only the distance, not the direction of the number on a number line, it is always expressed as a positive number or 0.

For example, -4 and 4 both have an absolute value of 4 because they are each 4 units from 0 on a number line—though they are located in opposite directions from 0 on the number line.

When solving absolute value **equations** and **inequalities**, you have to consider both the behavior of absolute value and the properties of equality and inequality.

Solving Equations Containing Absolute Values

Because both positive and negative values have a positive absolute value, solving absolute value equations means finding the solution for both the positive and the negative values.

Let’s first look at a very basic example.

$$|x| = 5$$

This equation is read “the absolute value of x is equal to five.” The solution is the value(s) that are five units away from 0 on a number line.

You might think of 5 right away; that is one solution to the equation. Notice that -5 is also a solution because -5 is 5 units away from 0 in the opposite direction. So, the solution to this equation $|x| = 5$ is $x = -5$ or $x = 5$.

A more complex absolute value problem is solved in a similar fashion. Consider $|x+5| = 15$. This equation asks you to find what number plus 5 has an absolute value of 15. Since 15 and -15 both have an absolute value of 15, the absolute value equation is true when the quantity $x+5$ is 15 or $x+5$ is -15, since $|15| = 15$ and $|-15| = 15$. So, you need to find out what value for x will make this expression equal to 15 as well as what value for x will make the expression equal to -15. Solving the two equations you get:

$$\begin{array}{rlrl} x+5 & =15 & \text{or } x+5 & =-15 \\ -5 & -5 & -5 & -5 \\ \hline x & =10 & \text{or } x & =-20 \end{array}$$

You can check these two solutions in the absolute value equation to see if $x = 10$ and $x = -20$ are correct.

$$\begin{aligned} |x+5| &= 15 & |x+5| &= 15 \\ |10+5| &= 15 & |-20+5| &= 15 \\ |15| &= 15 & |-15| &= 15 \\ 15 &= 15 & 15 &= 15 \end{aligned}$$

Solving Equations of the Form $|x| = a$

For any positive number a, the solution of $|x| = a$ is

$x = a$ or $x = -a$

x can be a single variable or any algebraic expression.

Let’s look at another example.

Example

Solve for p.

$|2p-4|=26$

Solution

$2p-4=26 \text{ or } 2p-4=-26$	Write the two equations that will give an absolute value of 26.
$\begin{array}{rlcrl} 2p-4 & =26 & & 2p-4 & =-26 \\ +4 & +4 & & +4 & +4 \\ \hline \frac{2p}{2} & =\frac{30}{2} & & \frac{2p}{2} & =\frac{-22}{2} \\ p & =15 & \text{or} & p & =-11 \end{array}$	Solve each equation for p by isolating the variable.
Check $\begin{array}{rr} \|2p-4\|=26 & \|2p-4\|=26 \\ \|2(15)-4\|=26 & \|2(-11)-4\|=26 \\ \|30-4\|=26 & \|-22-4\|=26 \\ \|26\|=26 & \|-26\|=26 \end{array}$	Check the solutions in the original equation. Both solutions check!

$p=15 \text{ or } p=-11$

Sometimes, you have to isolate an absolute value before solving the equation. An example is shown below.

Example

Solve for w.

$3|4w-1|-5=10$

Solution

$\begin{array}{rl} 3\|4w-1\|-5 & =10 \\ +5 & +5 \\ \hline \frac{3\|4w-1\|}{3} & =\frac{15}{3} \end{array}$	Isolate the term with the absolute value by adding 5 to both sides. Divide both sides by 3.
$\|4w-1\|=5$	Now the absolute value is isolated.
$\begin{array}{rlcrl} 4w-1 & =5 & & 4w-1 & =-5 \\ +1 & +1 & \text{or} & +1 & +1 \\ \hline \frac{4w}{4} & =\frac{6}{4} & & \frac{4w}{4} & =\frac{-4}{4} \\ w & =\frac{3}{2} & & w & =-1 \end{array}$ $w=\frac{3}{2} \text{ or } -1$	Write the two equations that will give an absolute value of 5 and solve them.
Check $\begin{array}{rr} 3\|4w-1\|-5=10 & 3\|4w-1\|-5=10 \\ 3\left\|4\left(\frac{3}{2}\right)-1\right\|-5=10 & 3\|4w-1\|-5=10 \\ 3\left\|\frac{12}{2}-1\right\|-5=10 & 3\|4(-1)-1\|-5=10 \\ 3\|6-1\|-5=10 & 3\|-4-1\|-5=10 \\ 3(5)-5=10 & 3\|-5\|-5=10 \\ 15-5=10 & 15-5=10 \\ 10=10 & 10=10 \end{array}$	Check the solutions in the original equation. Both solutions check!

$w=-1 \text{ or } w=\frac{3}{2}$

Let’s take a look at one more example:

Solve for y:

$|3y-5|=-1$

Before removing the absolute value sign and making two equations, think about what this equation means. It reads “the absolute value of the quantity $3y$ minus 5 is equal to -1.” Remember that an absolute value is the distance from 0 on a number line, so it must be a positive number. Since it’s impossible to have an absolute value equal to -1, this equation has *no solution*. There are no values for y that will make this a true statement. There is no additional work needed to know this equation has no solutions.

Solving Inequalities Containing Absolute Values

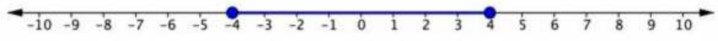
Let’s apply what you know about solving equations that contain absolute values and what you know about inequalities to solve inequalities that contain absolute values. Let’s start with a simple inequality.

$|x| \leq 4$

This inequality is read, “the absolute value of x is less than or equal to 4.” If you are asked to solve for x, you want to find out what values of x are 4 units or less away from 0 on a number line. You could start by thinking about the number line and what values of x would satisfy this equation.

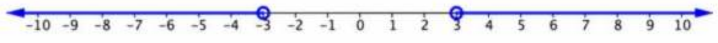
4 and -4 are both four units away from 0, so they are solutions. 3 and -3 are also solutions because each of these values is less than 4 units away from 0. So are 1 and -1, 0.5 and -0.5, and so on—there are an infinite number of values for x that will satisfy this inequality.

The graph of this inequality will have two closed circles, at 4 and -4. The distance between these two values on the number line is colored in blue because all of these values satisfy the equation.



The solution can be written this way: $-4 \leq x \leq 4$.

The situation is a little different when the inequality sign is “greater than” or “greater than or equal to.” Consider the simple inequality $|x|>3$. Again, you could think of the number line and what values of x are greater than 3 units away from zero. This time, 3 and -3 are not included in the solution, so there are open circles on both of these values. 2 and -2 would not be solutions because they are not more than 3 units away from 0. But, 5 and -5 would work and so would all of the values extending to the left of -3 and to the right of 3. The graph would look like the one below.



The solution to this inequality can be written this way: $x<-3$ or $x>3$.

Solving Absolute Value Inequalities

For any positive value of a:

$|x| \leq a$ is equivalent to $-a \leq x \leq a$ (this rule also applies for $|x|<a$)

$|x| \geq a$ is equivalent to $x \leq -a$ or $x \geq a$ (this rule also applies for $|x|>a$ |)

x can be a single variable or any algebraic expression.

Exercise

Solve the inequality: $|p| \leq 5$

A. $p \leq -5$ or $p \geq 5$
B. $-5 \leq p \leq 5$
C. $p \leq -5$
D. No solution

Answer

A. Incorrect. This is the solution to $|p| \geq 5$. For example, 7, which is greater than 5, is not a solution to the original inequality because $|7|$ is 7, which is not less than 5. The correct answer is $-5 \leq p \leq 5$.
B. Correct. The values of p that are greater than or equal to -5 and less than or equal to 5 satisfy this inequality.

C. Incorrect. If you substitute a value less than -5 into the original inequality, you get an untrue statement. For example, if $p=-10$, a value less than -5, then $|-10|\leq 5$. This is not true because the absolute value of -10 is 10 and 10 is not less than or equal to 5. The correct answer is $-5\leq p\leq 5$.

D. Incorrect. The values greater than or equal to -5 and less than or equal to 5 satisfy the inequality. The correct answer is $-5\leq p\leq 5$.

Let's look at a few more examples of inequalities containing absolute values.

Example

Solve for x.

$|x+3|>4$

Solution

$x+3<-4 \text{ or } x+3>4$ $\begin{array}{rrcrr} x+3< & -4 & & x+3> & 4 \\ -3 & -3 & & -3 & -3 \\ \hline x\quad < & -7 & \text{or} & x\quad > & 1 \end{array}$ $x<-7 \text{ or } x>1$	Since this is a "greater than" inequality, the solution can be rewritten according to the "greater than" rule. Solve each inequality.
$\begin{array}{rlrl} \vert x+3\vert > & 4 & \vert x+3\vert & >4 \\ \vert -7+3\vert = & 4 & \vert 1+3\vert & =4 \\ \vert -4\vert = & 4 & \vert 4\vert & =4 \\ 4= & 4 & 4 & =4 \end{array}$	Check the solutions in the original equation to be sure they work. Check the end point of the first related equation, -7. Try -10, a value less than -7, to check the inequality.
Check $\begin{array}{rlrl} \vert x+3\vert & >4 & \vert x+3\vert & >4 \\ \vert -10+3\vert & >4 & \vert 5+3\vert & >4 \\ \vert -7\vert & >4 & \vert 8\vert & >4 \\ 7 & >4 & 8 & >4 \end{array}$	Check the end point of the second related equation, 1. Try 5, a value greater than 1. Both solutions check!

$x<-7 \text{ or } x>1$

Example

Solve for y.

$3|2y+6|-9<27$

Solution

$\begin{array}{rrr} 3\vert 2y+6\vert -9< & & 27 \\ +9 & & +9 \\ \hline 3\vert 2y+6\vert & < & 36 \end{array} \quad (10.3.2.1)$	Begin to isolate the absolute value by adding 9 to both sides of the inequality.
$\frac{3\vert 2y+6\vert}{3}<\frac{36}{3}$	Divide both sides by 3 to isolate the absolute value.
$\vert 2y+6\vert <12$ $\begin{array}{rrrr} -12< & 2y+6< & & 12 \\ -6 & -6 & & -6 \\ \hline -18< & 2y & < & 6 \\ \frac{-18}{2}< & \frac{2y}{2} & < & \frac{6}{2} \\ -9< & y & < & 3 \end{array}$	Write the absolute value inequality using the "less than" rule. Subtract 6 from each part of the inequality. Divide by 2 to isolate the variable.

$-9<y<3$

As with equations, there may be instances in which there is no solution to an inequality.

Example

Solve for x.

$|2x+3|+9 \leq 7$

Solution

$\begin{array}{rr} \lvert 2x+3\rvert +9 \leq & 7 \\ -9 & -9 \\ \hline \lvert 2x+3\rvert \quad \leq & -2 \end{array}$	Isolate the absolute value by subtracting 9 from both sides of the inequality. The absolute value of a quantity can never be a negative number, so there is no solution to the inequality.

No solution

Summary

To solve an equation containing an absolute value, you want to isolate the absolute value expression. Once that is done, you can rewrite the absolute value equation as two equations, where one of the statements equates the value within the absolute value to the positive quantity on the other side of the equation and one that equates the value with the absolute value to the negative (or opposite) value.

Inequalities can also contain absolute values. Absolute inequalities also can be solved by rewriting them using compound inequalities.

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CHAPTER OVERVIEW

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SECTION OVERVIEW

11.1: Integer Exponents

Topic hierarchy

11.1.1: Exponential Notation

11.1.2: Simplify by Using the Product, Quotient, and Power Rules

11.1.3: Products and Quotients Raised to Powers

11.1.4: Scientific Notation

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11.1.1: Exponential Notation

Learning Objectives

- Evaluate expressions containing exponents.
- Evaluate exponential notations with exponents 0 of 1 and .
- Write an exponential expression involving negative exponents with positive exponents.

Introduction

A common language is needed in order to communicate mathematical ideas clearly and efficiently. **Exponential notation** is one example. It was developed to write repeated multiplication more efficiently. For example, growth occurs in living organisms by the division of cells. One type of cell divides 2 times in an hour. So in 12 hours, the cell will divide $2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2$ times. This can be written more efficiently as 2^{12}.

Exponential Vocabulary

We use exponential notation to write repeated multiplication, such as $10\cdot10\cdot10$ as 10^3. The 10 in 10^3 is called the **base**. The 3 in 10^3 is called the **exponent**. The expression 10^3 is called the exponential expression.

$\text{base} \rightarrow 10^{3 \leftarrow \text{exponent}}$

- 10^3 is read as “10 to the third power” or “10 raised to the power of 3” or “10 cubed.” It means $10\cdot10\cdot10$, or 1,000.
- 8^2 is read as “8 to the second power” or “8 raised to the power of 2” or “8 squared.” It means $8\cdot8$, or 64.
- 5^4 is read as “5 to the fourth power” or “5 raised to the power of 4.” It means $5\cdot5\cdot5\cdot5$, or 625.
- b^5 is read as “ b to the fifth power” or “ b raised to the power of 5.” It means $b\cdot b\cdot b\cdot b\cdot b$. Its value will depend on the value of b.

The exponent applies only to the number that it is next to. So in the expression xy^4, only the y is affected by the 4. xy^4 means $x\cdot y\cdot y\cdot y\cdot y$.

If the exponential expression is negative, such as -3^4, it means $-(3\cdot3\cdot3\cdot3)$ or -81.

If -3 is to be the base of the exponential expression, it must be written as $(-3)^4$, which means $-3\cdot-3\cdot-3\cdot-3$, or 81.

Likewise, $(-x)^4=(-x)\cdot(-x)\cdot(-x)\cdot(-x)=x^4$. while $-x^4=-(x\cdot x\cdot x\cdot x)$.

You can see that there is quite a difference, so you have to be very careful!

Evaluating Expressions Containing Exponents

Evaluating expressions containing exponents is the same as evaluating any expression. You substitute the value of the variable into the expression and simplify.

You can use PEMDAS to remember the order in which you should evaluate the expression. First, evaluate anything in Parentheses or grouping symbols. Next, look for Exponents, followed by Multiplication and Division (reading from left to right), and lastly, Addition and Subtraction (again, reading from left to right).

So, when you evaluate the expression $5x^3$ if $x=4$, first substitute the value 4 for the variable x. Then evaluate, using order of operations.

Example

Evaluate. $5x^3$ if $x=4$

Solution

$5\cdot4^3$	Substitute 4 for the variable x.
$5(4\cdot4\cdot4)=5\cdot64$	Evaluate 4^3.
320	Multiply.

$5x^3 = 320 \text{ when } x = 4$

Notice the difference between the example above and the one below.

✓ Example

Evaluate. $(5x)^3$ if $x = 4$

Solution

$(5 \cdot 4)^3$	Substitute 4 for the variable x.
20^3	Multiply.
$20 \cdot 20 \cdot 20 = 8,000$	Evaluate 20^3.

$(5x)^3 = 8,000 \text{ when } x = 4$

The addition of parentheses made quite a difference!

✓ Example

Evaluate. x^3 if $x = -4$

Solution

$(-4)^3$	Substitute -4 for the variable x.
$-4 \cdot -4 \cdot -4$	Evaluate.
$-4 \cdot -4 \cdot -4 = -64$	Multiply.

$x^3 = -64, \text{ when } x = -4$

? Exercise

Evaluate the expression $-(2x)^4, \text{ if } x = 3$.

A. 1,296
B. -1,296
C. 162
D. -162

Answer

A. Incorrect. Substitute the value of 3 for the variable x and evaluate $-(2 \cdot 3)^4$. Do not apply the negative sign until after you have evaluated the expression $(6)^4$. The correct answer is -1,296.
B. Correct. Substitute the value of 3 for the variable x and evaluate $-(2 \cdot 3)^4 = -6^4 = -1,296$.
C. Incorrect. Substitute the value of 3 for the variable x and evaluate $-(2 \cdot 3)^4$. Apply the exponent 4 to the product $2 \cdot 3$, or 6. Then apply the negative sign. The correct answer is -1,296.
D. Incorrect. Substitute the value of 3 for the variable x and evaluate $-(2 \cdot 3)^4$. Apply the exponent 4 to the product $2 \cdot 3$, or 6. Then apply the negative sign. The correct answer is -1,296.

Exponents of Zero and One

What does it mean when an exponent is 0 or 1? Let's consider 25^1. Any value raised to the power of 1 is just the value itself. This makes sense, because the exponent of 1 means the base is used as a **factor** only once. So the base stands alone, and 25^1 is simply 25.

But what about a value raised to the power of 0? Use what you know about powers of 10 to find out what the power of 0 means. Below is a list of powers of 10 and their equivalent values. Look at how the numbers change going down the left and right columns. Can you notice a pattern there?

Exponential Form	Expanded Form	Value
10^5	$10\cdot10\cdot10\cdot10\cdot10$	$100,000$
10^4	$10\cdot10\cdot10\cdot10$	$10,000$
10^3	$10\cdot10\cdot10$	$1,000$
10^2	$10\cdot10$	100
10^1	10	10

Moving down the table, each row drops one factor of 10 from the one above it. From row 1 to row 2, the exponential form goes from 10^5 to 10^4. The value drops from 100,000 to 10,000. Another way to put this is that each value is divided by 10 to produce the next value down the column.

Let's use this pattern of division by 10 to predict the value of 10^0.

Exponential Form	Expanded Form	Value
10^5	$10\cdot10\cdot10\cdot10\cdot10$	$100,000$
10^4	$10\cdot10\cdot10\cdot10$	$10,000$
10^3	$10\cdot10\cdot10$	$1,000$
10^2	$10\cdot10$	100
10^1	10	10
10^0	1	1

Following the pattern, you see that 10^0 is equal to 1. Would the pattern hold for a different base? Say a base of 3?

Exponential Form	Expanded Form	Value
3^5	$3\cdot3\cdot3\cdot3\cdot3$	243
3^4	$3\cdot3\cdot3\cdot3$	81
3^3	$3\cdot3\cdot3$	27
3^2	$3\cdot3$	9
3^1	3	3
3^0	1	1

Yes! And the same pattern would hold true for any non-zero number or variable raised to a power of 0, $n^0=1$.

There is a conflict when the base is 0. You know that $0^3=0$, $0^2=0$, and $0^1=0$, so you would expect 0^0 to also be equal to 0. However, the above pattern says that any base raised to the power of 0 is 1, so this leads you to believe that $0^0=1$. Notice the competing patterns: 0^0 cannot be both 0 and 1! In this case, mathematicians say that the value of 0^0 is *undefined*. (And remember that *undefined* is not the same as 0!)

Exponents of 0 or 1

Any number or variable raised to a power of 1 is the number itself. $n^1=n$

Any non-zero number or variable raised to a power of 0 is equal to 1. $n^0=1$

The quantity 0^0 is undefined.

✓ Example

Evaluate. $2x^0$ if $x=9$

Solution

$2\cdot9^0$	Substitute 9 for the variable x.
$2\cdot1$	Evaluate 9^0.
2	Multiply.

$2x^0 = 2, \text{ if } x = 9$

As done previously, to evaluate expressions containing exponents of 0 or 1, substitute the value of the variable into the expression and simplify.

? Exercise

Evaluate the expression $3x^0 - y^1, \text{ if } x = 12 \text{ and } y = -6$.

A. 42
B. -3
C. 9
D. 2

Answer

A. Incorrect. Substitute the value of 12 for the variable x and -6 for the variable y: $3 \cdot 12^0 - (-6)^1$. Remember that $12^0 = 1$. The correct answer is 9.
B. Incorrect. Substitute the value of 12 for the variable x and -6 for the variable y: $3 \cdot 12^0 - (-6)^1$. Remember that $(-6)^1 = -6$ and $-(-6) = 6$. The correct answer is 9.
C. Correct. $3 \cdot 12^0 - (-6)^1 = 3 \cdot 1 - (-6)^1 = 3 \cdot 1 + 6 = 9$
D. Incorrect. Substitute the value of 12 for the variable x and -6 for the variable y: $3 \cdot 12^0 - (-6)^1$. Remember that a base raised to the power of 1 is the base. The correct answer is 9.

Negative Exponents

What does it mean when an exponent is a negative integer? Let's use the powers of 10 pattern from earlier to find out. If you continue this pattern to add some more rows, beyond 10^0, you find the following:

Exponential Form	Expanded Form	Value
10^5	$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$	$100,000$
10^4	$10 \cdot 10 \cdot 10 \cdot 10$	$10,000$
10^3	$10 \cdot 10 \cdot 10$	$1,000$
10^2	$10 \cdot 10$	100
10^1	10	10
10^0	1	1
10^{-1}	$\frac{1}{10^1}$	$\frac{1}{10}$
10^{-2}	$\frac{1}{10^2}$	$\frac{1}{100}$

Following the pattern, you see that 10^0 is equal to 1. Then you get into negative exponents: 10^{-1} is equal to $\frac{1}{10^1}$, and 10^{-2} is the same as (\ \frac{1}{10^{2}}\).

Following this pattern, a number with a negative exponent can be rewritten as the reciprocal of the original number, with a positive exponent.

For example, $10^{-3} = \frac{1}{10^3}$ and $10^{-7} = \frac{1}{10^7}$.

To see if these patterns hold true for numbers other than 10, check out this table with powers of 3.

Exponential Form	Expanded Form	Value
3^5	$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$	243
3^4	$3 \cdot 3 \cdot 3 \cdot 3$	81
3^3	$3 \cdot 3 \cdot 3$	27
3^2	$3 \cdot 3$	9
3^1	3	3
3^0	1	1
3^{-1}	$\frac{1}{3}$	$\frac{1}{3}$
3^{-2}	$\frac{1}{3^2}$ or $\frac{1}{3 \cdot 3}$	$\frac{1}{9}$

The numbers are different but the patterns are the same. We are now ready to state the definition of a negative exponent.

Negative Exponent

For any non-zero number n and any integer x, $n^{-x}=\frac{1}{n^x}$. For example, $5^{-2}=\frac{1}{5^2}$.

Note that the definition above states that the base, n must be a "non-zero number."

Exercise

Evaluate the expression $(x^{-2})\cdot(x^0)$ when $x=6$.

A. $\frac{1}{36}$
B. $\frac{1}{6}$
C. 0
D. 36

Answer

A. Correct. Substitute the value of 6 for the variable x and evaluate. $6^{-2}\cdot 6^0=\frac{1}{6^2}\cdot 6^0=\frac{1}{36}\cdot 1=\frac{1}{36}$.
B. Incorrect. Substitute the value of 6 for the variable x and evaluate: $6^{-2}\cdot 6^0$. Remember that $(6)^0=1$. The correct answer is $\frac{1}{36}$.
C. Incorrect. Substitute the value of 6 for the variable x and evaluate: $6^{-2}\cdot 6^0$. Remember that $(6)^0=1$. The correct answer is $\frac{1}{36}$.
D. Incorrect. Substitute the value of 6 for the variable x and then evaluate: $6^{-2}\cdot 6^0$ Remember that $(6)^{-2}=\frac{1}{6^2}=\frac{1}{36}$. The correct answer is $\frac{1}{36}$.

Summary

Exponential notation is composed of a base and an exponent. It is a "shorthand" way of writing repeated multiplication, and indicates that the base is a factor and the exponent is the number of times the factor is used in the multiplication. The basic rules of exponents are as follows:

- An exponent applies only to the value to its immediate left.
- When a quantity in parentheses is raised to a power, the exponent applies to everything inside the parentheses.
- For any non-zero number n, $n^0=1$.
- For any non-zero number n and any integer x, $n^{-x}=\frac{1}{n^x}$.

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11.1.2: Simplify by Using the Product, Quotient, and Power Rules

Learning Objectives

- Use the product rule to multiply exponential expressions with like bases.
- Use the power rule to raise powers to powers.
- Use the quotient rule to divide exponential expressions with like bases.
- Simplify expressions using a combination of the properties.

Introduction

Exponential notation was developed to write repeated multiplication more efficiently. There are times when it is easier to leave the expressions in exponential notation when multiplying or dividing. Let's look at rules that will allow you to do this.

The Product Rule for Exponents

Recall that exponents are a way of representing repeated multiplication. For example, the notation 5^4 can be expanded and written as $5 \cdot 5 \cdot 5 \cdot 5$, or 625. And don't forget, the exponent only applies to the number immediately to its left, unless there are parentheses.

What happens if you multiply two numbers in exponential form with the same base? Consider the expression $\left(2^3\right)\left(2^4\right)$. Expanding each exponent, this can be rewritten as $(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2)$ or $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. In exponential form, you would write the product as 2^7. Notice, 7 is the sum of the original two exponents, 3 and 4.

What about $\left(x^2\right)\left(x^6\right)$? This can be written as $(x \cdot x)(x \cdot x \cdot x \cdot x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$ or x^8. And, once again, 8 is the sum of the original two exponents.

The Product Rule for Exponents

For any number x and any integers a and b, $\left(x^a\right)\left(x^b\right) = x^{a+b}$.

To multiply exponential terms with the same base, simply add the exponents.

Example

Simplify. $\left(a^3\right)\left(a^7\right)$

Solution

$\left(a^3\right)\left(a^7\right)$	The base of both exponents is a, so the product rule applies.
a^{3+7}	Add the exponents with a common base.

$\left(a^3\right)\left(a^7\right) = a^{10}$

When multiplying more complicated terms, multiply the coefficients and then multiply the variables.

Example

Simplify. $5a^4 \cdot 7a^6$

Solution

$35 \cdot a^4 \cdot a^6$	Multiply the coefficients.
$35 \cdot a^{4+6}$	The base of both exponents is a, so the product rule applies. Add the exponents.
$35 \cdot a^{10}$	Add the exponents with a common base.

$5a^4 \cdot 7a^6 = 35a^{10}$

? Exercise

Simplify the expression, keeping the answer in exponential notation.

$(4x^5)(2x^8)$

A. $8x^5 \cdot x^8$
B. $6x^{13}$
C. $8x^{13}$
D. $8x^{40}$

Answer

A. Incorrect. $8x^5 \cdot x^8$ is equivalent to $(4x^5)(2x^8)$, but it still is not in simplest form. Simplify $x^5 \cdot x^8$ by using the Product Rule to add exponents. The correct answer is $8x^{13}$.
B. Incorrect. $6x^{13}$ is not equivalent to $(4x^5)(2x^8)$. In this incorrect response, the correct exponents were added, but the coefficients were also added together. They should have been multiplied. The correct answer is $8x^{13}$.
C. Correct. $8x^{13}$ is equivalent to $(4x^5)(2x^8)$. Multiply the coefficients $(4 \cdot 2)$ and apply the Product Rule to add the exponents of the variables (in this case x) that are the same.
D. Incorrect. $8x^{40}$ is not equivalent to $(4x^5)(2x^8)$. Do not multiply the coefficients *and* the exponents. Remember, using the Product Rule add the exponents when the bases are the same. The correct answer is $8x^{13}$.

The Power Rule for Exponents

Let's simplify $(5^2)^4$. In this case, the base is 5^2 and the exponent is 4, so you multiply 5^2 four times: $(5^2)^4 = 5^2 \cdot 5^2 \cdot 5^2 \cdot 5^2 = 5^8$ (Using the Product Rule, add the exponents).

$(5^2)^4$ is a power of a power. It is the fourth power of 5 to the second power. And we saw above that the answer is 5^8. Notice that the new exponent is the same as the product of the original exponents: $2 \cdot 4 = 8$.

So, $(5^2)^4 = 5^{2 \cdot 4} = 5^8$ (which equals 390, 625, if you do the multiplication).

Likewise, $(x^4)^3 = x^{4 \cdot 3} = x^{12}$.

This leads to another rule for exponents: the **Power Rule for Exponents**. To simplify a power of a power, you multiply the exponents, keeping the base the same. For example, $(2^3)^5 = 2^{15}$.

The Power Rule for Exponents

For any positive number x and integers a and b:

$(x^a)^b = x^{a \cdot b}$

✓ Example

Simplify. $6(c^4)^2$

Solution

$6(c^4)^2$	Since you are raising a power to a power, apply the Power Rule and multiply exponents to simplify. The coefficient remains unchanged because it is outside of the parentheses.

$6(c^4)^2 = 6c^8$

Example

Simplify. $a^2\left(a^5\right)^3$

Solution

$a^2a^{5\cdot 3}$	Raise a^5 to the power of 3 by multiplying the exponents together (the Power Rule).
a^2a^{15} a^{2+15}	Since the exponents share the same base, a, they can be combined (the Product Rule).

$a^2\left(a^5\right)^3=a^{17}$

Exercise

Simplify: $-a\left(a^2\right)^4$

A. $-a\cdot a^8$
B. a^7
C. $-a^7$
D. $-a^9$

Answer

A. Incorrect. This expression is not simplified yet. Recall that $-a$ can also be written $-a^1$. Multiply $-a^1$ by a^8 to arrive at the correct answer. The correct answer is $-a^9$.
B. Incorrect. Do not add the exponents of 2 and 4 together. The Power Rule states that for a power of a power, you multiply the exponents. The correct answer is $-a^9$.
C. Incorrect. Do not add the exponents of 2 and 4 together. The Power Rule states that for a power of a power, you multiply the exponents. The correct answer is $-a^9$.
D. Correct. Using the Power Rule, $-a\left(a^2\right)^4=-a\cdot a^{2\cdot 4}=-a^1a^8=-a^{1+8}=-a^9$.

The Quotient Rule for Exponents

Let’s look at dividing terms containing exponential expressions. What happens if you divide two numbers in exponential form with the same base? Consider the following expression.

$\frac{4^5}{4^2}$

You can rewrite the expression as: $\frac{4\cdot 4\cdot 4\cdot 4\cdot 4}{4\cdot 4}$. Then you can cancel the common factors of 4 in the numerator and denominator: $\frac{\cancel{4}\cdot\cancel{4}\cdot 4\cdot 4\cdot 4}{\cancel{4}\cdot\cancel{4}}$ $=\frac{4\cdot 4\cdot 4}{1}$.

Finally, this expression can be rewritten as 4^3 using exponential notation. Notice that the exponent, 3, is the difference between the two exponents in the original expression, 5 and 2.

So, $\frac{4^5}{4^2}=4^{5-2}=4^3$.

Be careful that you subtract the exponent in the denominator from the exponent in the numerator.

$\frac{x^7}{x^9}=\frac{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}}{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot x\cdot x}=\frac{1}{x\cdot x}=\frac{1}{x^2}=x^{-2}$

or

$\frac{x^7}{x^9}=x^{7-9}=x^{-2}$

So, to divide two exponential terms with the same base, subtract the exponents.

The Quotient Rule for Exponents

For any non-zero number x and any integers a and b: $\frac{x^a}{x^b} = x^{a-b}$

Notice that $\frac{4^2}{4^2} = 4^0$. And we know that $\frac{4^2}{4^2} = \frac{16}{16} = 1$. So this may help to explain why $4^0 = 1$.

Example

Evaluate. $\frac{4^9}{4^4}$

Solution

4^{9-4}	These two exponents have the same base, 4. According to the Quotient Rule, you can subtract the power in the denominator from the power in the numerator.

$\frac{4^9}{4^4} = 4^5$

When dividing terms that also contain coefficients, divide the coefficients and then divide variable powers with the same base by subtracting the exponents.

Example

Simplify. $\frac{12x^4}{2x}$

Solution

$\left(\frac{12}{2}\right)\left(\frac{x^4}{x}\right)$	Separate into numerical and variable factors.
$6 \cdot x^{4-1}$	Since the bases of the exponents are the same, you can apply the Quotient Rule. Divide the coefficients and subtract the exponents of matching variables.

$\frac{12x^4}{2^x} = 6x^3$

Applying the Rules

All of these rules of exponents, the Product Rule, the Power Rule, and the Quotient Rule, are helpful when evaluating expressions with common bases.

Example

Evaluate $\frac{24x^8}{2x^5}$ when $x = 4$.

Solution

$\left(\frac{24}{2}\right)\left(\frac{x^8}{x^5}\right)$	Separate into numerical and variable factors.
$12 \cdot x^{8-5}$	Divide coefficients, and subtract the exponents of the variables.
$12x^3$	Simplify.
$(12)\ (4^3) = 12 \cdot 64$	Substitute the value 4 for the variable x.

$\frac{24x^8}{2x^5} = 768$

Usually, it is easier to simplify the expression before substituting any values for your variables, but you will get the same answer either way.

Example

Simplify. $\frac{a^2(a^5)^3}{8a^8}$

Solution

$\frac{a^2 \cdot a^{5 \cdot 3}}{8a^8}$	Use the order of operations with PEMDAS: E: Evaluate exponents. Use the Power Rule to simplify $(a^5)^3$
$\frac{a^2 a^{15}}{8a^8}$ $\frac{a^{2+15}}{8a^8}$ $\frac{a^{17}}{8a^8}$ $\frac{a^{17-8}}{8}$	M: Multiply, using the Product Rule as the bases are the same. D: Divide using the Quotient Rule.

$\frac{a^2(a^5)^3}{8a^8} = \frac{a^9}{8}$

Summary

There are rules that help when multiplying and dividing exponential expressions with the same base. To multiply two exponential terms with the same base, add their exponents. To raise a power to a power, multiply the exponents. To divide two exponential terms with the same base, subtract the exponents.

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11.1.3: Products and Quotients Raised to Powers

Learning Objectives

- Raise a product to a power.
- Raise a quotient to a power.
- Simplify expressions using a combination of the properties.

Introduction

The rules of **exponents** are very useful when simplifying and evaluating expressions. When multiplying, dividing, or raising a power to a power, using the rules for exponents helps to make the process more efficient. Now let’s look at rules for taking a product or a quotient to a power.

A Product Raised to a Power

Once the rules of exponents are understood, you can begin solving more complicated expressions more easily. Recall that when you take a power to a power, you multiply the exponents, $(x^a)^b = x^{a \cdot b}$.

What happens when you raise an entire *expression* inside parentheses to a power? You can use the techniques you already know to simplify this expression.

$$(2a)^4 = (2a)(2a)(2a)(2a) = (2 \cdot 2 \cdot 2 \cdot 2)(a \cdot a \cdot a \cdot a) = \left(2^4\right)\left(a^4\right) = 16a^4$$

Notice that the exponent is applied to each factor of $2a$. So, we can eliminate the middle steps.

$$\begin{aligned} (2a)^4 & = \left(2^4\right)\left(a^4\right) \text{, applying the } 4 \text{ to each factor, } 2 \text{ and } a \\ & = 16a^4 \end{aligned}$$

The product of two or more numbers raised to a power is equal to the product of each number raised to the same power.

A Product Raised to a Power

For any nonzero numbers a and b and any integer x, $(ab)^x = a^x \cdot b^x$

Caution! Do not try to apply this rule to sums. Think about the expression $(2+3)^2$. Does $(2+3)^2$ equal 2^2+3^2? No, it does not. $(2+3)^2 = 5^2 = 25$ and $2^2+3^2 = 4+9 = 13$. So, you can only use this rule when the numbers inside the parentheses are being multiplied (or divided, as we will see next).

Example

Simplify. $(2yz)^6$

Solution

$2^6y^6z^6$	Apply the exponent to each number in the product.

$(2yz)^6 = 64y^6z^6$

If the variable has an exponent with it, use the Power Rule: multiply the exponents.

Example

Simplify. $\left(-7a^4b\right)^2$

Solution

$(-7)^2\left(a^4\right)^2(b)^2$	Apply the exponent 2 to each factor within the parentheses.
$49a^{4 \cdot 2}b^2$	Square the coefficient and use the Power Rule to square $\left(a^4\right)^2$.
$49a^8b^2$	Simplify.

$(-7a^4b)^2 = 49a^8b^2$

Exercise

Simplify the expression. $(-3x^3y^2)^4$

A. x^4y^4
B. $-81xy^8$
C. $81x^{12}y^8$
D. $81x^7y^6$

Answer

A. Incorrect. Remember to apply the exponent to all terms within the parentheses, including the coefficient. $(-3x^3y^2)^4 = (-3)^4(x^3)^4(y^2)^4$; the correct answer is $81x^{12}y^8$.
B. Incorrect. Remember to apply the exponent to all terms within the parentheses, including the variable x. $(-3x^3y^2)^4 = (-3)^4(x^3)^4(y^2)^4$; the correct answer is $81x^{12}y^8$.
C. Correct. $(3x^3y^2)^4 = (-3)^4(x^3)^4(y^2)^4 = 81x^{12}y^8$.
D. Incorrect. Remember to multiply exponents (not add them) when you are raising a power to a power. In this case, $(-3x^3y^2)^4 = (-3)^4(x^3)^4(y^2)^4 = (-3)^4 \cdot x^{3\cdot 4} \cdot y^{2\cdot 4}$. The correct answer is $81x^{12}y^8$.

A Quotient Raised to a Power

Now let’s look at what happens if you raise a quotient to a power. Suppose you have $\frac{3}{4}$ and raise it to the 3rd power.

$$\left(\frac{3}{4}\right)^3 = \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = \frac{3\cdot 3\cdot 3}{4\cdot 4\cdot 4} = \frac{3^3}{4^3}$$

You can see that raising the quotient to the power of 3 can also be written as the numerator (3) to the power of 3, and the denominator (4) to the power of 3.

Similarly, if you are using variables, the quotient raised to a power is equal to the numerator raised to the power over the denominator raised to power.

$$\left(\frac{a}{b}\right)^4 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{a\cdot a\cdot a\cdot a}{b\cdot b\cdot b\cdot b} = \frac{a^4}{b^4}$$

When a quotient is raised to a power, you can apply the power to the numerator and denominator individually, as shown below.

$$\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$$

A Quotient Raised to a Power

For any number a, any non-zero number b, and any integer x, $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$.

Example

Simplify. $\left(\frac{2xy}{x}\right)^3$

Solution

$\frac{2^3x^3y^3}{x^3}$	Apply the power to each factor individually.
$2^3 \cdot \frac{x^3}{x^3} \cdot \frac{y^3}{1}$	Separate into numerical and variable factors.
$8 \cdot x^{(3-3)} \cdot y^3$	Simplify by taking 2 to the third power and applying the Quotient Rule for exponents: subtract the exponents of matching variables.
$8x^0y^3$	Simplify. Remember that $x^0 = 1$.

$$\left(\frac{2xy}{x}\right)^3 = 8y^3$$

Exercise

Simplify the expression. $\frac{(abc)^2}{(ab)^2}$

A. 1
B. c^2
C. $(abc)^2(ab)^{-2}$
D. $\frac{a^2b^2c^2}{a^2b^2}$

Answer

A. Incorrect. Follow the order of operations. Apply the exponent to each variable in the product first, and then use the Quotient Rule: subtract the exponents in terms that have the same base. The correct answer is c^2.
B. Correct. Apply the exponent to each variable in the product and then use the Quotient Rule: subtract the exponents in terms that have the same base.
C. Incorrect. Although $(abc)^2(ab)^{-2}$ is equivalent to $\frac{(abc)^2}{(ab)^2}$, it is not in simplest form. Apply the exponent to each variable in the product and then use the Quotient Rule: subtract the exponents in terms that have the same base. The correct answer is c^2.
D. Incorrect. This is an equivalent expression but not in simplest form. Use the Quotient Rule: subtract the exponents in terms that have the same base. The correct answer is c^2.

Simplifying and Evaluating Expressions with Exponents

Now let’s look at some more complex expressions and see how all the rules can help make working with exponents easier. While these expressions may look complicated at first, just keep following the rules of exponents to simplify them!

Example

Simplify. $(4x^3)^5 \cdot (2x^2)^{-4}$

Solution

$\frac{(4x^3)^5}{(2x^2)^4}$	Recall that with negative exponents, you can use the reciprocal, and move it to the denominator with a positive exponent.
$\frac{4^5(x^{3\cdot5})}{2^4(x^{2\cdot4})}$ $\frac{1{,}024x^{15}}{16x^8}$	Separate into numerical and variable factors and apply the exponent to each term.
$64x^{15-8}$	Divide 1,024 by 16, and subtract exponents using the Quotient Rule.

$64x^7$

By using the rules of exponents, you’ve now got an equivalent expression, $64x^7$, which is much easier to work with than $(4x^3)^5 \cdot (2x^2)^{-4}$.

The rules of exponents are also helpful when evaluating expressions for a certain value of a variable.

Example

Evaluate $\frac{24x^8y^2}{(2x^3y)^2}$ when $x = 4$ and $y = -2$.

Solution

$\frac{24x^8y^2}{2^2x^{3\cdot2}y^2}$	In the denominator, notice that a product is being raised to a power. Use the rules of exponents to simplify the denominator.
$\frac{24x^8y^2}{4x^6y^2}$	Simplify 2^2 and multiply the exponents of x.
$\left(\frac{24}{4}\right)\left(\frac{x^8}{x^6}\right)\left(\frac{y^2}{y^2}\right)$	Separate into numerical and variable factors.

$6\left(x^{8-6}\right)\left(y^{2-2}\right)$	Divide coefficients, use the Quotient Rule to divide the variables: subtract the exponents.
$6x^2y^0 = 6x^2$	Simplify. Remember that y^0 is 1.
$(6)\ \left(4^2\right) = 6 \cdot 16$	Substitute the value 4 for the variable x.

$\frac{24x^8y^2}{\left(2x^3y\right)^2} = 96$ when $x = 4$ and $y = -2$

Notice that you could have worked this problem by substituting 4 for x and 2 for y in the original expression. You would still get the answer of 96, but the computation would be much more complex. Notice that you didn't even need to use the value of y to evaluate the above expression.

? Exercise

Simplify. $\left(3a^2b^3\right)^2(-5ab)$

A. $\left(9a^4b^6\right)(-5ab)$
B. $-15a^5b^7$
C. $-45a^5b^6$
D. $-45a^5b^7$

Answer

A. Incorrect. The first term has been raised to a power, but you can simplify further. The two factors haven't been multiplied together. The correct answer is $-45a^5b^7$.
B. Incorrect. The coefficient of the first term must also be raised to a power. The correct answer is $-45a^5b^7$.
C. Incorrect. When you take a power of a power, the exponents are multiplied. The correct answer is $-45a^5b^7$.
D. Correct. In your answer, each variable appears only once, coefficients have been multiplied, and there are no powers of powers. $\left(3a^2b^3\right)^2(-5ab) = \left(3^2a^{2\cdot 2}b^{3\cdot 2}\right)(-5ab) = \left(9a^4b^6\right)(-5ab)$, which simplifies to $-45a^5b^7$.

Summary

There are many rules to use when working with exponential expressions. You can use these rules, as well as the definition of exponents, to simplify complex expressions. Simplifying an expression before evaluating the expression can often make the computation easier.

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11.1.4: Scientific Notation

Learning Objectives

- Convert between scientific and decimal notation.
- Multiply and divide using scientific notation.
- Solve application problems.

Introduction

When working with very large or very small numbers, scientists, mathematicians, and engineers often use **scientific notation** to express those quantities. Scientific notation uses exponential notation. The following are examples of scientific notation.

Light year: number of miles light travels in one year, about 5,880,000,000,000

Scientific notation is 5.88×10^{12} miles.

hydrogen atom: has a diameter of about 0.00000005 millimeters

Scientific notation is 5×10^{-8} millimeters

Computation with very large numbers is made easier with scientific notation.

Learning to Use Scientific Notation

When a number is written in scientific notation, the **exponent** tells you if the term is a large or a small number. A positive exponent indicates a large number and a negative exponent indicates a small number that is between 0 and 1.

Since it's so useful, let's look more closely at the details of scientific notation format.

Scientific Notation

A positive number is written in scientific notation if it is written as $a \times 10^n$ where the coefficient a has a value such that $1 \leq a < 10$ and n is an integer.

Look at the numbers below. Which of the numbers is written in scientific notation?

Number	Scientific Notation?	Explanation
1.85×10^{-2}	yes	$1 \leq 1.85 < 10$ -2 is an integer
$1.083 \times 10^{\frac{1}{2}}$	no	$\frac{1}{2}$ is not an integer
0.82×10^{14}	no	0.82 is not ≥ 1
10×10^3	no	10 is not < 10

Exercise

Which number below is written in scientific notation?

A. $4.25 \times 10^{0.08}$
B. 0.425×10^7
C. 42.5×10^5
D. 4.25×10^6

Answer

A. Incorrect. The exponent must be an integer and 0.08 is not an integer. The correct answer is 4.25×10^6.
B. Incorrect. This is not in scientific notation because 0.425 is less than 1. The correct answer is 4.25×10^6.
C. Incorrect. This is not in scientific notation because 42.5 is greater than 10. The correct answer is 4.25×10^6.

D. Correct. This is scientific notation. 4.25 is greater than 1 and less than 10, and 6 is an integer.

Writing Decimal Notation in Scientific Notation

Now let’s compare some numbers expressed in both scientific notation and standard decimal notation in order to understand how to convert from one form to the other. Take a look at the tables below. Pay close attention to the exponent in the scientific notation and the position of the decimal point in the decimal notation.

Large Numbers

Decimal Notation	**Scientific Notation**
500.0	5×10^{2}
80,000.0	8×10^{4}
$43,000,000.0$	4.3×10^{7}
$62,500,000,000.0$	6.25×10^{10}

Small Numbers

Decimal Notation	**Scientific Notation**
0.05	5×10^{-2}
0.0008	8×10^{-4}
0.00000043	4.3×10^{-7}
0.000000000625	6.25×10^{-10}

To write a *large* number in scientific notation, move the decimal point to the left to obtain a number between 1 and 10. Since moving the decimal point changes the value, you have to multiply the decimal by a power of 10 so that the expression has the same value.

Let’s look at an example.

$$\begin{aligned}180,000. &= 18,000.0\times10^{1}\\ &\quad 1,800.00\times10^{2}\\ &\quad 180.000\times10^{3}\\ &\quad 18.0000\times10^{4}\\ &\quad 1.80000\times10^{5}\end{aligned}$$

Notice that the decimal point was moved 5 places to the left, and the exponent is 5.

? Exercise

The world population is estimated to be about 6,800,000,000 people. Which answer expresses this number in scientific notation?

A. 7×10^{9}
B. 0.68×10^{10}
C. 6.8×10^{9}
D. 68×10^{8}

Answer

A. Incorrect. Scientific notation rewrites numbers, it doesn’t round them. The correct answer is 6.8×10^{9}.
B. Incorrect. Although 0.68×10^{10} is equivalent to 6,800,000,000, 0.68 is not the form for scientific notation as 0.68 is not a number between 1 and 10. The correct answer is 6.8×10^{9}.
C. Correct. The number 6.8×10^{9} is equivalent to 6,800,000,000 and uses the proper format for each factor.

D. Incorrect. Although 68×10^8 is equivalent to 6,800,000,000, it is not written in scientific notation as 68 is not between 1 and 10. The correct answer is 6.8×10^9.

? Advanced Question

Represent 1.00357×10^{-6} in decimal form.

A. 1.00357000000
B. 0.000100357
C. 0.000001357
D. 0.00000100357

Answer

A. Incorrect. You added six 0s to the end of the decimal. This does not change its value at all; 1.00357000000=1.00357, which is not equal to 1.00357×10^{-6}. The correct answer is 0.00000100357.
B. Incorrect. You moved the decimal point in the correct direction, but you did not move it enough places. The correct answer is 0.00000100357.
C. Incorrect. You moved the decimal point the correct number of spaces, but the number you created is different than the number you started with: $1.00357 \times 10^{-6} \neq 0.000001357$. Remember that the zeros in between 1 and 3 must also be included in the final number. The correct answer is 0.00000100357.
D. Correct. The exponent is -6. You moved the decimal point 6 spots to the left, creating the decimal 0.00000100357.

To write a small number (between 0 and 1) in scientific notation, you move the decimal to the *right* and the exponent will have to be *negative*.

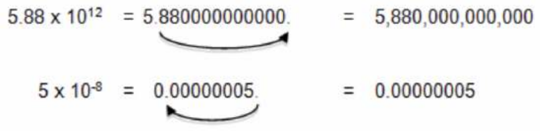
$$\begin{aligned} 0.00004 &= 00.0004 \times 10^{-1} \\ &\quad 000.004 \times 10^{-2} \\ &\quad 0000.04 \times 10^{-3} \\ &\quad 00000.4 \times 10^{-4} \\ &\quad 000004. \times 10^{-5} \end{aligned}$$

$$0.00004 = 4 \times 10^{-5}$$

You may notice that the decimal point was moved five places to the *right* until you got the number 4, which is between 1 and 10. The exponent is -5.

Writing Scientific Notation in Decimal Notation

You can also write scientific notation as decimal notation. For example, the number of miles that light travels in a year is 5.88×10^{12}. 5.88×10^{12}, and a hydrogen atom has a diameter of 5×10^{-8}mm. To write each of these numbers in decimal notation, you move the decimal point the same number of places as the exponent. If the exponent is *positive*, move the decimal point to the right. If the exponent is *negative*, move the decimal point to the left.



$5.88 \times 10^{12} = 5.880000000000. = 5{,}880{,}000{,}000{,}000$

$5 \times 10^{-8} = 0.00000005. = 0.00000005$

For each power of 10, you move the decimal point one place. Be careful here and don't get carried away with the zeros; the number of zeros after the decimal point will always be 1 *less* than the exponent because it takes one power of 10 to shift that first number to the left of the decimal.

? Exercise

Rewrite 1.57×10^{-10} in decimal notation.

A. 15,700,000,000

B. 0.000000000157
C. 0.0000000000157
D. 157×10^{-12}

Answer

A. Incorrect. You moved the decimal point in the wrong direction. The exponent is negative, so to convert to decimal format, move the decimal point to the left, not the right. The correct answer is 0.000000000157.
B. Correct. The expression has a negative exponent, so you move the decimal point 10 places to the left to convert it to decimal notation, 0.000000000157.
C. Incorrect. You inserted 10 zeros between the number and the decimal point. Move the decimal point 10 places to the left instead. The correct answer is 0.000000000157.
D. Incorrect. This number is equivalent to the original number, but it is not in decimal notation. The correct answer is 0.000000000157.

Multiplying and Dividing Numbers Expressed in Scientific Notation

Numbers that are written in scientific notation can be multiplied and divided rather simply by taking advantage of the properties of numbers and the rules of exponents that you may recall. To multiply numbers in scientific notation, first multiply the numbers that aren't powers of 10 (the a in $a \times 10^{n}$). Then multiply the powers of ten by adding the exponents.

This will produce a new number times a different power of 10. All you have to do is check to make sure this new value is in scientific notation. If it isn't, you convert it.

Let's look at some examples.

Example

$\left(3 \times 10^{8}\right)\left(6.8 \times 10^{-13}\right)$.

Solution

$(3 \times 6.8)\left(10^{8} \times 10^{-13}\right)$	Regroup, using the commutative and associative properties.
$(20.4)\left(10^{8} \times 10^{-13}\right)$	Multiply the coefficients.
20.4×10^{-5}	Multiply the powers of 10 using the Product Rule: add the exponents.
$\left(2.04 \times 10^{1}\right) \times 10^{-5}$	Convert 20.4 into scientific notation by moving the decimal point one place to the left and multiplying by 10^{1}.
$2.04 \times \left(10^{1} \times 10^{-5}\right)$	Group the powers of 10 using the associative property of multiplication.
$2.04 \times 10^{1+(-5)}$	Multiply using the Product Rule: add the exponents.

$\left(3 \times 10^{8}\right)\left(6.8 \times 10^{-13}\right)=2.04 \times 10^{-4}$

Advanced Example

$\left(8.2 \times 10^{6}\right)\left(1.5 \times 10^{-3}\right)\left(1.9 \times 10^{-7}\right)$

Solution

$(8.2 \times 1.5 \times 1.9)\left(10^{6} \times 10^{-3} \times 10^{-7}\right)$	Regroup, using the commutative and associative properties.
$(23.37)\left(10^{6} \times 10^{-3} \times 10^{-7}\right)$	Multiply the numbers.
23.37×10^{-4}	Multiply the powers of 10 using the Product Rule: add the exponents.

$\left(2.337 \times 10^{1}\right) \times 10^{-4}$	Convert 23.37 into scientific notation by moving the decimal point one place to the left and multiplying by 10^1.
$2.337 \times \left(10^{1} \times 10^{-4}\right)$	Group the powers of 10 using the associative property of multiplication.
$2.337 \times 10^{1+(-4)}$	Multiply using the Product Rule and add the exponents.

$$\left(8.2 \times 10^{6}\right)\left(1.5 \times 10^{-3}\right)\left(1.9 \times 10^{-7}\right) = 2.337 \times 10^{-3}$$

In order to divide numbers in scientific notation, you once again apply the properties of numbers and the rules of exponents. You begin by dividing the numbers that aren't powers of 10 (the a in $a \times 10^{n}$). Then you divide the powers of 10 by subtracting the exponents.

This will produce a new number times a different power of 10. If it isn't already in scientific notation, you convert it, and then you're done.

Let's look at some examples.

✓ Example

$$\frac{2.829 \times 10^{-9}}{3.45 \times 10^{-3}}$$

Solution

$\left(\frac{2.829}{3.45}\right)\left(\frac{10^{-9}}{10^{-3}}\right)$	Regroup, using the associative property.
$(0.82)\left(\frac{10^{-9}}{10^{-3}}\right)$	Divide the coefficients.
$0.82 \times 10^{-9-(-3)}$ 0.82×10^{-6}	Divide the powers of 10 using the Quotient Rule: subtract the exponents.
$\left(8.2 \times 10^{-1}\right) \times 10^{-6}$	Convert 0.82 into scientific notation by moving the decimal point one place to the right and multiplying by 10^{-1}.
$8.2 \times \left(10^{-1} \times 10^{-6}\right)$	Group the powers of 10 together using the associative property.
$8.2 \times 10^{-1+(-6)}$	Multiply the powers of 10 using the Product Rule: add the exponents.

$$\frac{2.829 \times 10^{-9}}{3.45 \times 10^{-3}} = 8.2 \times 10^{-7}$$

✓ Advanced Example

$$\frac{\left(1.37 \times 10^{4}\right)\left(9.85 \times 10^{6}\right)}{5.0 \times 10^{12}}$$

Solution

$\frac{(1.37 \times 9.85)\left(10^{6} \times 10^{4}\right)}{5.0 \times 10^{12}}$	Regroup the terms in the numerator according to the associative and commutative properties.
$\frac{13.4945 \times 10^{10}}{5.0 \times 10^{12}}$	Multiply.
$\left(\frac{13.4945}{5.0}\right)\left(\frac{10^{10}}{10^{12}}\right)$	Regroup, using the associative property.
$(2.6989)\left(\frac{10^{10}}{10^{12}}\right)$	Divide the numbers.
$(2.6989)\left(10^{10-12}\right)$ 2.6989×10^{-2}	Divide the powers of 10 using the Quotient Rule: subtract the exponents.

$$\frac{\left(1.37 \times 10^{4}\right)\left(9.85 \times 10^{6}\right)}{5.0 \times 10^{12}} = 2.6989 \times 10^{-2}$$

Notice that when you divide exponential terms, you subtract the exponent in the denominator from the exponent in the numerator.

? Exercise

Evaluate $\left(4\times10^{-10}\right)\left(3\times10^{5}\right)$ and express the result in scientific notation.

A. 1.2×10^{-4}
B. 12×10^{-5}
C. 7×10^{-5}
D. 1.2×10^{-50}

Answer

A. Correct. 1.2×10^{-4} is an accurate computation and correct scientific notation.
B. Incorrect. Almost correct, but now you have to convert the coefficient 12 into scientific notation. 12 is greater than 10 and scientific notation requires this number to be greater than or equal to 1 and less than 10. The correct answer is 1.2×10^{-4}.
C. Incorrect. Multiply, not add, the numbers 4 and 3. The correct answer is 1.2×10^{-4}.
D. Incorrect. Add, not multiply, exponents. The correct answer is 1.2×10^{-4}.

? Advanced Question

Evaluate $\left(3.15\times10^{4}\right)\left(5.15\times10^{-7}\right)$ and express the result as a decimal.

A. 0.0162225
B. 162225
C. 0.000162225
D. 16.2225

Answer

A. Correct. $3.15\times5.15=16.2225$, and $10^{4}\times10^{-7}=10^{-3}$. The exponent is negative, so to convert to decimal format, move the decimal point three spaces to the left for a value of 0.0162225.
B. Incorrect. The expression has a negative exponent, so you move the decimal point 3 places to the left, not to the right, to convert it to decimal notation. The correct answer is 0.0162225.
C. Incorrect. You moved the decimal point in the correct direction, but not the correct number of places. Remember that $10^{4}\times10^{-7}=10^{-3}$, so you have to move the decimal point three places to the left. The correct answer is 0.0162225.
D. Incorrect. $3.15\times5.15=16.2225$, but it looks like you forgot to multiply $10^{4}\times10^{-7}$. The correct answer is 0.0162225.

Summary

Scientific notation was developed to assist mathematicians, scientists, and others when expressing and working with very large and very small numbers. Scientific notation follows a very specific format in which a number is expressed as the product of a number greater than or equal to 1 and less than 10, and a power of 10. The format is written $a\times10^{n}$, where $1\leq a<10$ and n is an integer.

To multiply or divide numbers in scientific notation, you can use the commutative and associative properties to group the exponential terms together and apply the rules of exponents.

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SECTION OVERVIEW

11.2: Polynomials with Single Variables

Topic hierarchy

11.2.1: Introduction to Single Variable Polynomials

11.2.2: Adding and Subtracting Polynomials

11.2.3: Multiplying Polynomials

11.2.4: Multiplying Special Cases

11.2.5: Dividing by a Monomial

11.2.6: Dividing by Binomials and Polynomials

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11.2.1: Introduction to Single Variable Polynomials

Learning Objectives

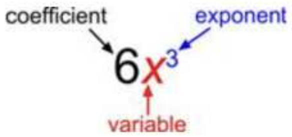
- Identify the terms, the coefficients and the exponents of a polynomial.
- Evaluate a polynomial for given values of the variable.
- Simplify polynomials by collecting like terms.

Introduction

Algebraic expressions are created by combining numbers and variables using arithmetic operations: addition, subtraction, multiplication, division, and exponentiation. Using all but division, you can create an expression called a **polynomial** by adding or subtracting terms. Polynomials are very useful in applications from science and engineering to business. Monomials (and polynomials in general) may have more than one variable, but in this unit, you will only work with single variable polynomials.

Monomials

The basic building block of a polynomial is a **monomial**. A monomial is one term and can be a number, a variable, or the product of a number and variables with an exponent. The number part of the term is called the **coefficient**.



The coefficient can be any real number, including 0. The exponent of the variable must be a whole number: 0, 1, 2, 3, and so on. A monomial cannot have a variable in the denominator or a negative exponent.

The value of the exponent is the **degree** of the monomial. Remember that a variable that appears to have no exponent really has an exponent of 1. And a monomial with no variable has a degree of 0. (x^0 has the value of 1 if $x \neq 0$. So a number such as 3 could be written as $3x^0$, if $x \neq 0$. When $x \neq 0$ it can be written as $3x^0 = 3 \cdot 1 = 3$.)

Example

Identify the coefficient, variable, and exponent of the monomial $\frac{3}{5}k^8$.

Solution

The variable is k.

The exponent of k is 8.

The coefficient of k^8 is $\frac{3}{5}$.

Example

Identify the coefficient, variable, and exponent of x.

Solution

The exponent of x is 1.	$x = x^1$, so the exponent is 1.
The coefficient of x is 1.	$x = 1x^1$ so the coefficient is also 1.

The variable is x.

? Exercise

Identify the coefficient, variable, and exponent of $3y$.

A. The variable is y, the exponent is 3, and the coefficient is 1.
B. The variable is y, the exponent is 0, and the coefficient is 3.
C. The variable is y, the exponent is 1, and the coefficient is 3.
D. The variable is y, the exponent is 3, and the monomial has no coefficient.

Answer

A. The variable is y, the exponent is 3, and the coefficient is 1.

Incorrect. The exponent is the power of the variable and the coefficient is the number before the variable. The coefficient in this case is 3, and the exponent is 1 because $3y = 3y^1$.

B. The variable is y, the exponent is 0, and the coefficient is 3.

Incorrect. The coefficient is 3, but the exponent is 1 because $3y = 3y^1$. An exponent of 0 would be $3y^0$, and since $y^0 = 1$ when $y \neq 0$, this would be equal to 3 (when $y \neq 0$).

C. The variable is y, the exponent is 1, and the coefficient is 3.

Correct. The coefficient is 3, the variable is y, and the exponent is 1 because $3y = 3y^1$.

D. The variable is y, the exponent is 3, and the monomial has no coefficient.

Incorrect. The exponent is the power of the variable and the coefficient is the number before the variable. The coefficient in this case is 3, and the exponent is 1 because $3y = 3y^1$.

Naming and Writing Polynomials

A polynomial is a monomial or the sum or difference of two or more polynomials. Each monomial is called a **term** of the polynomial.

Some polynomials have specific names indicated by their prefix.

monomial - is a polynomial with exactly one term (“mono” - means one)

binomial - is a polynomial with exactly two terms (“bi” - means two)

trinomial - is a polynomial with exactly three terms (“tri” - means three)

The word “polynomial” has the prefix, “poly,” which means many. However, the word polynomial can be used for all numbers of terms, including only one term.

Because the exponent of the variable must be a whole number, monomials and polynomials cannot have a variable in the denominator.

Polynomials can be classified by the degree of the polynomial. The degree of a polynomial is the degree of its highest-degree term. So the degree of $2x^3 + 3x^2 + 8x + 5$ is 3.

A polynomial is said to be written in standard form when the terms are arranged from the highest degree to the lowest degree. When it is written in standard form, it is easy to determine the degree of the polynomial.

The tables below illustrate some examples of monomials, binomials, trinomials, and other polynomials. They are all written in standard form.

Monomials

15
$\frac{1}{2}x$
$-4y^3$
$16n^4$

Binomials

$3y+13$
$4p-7$
$3x^2+\frac{5}{8}x$
$14y^3+3y$

Trinomials

x^3-x^2+1
$3x^2+2x-9$
$3y^3+y^2-2$
$a^7+2a^5-3a^3$

Other Polynomials

$5x^4+3x^3-6x^2+2x$
$\frac{1}{3}x^5-2x^4+\frac{2}{9}x^3-x^2+4x-\frac{5}{6}$
$3t^3-3t^2-3t-3$
$q^7+2q^5-3q^3+q$

When the coefficient of a polynomial term is 0, you usually do not write the term at all (because 0 times anything is 0, and adding 0 doesn't change the value). The last binomial in the binomial table, $14y^3+3y$, could be written as a trinomial, $14y^3+0y^2+3y$.

A term without a variable is called a **constant** term, and the degree of that term is 0. For example 13 is the constant term in $3y+13$. You would usually say that $14y^3+3y$ has no constant term or that the constant term is 0.

Exercise

Which of the following expressions are polynomials?

$2x^4-3x^3$

14

$\frac{x+1}{x}$

A. Only $2x^4-3x^3$ and $\frac{x+1}{x}$ are polynomials.
B. Only $2x^4-3x^3$ and 14 are polynomials.
C. Only $2x^4-3x^3$ is a polynomial.
D. None of the expressions is a polynomial.

Answer

A. Only $2x^4-3x^3$ and $\frac{x+1}{x}$ are polynomials.

Incorrect. $\frac{x+1}{x}$ is not a polynomial because it has a variable in the denominator. The correct answer is $2x^4-3x^3$ and 14.

B. Only $2x^4-3x^3$ and 14 are polynomials.

Correct. $\frac{x+1}{x}$ is not a polynomial because it has a variable in the denominator.

C. Only $2x^4-3x^3$ is a polynomial.

Incorrect. 14 is also a monomial, a type of polynomial. The correct answer is $2x^4-3x^3$ and 14.

D. None of the expressions is a polynomial.

Incorrect. x^4-3y^3 and 14 are polynomials. $\frac{x+1}{x}$ is not a polynomial because it has a variable in the denominator.

Evaluating Polynomials

You can evaluate polynomials just as you have been evaluating expressions all along. To evaluate an expression for a value of the variable, you substitute the value for the variable *every time* it appears. Then use the order of operations to find the resulting value for the expression.

✓ Example

Evaluate $3x^2-2x+1$ and $x=-1$.

Solution

$3(-1)^2-2(-1)+1$	Substitute -1 for each x in the polynomial.
$3(1)-2(-1)+1$	Following the order of operations, evaluate exponents first.
$3+(-2)(-1)+1$	Multiply 3 times 1, and then multiply -2 times -1.
$3+2+1$	Change the subtraction to addition of the opposite.
$3x^2-2x+1=6,\ \text{for } x=-1$	Find the sum.

✓ Example

Evaluate $-\frac{2}{3}p^4+2p^3-p$ for $p=3$

Solution

$-\frac{2}{3}(3)^4+2(3)^3-3$	Substitute 3 for each p in the polynomial.
$-\frac{2}{3}(81)+2(27)-3$	Following the order of operations, evaluate exponents first and then multiply.
$-54+54-3$	Add and then subtract to get -3.

$$-\frac{2}{3}p^4+2p^3-p=-3,\ \text{for } p=3$$

? Exercise

Evaluate $3x^3-2x^2$ for $x=-2$.

A. $-24-2x^2$
B. -32
C. -16
D. 16

Answer

A. $-24-2x^2$

Incorrect. $x=-2$ for *all* appearances of x in the expression. You must substitute -2 into both instances: $3(-2)^3-2(-2)^2$. The correct answer is -32.

B. -32

Correct. Since $x=-2$: $3x^3-2x^2=3(-2)^3-2(-2)^2=3(-8)-2(4)=-24-8=-32$.

C. -16

Incorrect. You may have forgotten that $(-2)^2$ is positive, or made another mistake in your calculations. Since $x=-2$: $3x^3-2x^2=3(-2)^3-2(-2)^2=3(-8)-2(4)=-24-8=-32$.

D. 16

Incorrect. You may have forgotten that $(-2)^3$ is negative (the product of two negative values is positive, so multiplying by a third negative gives a negative product). Since $x=-2$: $3x^3-2x^2=3(-2)^3-2(-2)^2=3(-8)-2(4)=-24-8=-32$.

Simplifying Polynomials

A polynomial may need to be simplified. One way to simplify a polynomial is to combine the **like terms** if there are any. Two or more terms in a polynomial are like terms if they have the same variable (or variables) with the same exponent. For example, $3x^2$ and $-5x^2$ are like terms: They both have x as the variable, and the exponent is 2 for each. However, $3x^2$ and $3x$ are not like terms, because their exponents are different.

Here are some examples of terms that are like and some that are unlike.

Monomials	**Terms**	**Explanation**
$3x$ $14x$	like	same variables with same exponents
$16z^2$ $-5z^2$	like	same variables with same exponents
$3x$ $5y$	unlike	different variables (although the same exponents)
$-3z$ $-3z^2$	unlike	same variables but with different exponents

Example

Which of these terms are like terms?

$7x^3, 7x, 7y, -8x^3, 9y, -3x^2, 8y^2$

Solution

$x: 7x^3, 7x, -8x^3, -3x^2$ $y: 7y, 9y, 8y^2$	Like terms must have the same variables, so first identify which terms use the same variables.
The x terms $7x^3$ and $-8x^3$ have the same exponent.he y terms $7y$ and $9y$ have the same exponent.	Like terms must also have the same exponents. Identify which terms with the same variables also use the same exponents.

$7x^3$ and $-8x^3$ are like terms.

$7y$ and $9y$ are like terms.

Exercise

Which of these are like terms?

$-3a, 3a^2, 8b, -3b^3, 8a, 14b^2, 9a$

A. Only $-3a$, $8a$, and $9a$ are like terms.
B. Only $8a$ and $9a$ are like terms.
C. $8b$, $-3b^3$, and $14b^2$ are like terms, and $-3a$, $3a^2$, $8a$, and $9a$ are like terms.
D. $3a^2$ and $14b^2$ are like terms, and $-3a$, $8b$, $8a$, and $9a$ are like terms.

Answer

A. Only $-3a$, $8a$, and $9a$ are like terms.

Correct. These terms have the same variable, a, with an exponent of 1.

B. Only $8a$ and $9a$ are like terms.

Incorrect. $-3a$ also is a like term with $8a$ and $9a$, because it has the same variable with the same exponent.

C. $8b$, $-3b^3$, and $14b^2$ are like terms, and $-3a$, $3a^2$, $8a$, and $9a$ are like terms.

Incorrect. Each group has the same variable, but their exponents are different. The only terms with the same variable *and* exponent are $-3a$, $8a$, and $9a$.

D. $3a^2$ and $14b^2$ are like terms, and $-3a$, $8b$, $8a$, and $9a$ are like terms.

Incorrect. Each group has the same exponent, but their variables are different. The only terms with the same variable *and* exponent are $-3a$, $8a$, and $9a$.

You can use the distributive property to simplify the sum of like terms. Recall that the distributive property of addition states that the product of a number and a sum (or difference) is equal to the sum (or difference) of the products.

$$2(3+6)=2(3)+2(6)$$

Both expressions equal 18. So you can write the expression in whichever form is the most useful.

Let’s see how we can use this property to combine like terms.

✓ Example

Simplify $3x^2-5x^2$.

Solution

$3\left(x^2\right)-5\left(x^2\right)$	$3x^2$ and $5x^2$ are like terms.
$(3-5)\left(x^2\right)$	We can rewrite the expression as the product of the difference.
$(-2)\left(x^2\right)$	Calculate $3-5$.
$3x^2-5x^2=-2x^2$	Write the difference of $3-5$ as the new coefficient.

You may have noticed that combining like terms involves combining the coefficients to find the new coefficient of the like term. You can use this as a shortcut.

✓ Example

Simplify $6a^4+4a^4$.

Solution

$6a^4+4a^4$	Notice that both terms have a number multiplied by a^4. This makes them like terms.
$(6+4)\left(a^4\right)$	Combine the coefficients, 6 and 4.
$(10)\left(a^4\right)$	Calculate the sum.
$6a^4+4a^4=10a^4$	Write the sum as the new coefficient.

When you have a polynomial with more terms, you have to be careful that you combine *only* like terms. If two terms are not like terms, you can’t combine them.

✓ Example

Simplify $3x^2+3x+x+1+5x$.

Solution

$3x,\ x$, and $5x$ are like terms	First identify which terms are *like terms*: only $3x,\ x$, and $5x$ are like terms.
$3x^2+3x+x+1+5x=3x^2+(3x+x+5x)+1$	Use the commutative and associative properties to group the like terms together.
$3x^2+(3+1+5)x+1$	Add the coefficients of the like terms. Remember that the coefficient of x is 1 because $x=1x$.
$3x^2+(9)x+1$	
$3x^2+3x+x+1+5x=3x^2+9x+1$	Write the sum as the new coefficient.

? Exercise

Simplify by combining like terms.

$-3a+3a^2+8a+9a-3$

A. $3a^2+17a-3$
B. $3a^2+14a-3$
C. $17a-3$
D. $17a-3a-3$

Answer

A. Incorrect. There are three terms with a (remember that a^1) is written as a): $-3a,\ 8a$, and $9a$. Combining them gives $-3a+8a+9a=(-3+8+9)a=14a$, so the simplified polynomial is $3a^2+14a-3$.
B. Correct. There are three terms with a (remember that a^1 is written as a): $-3a,\ 8a$, and $9a$. Combining them gives $-3a+8a+9a=(-3+8+9)a=14a$, so the simplified polynomial is $3a^2+14a-3$.
C. Incorrect. You may have incorrectly combined the first two terms, $-3a+3a^2$. These are not like terms, so you can't combine them. There are three terms with a (remember that a^1 is written as a): $-3a,\ 8a$, and $9a$. Combining them gives $-3a+8a+9a=(-3+8+9)a=14a$, so the simplified polynomial is $3a^2+14a-3$.
D. Incorrect. There are three terms with a (remember that a^1 is written as a): $-3a,\ 8a$, and $9a$. Combining them gives $-3a+8a+9a=(-3+8+9)a=14a$, so the simplified polynomial is $3a^2+14a-3$.

Summary

Polynomials are algebraic expressions that contain any number of terms combined by using addition or subtraction. A term is a number, a variable, or a product of a number and one or more variables with exponents. Like terms (same variable or variables raised to the same power) can be combined to simplify a polynomial. The polynomials can be evaluated by substituting a given value of the variable into *each* instance of the variable, then using order of operations to complete the calculations.

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11.2.2: Adding and Subtracting Polynomials

Learning Objectives

- Add polynomials.
- Find the opposite of a polynomial.
- Subtract polynomials.

Introduction

Adding and subtracting **polynomials** may sound complicated, but it's really not much different from the addition and subtraction that you do every day. The main thing to remember is to look for and combine **like terms**.

Adding Polynomials

You can add two (or more) polynomials as you have added algebraic expressions. You can remove the parentheses and combine like terms.

Example

Add. $(3b+5)+(2b+4)$

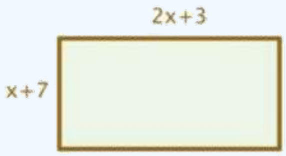
Solution

$(3b+2b)+(5+4)$	Regroup using the **commutative property of addition** and the **associative property of addition**.
$5b+9$	Combine like terms.

$(3b+5)+(2b+4)=5b+9$

Example

A rectangular garden has one side with a length of $x+7$ and another with a length $2x+3$. Find the perimeter of the garden.



Solution

$(x+7)+(2x+3)+(x+7)+(2x+3)$	The perimeter of a rectangle is the sum of its side lengths.
$(x+2x+x+2x)+(7+3+7+3)$	Regroup by like terms using commutative and associative properties.
$6x+20$	Add like terms.

The perimeter is $6x+20$.

The procedure is the same when you add polynomials that contain negative coefficients or subtraction:

Example

Add.

$(-5x^2-10x+2)+(3x^2+7x-4)$

Solution

$-5x^2+(-10x)+2+3x^2+7x+(-4)$	Rewrite subtraction as addition of the opposite.
$(-5x^2+3x^2)+(-10x+7x)+(2-4)$	Regroup using commutative and associative properties.
$-2x^2+(-3x)+(-2)$	Combine like terms.

$(-5x^2-10x+2)+(3x^2+7x-4)=-2x^2-3x-2$

The above examples show addition of polynomials horizontally, by reading from left to right along the same line. Some people like to organize their work vertically instead, because they find it easier to be sure that they are combining like terms. The example below shows this “vertical” method of adding polynomials:

Example

Add.

$(3x^2+2x-7)+(7x^2-4x+8)$.

Solution

$\begin{array}{rr} & 3x^2+2x-7 \\ + & 7x^2-4x+8 \end{array}$	Write one polynomial below the other, making sure to line up like terms.
$\begin{array}{rr} & 3x^2+2x-7 \\ + & 7x^2-4x+8 \\ \hline & 10x^2-2x+1 \end{array}$	Combine like terms, paying close attention to the signs.

$(3x^2+2x-7)+(7x^2-4x+8)=10x^2-2x+1$

Sometimes in a vertical arrangement, you can line up every term beneath a like term, as in the example above. But sometimes it isn't so tidy. When there isn't a matching like term for every term, there will be empty places in the vertical arrangement.

Example

Add.

$(4x^3+5x^2-6x+2)+(-4x^2+10)$

Solution

$\begin{array}{rrrrr} & 4x^3 & +5x^2 & -6x & +2 \\ + & & & -4x^2 & +10 \\ \hline \end{array}$	Write one polynomial below the other, lining up like terms vertically. Leave a blank space above or below every term without a matching like term.
$\begin{array}{rrrrr} & 4x^3 & +5x^2 & -6x & +2 \\ + & & -4x^2 & & +10 \\ \hline & 4x^3+ & x^2 & -6x & +12 \end{array}$	Combine like terms, paying close attention to the signs.

$(4x^3+5x^2-6x+2)+(-4x^2+10)=4x^3+x^2-6x+12$

Exercise

Find the sum.

$(4a^2+5a+7)+(8a+2)$

A. $9a^2+8a+9$
B. $16a^2+10a$
C. $12a^2+5a+9$
D. $4a^2+13a+9$

Answer

A. Incorrect. $4a^2$ and $5a$ cannot be combined because they are not like terms. Regroup by like terms to get $4a^2+(5a+8a)+(7+2)$. The correct answer is $4a^2+13a+9$.
B. Incorrect. You must first regroup by like terms before you add coefficients. Regrouping gives $4a^2+(5a+8a)+(7+2)$. The correct answer is $4a^2+13a+9$.
C. Incorrect. You can't combine $4a^2$ and $8a$ because they aren't like terms. Regroup by like terms to get $4a^2+(5a+8a)+(7+2)$. The correct answer is $4a^2+13a+9$.
D. Correct. Regroup by like terms to get $4a^2+(5a+8a)+(7+2)$, which is $4a^2+13a+9$.

Finding the Opposite of a Polynomial

When you subtract polynomials, you will add the opposite, as you have done with real numbers. So how do you find the opposite of a polynomial? Recall that the opposite of 3 is -3, and the opposite of -3 is 3. Just as the opposite of a number is found by multiplying the number by -1, we can find the opposite of a polynomial by multiplying it by -1.

Example

Find the opposite of $9x^2+10x+5$.

Solution

$(-1)(9x^2+10x+5)$	Find the opposite by multiplying by -1
$(-1)9x^2+(-1)10x+(-1)5$	Distribute -1 to each term in the polynomial.
$-9x^2+(-10x)+(-5)$	Multiply each coefficient by -1.
The opposite of $9x^2+10x+5$ is $-9x^2-10x-5$.	Rewrite addition of a negative term as subtraction.

Be careful when there are negative values or subtractions in the polynomial already.

Example

Find the opposite of $3p^2-5p+7$.

Solution

$(-1)(3p^2-5p+7)$	Find the opposite by multiplying by -1.
$(-1)[3p^2+(-5)p+7]$	Change the subtraction to adding the opposite.
$(-1)3p^2+(-1)(-5)p+(-1)7$	Distribute -1 to each term in the polynomial and multiply each coefficient by -1.
$-3p^2+5p+(-7)$	Rewrite addition of a negative term as subtraction.

The opposite of $3p^2-5p+7$ is $-3p^2+5p-7$.

Notice that in finding the opposite of a polynomial, you change the sign of *each term* in the polynomial.

? Exercise

Find the opposite of the polynomial.

$8a^3 - 3a - 2.$

A. $-8a^3 - 3a - 2$
B. $8a^3 + 3a - 2$
C. $8a^3 + 3a + 2$
D. $-8a^3 + 3a + 2$

Answer

A. Incorrect. Each term must have the sign changed. The correct answer is $-8a^3 + 3a + 2$.
B. Incorrect. Each term must have the sign changed. The correct answer is $-8a^3 + 3a + 2$.
C. Incorrect. Each term must have the sign changed. The correct answer is $-8a^3 + 3a + 2$.
D. Correct. Each term must have the sign changed, so the answer is $-8a^3 + 3a + 2$.

Subtracting Polynomials

Just as subtracting real numbers is the same as adding the opposite, you can subtract polynomials by adding the opposite of the second polynomial. Let's look at an example:

✓ Example

Subtract. $(15x^2 + 12x + 20) - (9x^2 + 10x + 5)$

Solution

$(15x^2 + 12x + 20) + (-9x^2 - 10x - 5)$	Change the subtraction to adding the opposite. *Be careful* to change the sign of *each* term!
$(15x^2 + -9x^2) + (12x - 10x) + (20 - 5)$	Regroup to match like terms.
$6x^2 + 2x + 15$	Combine like terms.

$(15x^2 + 12x + 20) - (9x^2 + 10x + 5) = 6x^2 + 2x + 15$

When polynomials include a lot of terms, it can be easy to lose track of the signs. Be careful to transfer them correctly, especially when subtracting a negative term.

✓ Example

Subtract. $(14x^3 + 3x^2 - 5x + 14) - (7x^3 + 5x^2 - 8x + 10)$

Solution

$(14x^3 + 3x^2 - 5x + 14) + (-7x^3 - 5x^2 + 8x - 10)$	Rewrite as adding the opposite.
$14x^3 + 3x^2 + (-5)x + 14 + (-7)x^3 + (-5)x^2 + 8x + (-10)$	You may want to rewrite all subtractions as adding the opposite.
$14x^3 + (-7)x^3 + 3x^2 + (-5)x^2 + (-5)x + 8x + 14 + (-10)$	Regroup to put like terms together.
$7x^3 + (-2)x^2 + 3x + 4$	Combine like terms.

$(14x^3 + 3x^2 - 5x + 14) - (7x^3 + 5x^2 - 8x + 10) = 7x^3 - 2x^2 + 3x + 4$

Complex problems, like the one above, may be more easily solved using the vertical approach (shown below). However you choose to combine polynomials is up to you; the key point is to identify like terms, and be able to organize them accurately.

Example

$(14x^3+3x^2-5x+14)-(7x^3+5x^2-8x+10)$

Solution

$\begin{array}{r} 14x^3+3x^2-5x+14 \\ \underline{-(7x^3+5x^2-8x+10)} \end{array}$	Reorganizing using the vertical approach.
$\begin{array}{r} 14x^3+3x^2-5x+14 \\ \underline{-7x^3-5x^2+8x-10} \\ 7x^3-2x^2+3x+\quad 4 \end{array}$	Change the subtraction to adding the opposites, and combine like terms.

$(14x^3+3x^2-5x+14)-(7x^3+5x^2-8x+10)=7x^3-2x^2+3x+4$

As with integer operations, experience and practice makes it easier to add and subtract polynomials.

Exercise

Subtract. $(4a^3-5a+7)-(8a^3-3a-2)$

A. $-4a^3-8a+5$
B. $-(-4a^3)+3a+9$
C. $-4a^3-2a+9$
D. $-4a^3+2a+5$

Answer

A. Incorrect. *Each* term in the second polynomial must be subtracted from the first polynomial: $4a^3-5a+7-8a^3+3a+2$. The correct answer is $-4a^3-2a+9$.
B. Incorrect. The subtracted $5a$ in the first polynomial must still be subtracted: $4a^3-5a+7-8a^3+3a+2$. The correct answer is $-4a^3-2a+9$.
C. Correct. $4a^3-5a+7-8a^3+3a+2=-4a^3-2a+9$.
D. Incorrect. Each term in the second polynomial must be subtracted from the first polynomial: $4a^3-5a+7-8a^3+3a+2$. The correct answer is $-4a^3-2a+9$.

Summary

When adding or subtracting polynomials, use the commutative and associative properties to regroup the terms in a polynomial into groups of like terms. Change subtraction, including subtraction of the second polynomial, to addition of the opposite. When finding the opposite of a polynomial, be sure to change the sign of *each* term. Then you can combine the like terms.

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11.2.3: Multiplying Polynomials

Learning Objectives

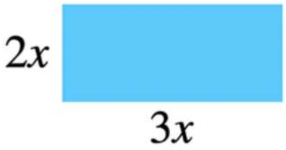
- Multiply monomials.
- Multiply monomials times polynomials.
- Multiply two binomials.
- Multiply any two polynomials.

Introduction

Multiplying **polynomials** involves applying the rules of exponents and the distributive property to simplify the product. This multiplication can also be illustrated with an area model, and can be useful in modeling real world situations. Understanding polynomial products is an important step in learning to solve algebraic equations involving polynomials.

Multiplying Monomials

Let's begin by multiplying two simple **monomials** together. Consider a rectangle whose length is $2x$ and whose width is $3x$. To find the area of this rectangle, multiply the length by the width



$$\text{Area of rectangle } = (2x)(3x) = (2x)(3x) = 2\cdot 3\cdot x\cdot x = 6x^2$$

Note that the commutative and associative properties of multiplication are used to rearrange the factors, putting the coefficients together and the variables together.

The area, $6x^2$, is a product that includes a coefficient (6) and a variable with a whole number exponent (x^2). In other words, it's a monomial, too. So the result of multiplying two monomials is another monomial!

Let's try a slightly more complex problem: $-9x^3\cdot 3x^2$

Example

Multiply. $-9x^3\cdot 3x^2$

Solution

$-9\cdot 3\cdot x^3\cdot x^2$	Use commutative and associative properties of multiplication to rearrange the factors.
$-27\cdot x^3\cdot x^2$	Multiply constants. Remember that a positive number times a negative number yields a negative number.
$-27\cdot x^{3+2}$ $-27\cdot x^5$	Multiply variable terms. Remember to add the exponents when multiplying exponents with the same base.

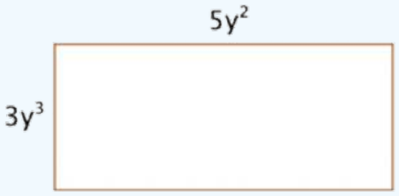
$-9x^3\cdot 3x^2 = -27x^5$

That’s it! When multiplying monomials, multiply the coefficients together, and then multiply the variables together. If two variables have the same base, follow the rules of exponents, like this:

$5a^4\cdot 7a^6 = 35a^{10}$

Exercise

Find the area of the rectangle:



A. $8y^3$
B. $15y^5$
C. $15y^{10}$
D. $8y^5$

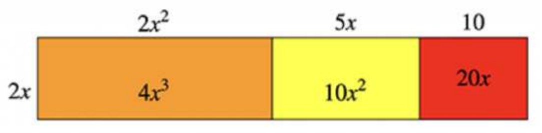
Answer

A. Incorrect. Multiply the two coefficients to get 15, then multiply the variables. Use the rules of exponents: $y^3 \cdot y^2 = y^{3+2} = y^5$. The correct answer is $15y^5$.
B. Correct. Multiply $3 \cdot 5$ and $y^3 \cdot y^2$, using the rules of exponents to get $15y^5$.
C. Incorrect. When you multiply with exponents, if the bases are the same, you add the exponents: $y^3 \cdot y^2 = y^{3+2} = y^5$. The correct answer is $15y^5$.
D. Incorrect. You correctly multiplied the variables, $y^3 \cdot y^2 = y^{3+2} = y^5$, but you appear to have added the coefficients rather than multiplied. The correct answer is $15y^5$.

The Product of a Monomial and a Polynomial

The distributive property can be used to multiply a polynomial by a monomial. Just remember that the monomial must be multiplied by each term in the polynomial. Consider the expression $2x\left(2x^2+5x+10\right)$.

This expression can be modeled with a sketch like the one below.



The model above illustrates the distributive property.

$$2x\left(2x^2+5x+10\right) = 2x \cdot 2x^2 + 2x \cdot 5x = 2x \cdot 10 = 4x^3 + 10x^2 + 20x$$

Here’s an example:

Example

Simplify. $5x\left(4x^2+3x+7\right)$

Solution

$5x \cdot 4x^2 + 5x \cdot 3x + 5x \cdot 7$	Distribute the monomial to each term of the polynomial.
$20x^3 + 15x^2 + 35x$	Multiply.

$$5x\left(4x^2+3x+7\right) = 20x^3 + 15x^2 + 35x$$

You may need to rewrite subtraction as adding the opposite.

Example

Simplify. $7x^2\left(2x^2-5x+1\right)$

Solution

$7x^2\left[2x^2+(-5x)+1\right]$	Rewrite the subtraction as adding the opposite.
$7x^2\cdot 2x^2+7x^2\cdot -5x+7x^2\cdot 1$	Distribute the monomial to each term of the polynomial.
$14x^4+(-35)x^3+7x^2$	Multiply.
$7x^2\left(2x^2-5x+1\right)=14x^4-35x^3+7x^2$	Rewrite addition of terms with negative coefficients as subtraction.

Exercise

Find the product. Watch the signs!

$-3t^2\left(7t^3+3t^2-t\right)$

A. $-21t^5-9t^4+3t^3$

B. $-21t^5+9t^4-3t^3$

C. $-21t^6-9t^4+3t^2$

D. $-21t^5+3t^2-t$

Answer

A. Correct. Rewriting the subtraction as adding the opposite gives $-3t^2\left[7t^3+3t^2+(-t)\right]$. Distributing the monomial $-3t^2$ gives $-3t^2\cdot 7t^3+\left(-3t^2\right)\cdot 3t^2+\left(-3t^2\right)\cdot(-t)$, which is $-21t^5+\left(9t^4\right)+\left(3t^3\right)$. Rewriting addition of terms with negative coefficients as subtraction gives $-21t^5-9t^4+3t^3$.

B. Incorrect. The negative must be distributed to all terms along with the $3t^2$. This changes the sign of the middle and last terms. The correct answer is $-21t^5-9t^4+3t^3$.

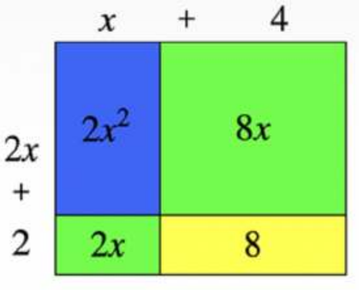
C. Incorrect. By the laws of exponents, you add (not multiply) exponents when multiplying: $-3t^2\cdot 7t^3+\left(-3t^2\right)\cdot 3t^2+\left(-3t^2\right)\cdot(-t)$ is $-21t^5+\left(-9t^4\right)+\left(3t^3\right)$. The correct answer is $-21t^5-9t^4+3t^3$.

D. Incorrect. You must distribute the monomial to all three terms in the polynomial, not just the first one: $-3t^2\cdot 7t^3+\left(-3t^2\right)\cdot 3t^2+\left(-3t^2\right)\cdot(-t)$. The correct answer is $-21t^5-9t^4+3t^3$.

Product of Two Binomials

Now let's explore multiplying two binomials. Once again, you can draw an area model to help make sense of the process. You'll use each binomial as one of the dimensions of a rectangle, and their product as the area.

The model below shows $(x+4)(2x+2)$:



Each binomial is expanded into variable terms and constants, $x+4$, along the top of the model and $2x+2$ along the left side. The product of each pair of terms is a colored rectangle. The total area is the sum of all of these small rectangles, $2x^2+8x+2x+8$. If you combine all the **like terms**, you can write the product, or area, as $2x^2+10x+8$.

You can use the distributive property to determine the product of two binomials.

✓ Example

$(x+4)(2x+2)$

Solution

$x\cdot 2x+x\cdot 2+4\cdot 2x+4\cdot 2$	Distribute the x over $2x+2$, then distribute 4 over $2x+2$.
$2x^2+2x+8x+8$	Multiply.
$2x^2+10x+8$	Combine like terms $(8x+2x)$.

$(x+4)(2x+2)=2x^2+10x+8$

Consider the model above to see where each piece of $2x^2+8x+2x+8$ comes from. Can you see where you multiply x by $2x+2$, and where you get $2x^2$ from $x\cdot 2x$?

Another way to look at multiplying binomials is to see that each term in one binomial is multiplied by each term in the other binomial. In the example above, the x in $x+4$ gets multiplied by both the $2x$ and the 2 from $2x+2$, and the 4 gets multiplied by both the $2x$ and the 2.

Some people use the FOIL method to keep track of which pairs have been multiplied. The letters in FOIL stand for First, Outer, Inner, Last:

First term in each binomial: $(\mathbf{x}+4)(\mathbf{2x}+2)\mathrm{x}\cdot 2\mathrm{x}=2\mathrm{x}^2$

Outer terms: $(\mathbf{x}+4)(2\mathrm{x}+\mathbf{2})\mathrm{x}\cdot 2=2\mathrm{x}$

Inner terms: $(\mathrm{x}+\mathbf{4})(\mathbf{2x}+2)4\cdot 2\mathrm{x}=8\mathrm{x}$

Last terms in each binomial: $(\mathrm{x}+\mathbf{4})(2\mathrm{x}+\mathbf{2})4\cdot 2=8$

When you add the four results, you get the same answer:

$2x^2+2x+8x+8=2x^2+10x+8$

Here is another example, using FOIL this time.

✓ Example

$(4x-10)(2x+3)$

Solution

$\begin{aligned} 4x\cdot 2x&=8x^2 \text{ (First)}\\ 4x\cdot 3&=12x \text{ (Outer)}\\ -10\cdot 2x&=-20x \text{ (Inner)}\\ -10\cdot 3&=-30 \text{ (Last)}\end{aligned}$	Be careful about including the negative sign on the -10, since 10 is subtracted.
$8x^2+12x-20x-30$	Combine like terms.

$(4x-10)(2x+3)=8x^2-8x-30$

Because multiplication is commutative, the terms can be multiplied in either order. The expression $(2x+2)(x+4)$ has the same product as $(x+4)(2x+2)$, $2x^2+10x+8$. (Work it out and see.) The order in which you multiply binomials does not matter. What matters is that you multiply each term in one binomial by each term in the other binomial.

The last step in multiplying polynomials is to combine like terms. Remember that a polynomial is simplified only when there are no like terms remaining.

Exercise

Find the product: $(a+10)(2a-7)$

A. $2a^2+19a-70$
B. $3a+3$
C. $2a^2-70$
D. $2a^2+13a-70$

Answer

A. Incorrect. The middle term should be $20a-7a$, which is $13a$. The correct answer is $2a^2+13a-70$.
B. Incorrect. Multiply, don't add, the terms in one binomial by the terms in the other binomial. The correct answer is $2a^2+13a-70$.
C. Incorrect. Multiply each term in one binomial by the terms in the other binomial. $(\text{a})\ (-7)$ and $(10)(2a)$ are missing. The correct answer is $2a^2+13a-70$.
D. Correct. Using the FOIL method, you find: $(a+10)(2a-7)=a\cdot 2a+a\cdot -7+10\cdot 2a+10\cdot -7=2a^2-7a+20a-70$ $=2a^2+13a-70$.

Product of a Binomial and a Trinomial

Another type of polynomial multiplication problem is the product of a binomial and trinomial. Although the FOIL method cannot be used exactly since there are more than two terms in a trinomial, you still use the distributive property to organize the individual products. Using the distributive property, each term in the binomial must be multiplied by each of the terms in the trinomial. Two examples are shown below.

Example

$(3x+6)\left(5x^2+3x+10\right)$

Solution

$3x\left(5x^2+3x+10\right)+6\left(5x^2+3x+10\right)$	Distribute the trinomial to each term in the binomial.
$3x\cdot 5x^2+3x\cdot 3x+3x\cdot 10+6\cdot 5x^2+6\cdot 3x+6\cdot 10$	Use the distributive property to distribute the monomials to each term in the trinomials.
$15x^3+9x^2+30x+30x^2+18x+60$	Multiply.
$15x^3+\left(9x^2+30x^2\right)+(30x+18x)+60$	Group like terms.
$(3x+6)\left(5x^2+3x+10\right)=15x^3+39x^2+48x+60$	Combine like terms.

As you can see, multiplying a binomial by a trinomial leads to a lot of individual terms! Some people prefer to set up these problems vertically and gather like terms as they multiply. This method is shown below, using the same problem as above.

Example

$(3x+6)\left(5x^2+3x+10\right)$

Solution

$\begin{array}{r} 3x+\quad 6 \\ \times\quad 5x^2+\ 3x+10 \\ \hline +30x+60 \end{array}$	Set up the problem in a vertical form, and begin by multiplying $3x+6$ by $+10$. Place the products underneath, as shown.
$\begin{array}{rr} & 3x+\quad 6 \\ & \times\quad 5x^2+\ \ {\color{blue}3x}+10 \\ \hline & +\ 30x+60 \\ {\color{blue}+\ 9x^2+18x} & \end{array}$	Now multiply $3x+6$ by $3x$. Notice that $(6)\ (3x)=18x$; since this term is like $30x$, place it directly beneath it.

$\begin{array}{rrrr} & & 3x + & 6 \\ \times & 5x^2 + & 3x + & 10 \\ \hline & & +30x + & 60 \\ & + \quad 9x^2 + & 18x & \\ +15x^3 + & 30x^2 & & \\ \hline \end{array}$	Finally, multiply $3x+6$ by $5x^2$. Notice that $30x^2$ is placed underneath $9x^2$.
$\begin{array}{rrrr} & & 3x + & 6 \\ \times & 5x^2 + & 3x + & 10 \\ \hline & & +30x + & 60 \\ & + \quad 9x^2 + & 18x & \\ +15x^3 + & 30x^2 & & \\ \hline +15x^3 + & 39x^2 + & 48x + & 60 \end{array}$	Now add like terms.

$15x^3+39x^2+48x+60$

Notice that although the two problems were solved using different strategies, the product is the same. Both the horizontal and vertical methods apply the distributive property to multiply a binomial by a trinomial.

The next example shows multiplication by a binomial and trinomial that each contains subtraction. The example completes the multiplication without rewriting each subtraction as addition of the opposite. Notice where you must be careful with the signs! (If you prefer, you can continue to rewrite subtraction as adding the opposite.)

✓ Example

$(2p-1)\left(3p^2-3p+1\right)$

Solution

$2p\left(3p^2-3p+1\right)-1\left(3p^2-3p+1\right)$	Distribute the trinomial to each term in the binomial.
$2p\cdot 3p^2+2p\cdot -3p+2p\cdot 1-1\left(3p^2\right)-1(-3p)-1(1)$	If you don't write subtraction as adding the opposite, then be sure to think of it that way. So you are distributing -1 and multiplying each term of the trinomial by -1.
$6p^3-6p^2+2p-3p^2+3p-1$	Multiply. (Notice that the subtracted 1 and the subtracted $3p$ have a positive product that is added.)
$6p^3-9p^2+5p-1$	Combine like terms.

$6p^3-9p^2+5p-1$

? Exercise

Find the product: $(3x-2)\left(2x^2+4x-11\right)$

A. $6x^3+8x^2-41x+22$
B. $6x^3+8x^2-41x-22$
C. $6x^3+12x+22$
D. $3x^3+8x^2+25x-22$

Answer

A. Correct. $3x\left(2x^2+4x-11\right)-2\left(2x^2+4x-11\right)=6x^3+12x^2-33x-4x^2-8x+22=6x^3+8x^2-41x+22$
B. Incorrect. $(-2)(-11)=22$. The correct answer is $6x^3+8x^2-41x+22$.
C. Incorrect. Each term in the binomial must be multiplied by each term in the trinomial. The correct answer is $6x^3+8x^2-41x+22$.
D. Incorrect. Multiply the coefficients $(3x)\left(2x^2\right)$; combine like terms by adding/subtracting coefficients according to their signs. The correct answer is $6x^3+8x^2-41x+22$.

Summary

Multiplication of binomials and polynomials requires use of the distributive property as well as the commutative and associative properties of multiplication. Whether the polynomials are monomials, binomials, or trinomials, carefully multiply each term in one polynomial by each term in the other polynomial. Be careful to watch the addition and subtraction signs and negative coefficients. A product is written in simplified form if all of its like terms have been combined.

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11.2.4: Multiplying Special Cases

Learning Objectives

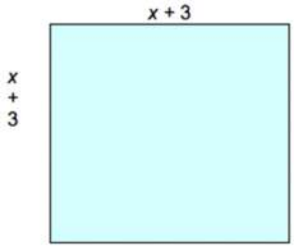
- Square a binomial.
- Multiply the sum and difference of the same two terms.

Introduction

While the distributive property can be used for all polynomial multiplication, some products with **binomials** can be found using short cuts. These methods are sometimes called special products.

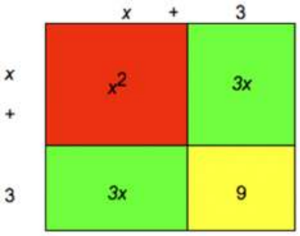
Square of a Binomial Sum

Multiplying a number by itself is often called squaring. You can represent this multiplication as a square. The number being squared is the length of the sides of the square and the product is represented by the area of that square. Consider a square whose side length is described by the binomial $x+3$:



The area of this square is $(x+3)(x+3)$ or $(x+3)^2$.

The following shows the same square, but separates the variable and constant terms:



From this area model, you can see that the area can be described by the sum of the red, green and yellow pieces. That is, the area is $x^2+3x+3x+9=x^2+6x+9$.

So $(x+3)^2=x^2+3x+3x+9=x^2+6x+9$.

You can also find the square of a sum of two terms by using the FOIL method.

Example

Square the binomial. $(x+5)^2$**.**

Solution

$(x+5)(x+5)$	Multiply the binomial by itself.
First: $x\cdot x=x^2$ Outer: $x\cdot 5=5x$ Inner: $5\cdot x=5x$ Last: $5\cdot 5=25$	
$x^2+5x+5x+25$	Add the terms.
$x^2+10x+25$	Combine like terms.

$(x+5)^2=x^2+10x+25$

Notice that the Outer and Inner terms are the same.

So far, we have two examples of squaring a sum of two terms, one from constructing an area model and one from algebraic calculations:

$(x+3)^2=x^2+6x+9$
$(x+5)^2=x^2+10x+25$

In both cases, the first term is squared to get the first term of the product. The two terms are multiplied and doubled for the middle term of the product, and the last term is squared for the last term of the product.

This pattern will hold true for the square of the sum of any two terms: Square the first term, add twice the product of the first and the last term, add the last term squared.

Squaring a Binomial Sum

To square a binomial, do the following:

- Square the first term.
- Add the product of the two middle terms, times two.
- Add the square of the last term.

This process is illustrated by the following case:

$(a+b)^2=a^2+2ab+b^2$

Example

Square the binomial. $(2x+6)^2$

Solution

$(2x)^2=4x^2$	Square the first term.
$(2x)(6)(2)=24x$	Multiply the two terms together and double the product.
$6^2=36$	Square the last term.
$4x^2+24x+36$	Combine the terms.

$(2x+6)^2=4x^2+24x+36$

You can check this answer using the distributive property or FOIL.

Exercise

Find the product: $(2s+9)^2$

A. $4s^2+81$
B. $4s+18$
C. $4s^2+36s+81$
D. $4s^2+18s+18$

Answer

A. Incorrect. You squared the first term, and you squared the last term. But you forgot the middle term, which is 2 times the product of 2s and 9, which is 36s. The correct answer is $4s^2+36s+81$.
B. Incorrect. You must square $(2s+9)$, not double it. $(a+b)^2=a^2+2ab+b^2$; in this example, $a=2s$ and $b=9$, so $(2s+9)^2=4s^2+36s+81$.
C. Correct. $(a+b)^2=a^2+2ab+b^2$; in this example, $a=2s$ and $b=9$ so $(2s+9)^2=4s^2+2(18s)+81=4s^2+36s+81$.

D. Incorrect. The terms $18s$ and 18 are incorrect. $(a+b)^2=a^2+2ab+b^2$; in this example, $a=2s$ and $b=9$, so $(2s+9)^2=4s^2+36s+81$.

Square of a Binomial Difference

Is there also a pattern when you square the *difference* between two terms? Yes! Since subtraction can be expressed as adding the opposite, a similar pattern does occur.

Consider the square of the binomial $(x-7)$. You can use FOIL.

✓ Example

Square the binomial. $(x-7)^2$

Solution

$(x-7)(x-7)$	Rewrite as multiplication.
$\begin{aligned}\text{First: }& x\cdot x=x^2\\ \text{Outer: }& x\cdot -7=-7x\\ \text{Inner: }& -7\cdot x=-7x\\ \text{Last: }& -7\cdot -7=49\end{aligned}$	
$x^2+(-7x)+(-7x)+49$	Add the terms.
$x^2-14x+49$	Combine the like terms.

$(x-7)^2=x^2-14x+49$

Notice that the pattern is similar to when you square the sum of a binomial.

Squaring a Binomial Difference

To square a binomial difference, do the following:

- Square the first term
- Subtract the product of the two middle terms, times two
- Add the square of the last term.

$(a-b)^2=a^2-2ab+b^2$

✓ Example

Square the binomial. $(4s-3)^2$

Solution

$(4s)^2=16s^2$	Square the first term, including the coefficient.
$(4s)(-3)(2)=-24s$	Multiply the two terms together and double the product.
$(-3)^2=9$	Square the last term.
$16s^2-24s+9$	Add the terms.

$(4s-3)^2=16s^2-24s+9$

? Exercise

Square the binomial. $(2r-9)^2$

A. $4r^2-81$

B. $4r-18$
C. $4r^2-36r+81$
D. $4r^2+18r-18$

Answer

A. Incorrect. You squared the first term correctly, and you squared the last term correctly, but you forgot that there is a middle term, which is $(2r)(-9)(2)=-36r$. The correct answer is $4r^2-36r+81$.
B. Incorrect. The exponent 2 means the binomial is to be squared, not doubled. The correct answer is $4r^2-36r+81$.
C. Correct. $(a-b)^2=a^2-2ab+b^2$, and in this case $a=2r$ and $b=9$. So $(2r-9)^2=4r^2-36r+81$.
D. Incorrect. The terms $18r$ and -18 are incorrect. $(a-b)^2=a^2-2ab+b^2$, and in this case $a=2r$ and $b=9$. So $(2r-9)^2$ $=4r^2-36r+81$.

Product of a Sum and a Difference

There is a third special product to consider among binomials: the product of the sum of two terms and the difference of the same two terms. In this case, there is also a pattern. Here is an example:

✓ Example

Multiply the binomials. $(x+8)(x-8)$

Solution

$\begin{aligned}\text{First: } & x\cdot x=x^2\\ \text{Outer: } & x\cdot -8=-8x\\ \text{Inner: } & 8\cdot x=+8x\\ \text{Last: } & 8\cdot -8=-64\end{aligned}$	
$x^2-8x+8x-64$	Add the terms.

$(x+8)(x-8)=x^2-64$

Note that the answer to this binomial product is a binomial itself: the difference of two perfect squares. There is no middle term in this case. Why does this happen? The two terms are opposites and therefore add to zero.

Product of a Sum and a Difference

The product of the sum of two terms $(a+b)$ and the difference of the same terms $(a-b)$ is the difference of the squares of the two terms.

$(a+b)(a-b)=(a-b)(a+b)=a^2-b^2$

✓ Example

Multiply the binomials. $(2n-5)(2n+5)$

Solution

$(2n)^2=4n^2$	Square the first term, including the coefficient.
$(5)^2=25$	Square the last term.
$4n^2-25$	Take the difference.

$(2n-5)(2n+5)=4n^2-25$

Exercise

Find the product: $(2r-9)(2r+9)$

A. $4r^2-81$
B. $4r-18$
C. $4r^2-36r+81$
D. $4r^2-36r-81$

Answer

A. Correct. $(a-b)(a+b)=a^2-b^2$, and in this case $a=2r$ and $b=9$. So $(2r-9)(2r+9)=4r^2-81$.
B. Incorrect. Each term must be squared, not doubled. $(2r-9)(2r+9)=4r^2-81$.
C. Incorrect. This is not the square of a difference. One binomial is a difference but the other is a sum. $(a-b)(a+b)=a^2-b^2$, and in this case $a=2r$ and $b=9$. So $(2r-9)(2r+9)=4r^2-81$.
D. Incorrect. $(a-b)(a+b)=a^2-b^2$, and in this case $a=2r$ and $b=9$. So $(2r-9)(2r+9)=4r^2-81$.

Summary

Some products of multiplying binomials follow a predictable pattern that makes it easy to multiply them. These are known as special products. There are three special products of binomials that each follow a specific formula: squaring the sum of a binomial, squaring the difference of a binomial, and the product of a sum and a difference.

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11.2.5: Dividing by a Monomial

Learning Objectives

- Divide a monomial by a monomial.
- Divide a polynomial by a monomial.

Introduction

The fourth arithmetic operation is division, the inverse of multiplication. Division of polynomials isn't much different from division of numbers. Let's start with dividing a **monomial** by another monomial, which is the basis for dividing a polynomial by a monomial.

Dividing Monomials by Monomials

When you multiply two monomials, you multiply the coefficients together and then you multiply the variables together. Similarly, when dividing monomials, you divide the coefficients and then divide variables. When there are exponents with the same base, the law of exponents says you divide by subtracting the exponents. Consider this example:

Example

Divide. $\frac{10y^5}{2y^2}$

Solution

$\left(\frac{10}{2}\right)\left(\frac{y^5}{y^2}\right)$	Group the monomial into numerical and variable factors.
$5\left(y^{5-2}\right)$	Divide the coefficients, and divide the variables by subtracting the exponents of each y term.

$\frac{10y^5}{2y^2}=5y^3$

Here's another example:

Example

A rectangle has an area of $8x^2$ and a length of $4x$. Find the width of the rectangle using the formula: $\frac{\text{Area}}{\text{length}}=\text{width}$

Solution

$\frac{8x^2}{4x}$	Substitute known values.
$2x^{2-1}$	Divide coefficients, and divide the variables by subtracting the exponents of each x term.
$2x$	

$\text{width}=2x$ units

Sometimes division requires simplification.

Example

Divide. $\frac{-6r^3}{4r^4}$

Solution

$\left(\frac{-6}{4}\right)\left(\frac{r^3}{r^4}\right)$	Group the monomial into numerical and variable factors.
$\left(\frac{-3}{2}\right)\left(\frac{r^3}{r^4}\right)$	Simplify $\left(\frac{-6}{4}\right)$ to $\left(\frac{-3}{2}\right)$.

$\frac{-3}{2}r^{-1}$	Divide the variables by subtracting the exponents of r. Note that the variable has a negative exponent.
$\frac{-3}{2}\cdot\frac{1}{r}$	Simplify r^{-1} by rewriting it as the inverse of r.
$\frac{-3}{2r}$	Multiply.

$$\frac{-6r^3}{4r^4}=\frac{-3}{2r}$$

Remember that a term is not considered simplified if it contains a negative exponent; this is why $\frac{-3}{2}r^{-1}$ was rewritten as $\frac{-3}{2r}$.

? Exercise

Divide: $\frac{22x^4}{2x}$

A. $11x^4$
B. $22x^3$
C. $11x^3$
D. $22x^4$

Answer

A. Incorrect. You divided 22 by 2, but you must subtract the exponents of the variable x. Since $x=x^1$, this is $x^{4-1}=x^3$. The correct answer is $11x^3$.
B. Incorrect. You correctly divided the variables, but you must also divide 22 by 2. The correct answer is $11x^3$.
C. Correct. $\frac{22}{2}=11$ and $x^{4-1}=x^3$ so the correct answer is $11x^3$.
D. Incorrect. Divide the coefficients to get $\frac{22}{2}=11$ for the coefficient. Divide the variables by subtracting the exponents. Since $x=x^1$, this is $x^{4-1}=x^3$. The correct answer is $11x^3$.

Dividing Polynomials by Monomials

The distributive property states that you can distribute a factor that is being multiplied by a sum or difference, and likewise you can distribute a divisor that is being divided into a sum or difference (as division can be changed to multiplication.)

$$\frac{8+4+10}{2}=\frac{22}{2}=11$$

Or you can distribute the 2, and divide each term by 2.

$$\frac{8}{2}+\frac{4}{2}+\frac{10}{2}=4+2+5=11$$

Let’s try something similar with a polynomial.

✓ Example

Divide. $\frac{14x^3-6x^2+2x}{2x}$

Solution

$\frac{14x^3}{2x}-\frac{6x^2}{2x}+\frac{2x}{2x}$	Distribute $2x$ over the polynomial by dividing each term by $2x$.
$7x^2-3x+1$	Divide each term, a monomial divided by another monomial.

$$\frac{14x^3-6x^2+2x}{2x}=7x^2-3x+1$$

Let’s try one more example. Watch the signs.

✓ Example

Divide. $\frac{27y^4+6y^2-18}{-6y}$

Solution

$\frac{27y^4}{-6y}+\frac{6y^2}{-6y}-\frac{18}{-6y}$	Divide each term in the polynomial by the monomial.
$-\frac{9}{2}y^3-y+3y^{-1}$	Simplify. Remember that 18 can be written as $18y^0$. So the exponents are $0-1=-1$.
$-\frac{9}{2}y^3-y+\frac{3}{y}$	Write the final answer without any negative exponents.

$\frac{27y^4+6y^2-18}{-6y}=-\frac{9}{2}y^3-y+\frac{3}{y}$

? Exercise

Divide: $\frac{30t^4-10t^3+t^2-20}{10t^2}$

A. $3t^2-t+\frac{1}{10}-\frac{2}{t^2}$
B. $3t^2-10t^3+t^2-20$
C. $30t^2-10t^3-20$
D. $3t^8-t^6+\frac{1}{10}-20t^2$

Answer

A. Correct. Divide each term in the polynomial by the monomial: $\frac{30t^4}{10t^2}-\frac{10t^3}{10t^2}+\frac{t^2}{10t^2}-\frac{20}{10t^2}$, which gives $3t^2-t+\frac{1}{10}-\frac{2}{t^2}$.
B. Incorrect. You only divided the first term. Divide each term in the polynomial by the monomial: $\frac{30t^4}{10t^2}-\frac{10t^3}{10t^2}+\frac{t^2}{10t^2}-\frac{20}{10t^2}$. The correct answer is $3t^2-t+\frac{1}{10}-\frac{2}{t^2}$.
C. Incorrect. You removed the two t^2 terms, but did not divide. Divide each term in the polynomial by the monomial: $\frac{30t^4}{10t^2}-\frac{10t^3}{10t^2}+\frac{t^2}{10t^2}-\frac{20}{10t^2}$. The correct answer is $3t^2-t+\frac{1}{10}-\frac{2}{t^2}$.
D. Incorrect. Divide each term in the polynomial by the monomial: $\frac{30t^4}{10t^2}-\frac{10t^3}{10t^2}+\frac{t^2}{10t^2}-\frac{20}{10t^2}$. The correct answer is $3t^2-t+\frac{1}{10}-\frac{2}{t^2}$.

Summary

To divide a monomial by a monomial, divide the coefficients (or simplify them as you would a fraction) and divide the variables with like bases by subtracting their exponents. To divide a polynomial by a monomial, divide each term of the polynomial by the monomial. Be sure to watch the signs! Final answers should be written without any negative exponents.

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11.2.6: Dividing by Binomials and Polynomials

Learning Objectives

- Divide a polynomial by a binomial.
- Divide a polynomial by another polynomial.

Introduction

Dividing a **polynomial** by a monomial can be handled by dividing each term in the polynomial separately. This can't be done when the divisor has more than one term. However, the process of long division can be very helpful with polynomials.

Dividing by a Binomial

First, recall how you can use long division to divide two whole numbers, say 900 divided by 37.

37\longdiv900

First, you would think about how many 37s are in 90, as 9 is too small. (Note: you could also think, how many 40s are there in 90.)

2 37\longdiv900 74	There are two 37s in 90, so write 2 above the last digit of 90. Two 37s is 74; write that product below the 90.
2 37\longdiv900 -74 16	Subtract: $90-74$ is 16. (If the result is larger than the divisor, which is 37, then you need to use a larger number for the quotient.)
2 37\longdiv900 -74 160	Bring down the next digit (0) and consider how many 37s are in 160.
24 37\longdiv900 -74 160 -148	There are four 37s in 160, so write the 4 next to the two in the quotient. Four 37s is 148; write that product below the 160.
24 37\longdiv900 -74 160 -148 12	Subtract: $160-148$ is 12. This is less than 37 so the 4 is correct. Since there are no more digits in the dividend to bring down, you're done.

The final answer is 24R12, or $24\frac{12}{37}$. You can check this by multiplying the quotient (without the remainder) by the divisor, and then adding in the remainder. The result should be the dividend:

$$24\cdot 37+12=888+12=900$$

To divide polynomials, use the same process. This example shows how to do this when dividing by a **binomial**.

Example

Divide: $(x^2-4x-12)\div(x+2)$

Solution

$x+2$\longdiv$x^2-4x-12$	How many x 's are there in x^2? That is, what is $\frac{x^2}{x}$?

x $x+2$\longdiv$x^2-4x-12$ $-(x^2+2x)$	$\frac{x^2}{x}=x$. Put x in the quotient above the $-4x$ term. (These are like terms, which helps to organize the problem.) Write the product of the divisor and the part of the quotient you just found under the dividend.
x $x+2$\longdiv$x^2-4x-12$ $-x^2-2x$	Since $x(x+2)=x^2+2x$, write this underneath, and get ready to subtract.
x $x+2$\longdiv$x^2-4x-12$ $-x^2-2x$	Rewrite $-(x^2+2x)$ as its opposite $-x^2-2x$ so that you can add the opposite. (Adding the opposite is the same as subtracting, and it is easier to do.)
x $x+2$\longdiv$x^2-4x-12$ $-x^2-2x$ $-6x$	Add $-x^2$ to x^2, and $-2x$ to $-4x$.
x $x+2$\longdiv$x^2-4x-12$ $-x^2-2x$ $-6x-12$	Bring down -12.
x $x+2$\longdiv$x^2-4x-12$ $-x^2-2x$ $-6x-12$	Repeat the process. How many times does x go into $-6x$? In other words, what is $\frac{-6x}{x}$?
$x-\ 6$ $x+2$\longdiv$x^2-4x-12$ $-x^2-2x$ $-6x-12$ $-(-6x-12)$	Since $\frac{-6x}{x}=-6$, write -6 in the quotient. Multiply -6 and $x+2$ and prepare to subtract the product.
$x\ -6$ $x+2$\longdiv$x^2-4x-12$ $-x^2-2x$ $-6x-12$ $6x+12$	Rewrite $-(-6x-12)$ as $6x+12$, so that you can add the opposite.
$x-\ 6$ $x+2$\longdiv$x^2-4x-12$ $-x^2-2x$ $-6x\ -12$ $6x\ +12$ 0	Add. In this case, there is no remainder, so you’re done.

$$(x^2-4x-12)\div(x+2)=x-6$$

Check this by multiplying:

$$(x-6)(x+2)=x^2+2x-6x-12=x^2-4x-12$$

Let’s try another example. In this example, a term is “missing” from the dividend.

✓ Example

Divide: $(x^3-6x-10)\div(x-3)$

Solution

$x-3\backslash\text{longdiv}x^3+0x^2-6x-10$	In setting up this problem, notice that there is an x^3 term but no x^2 term. Add $0x^2$ as a “place holder” for this term. (Since 0 times anything is 0, you’re not changing the value of the dividend.)
x^2 $x-3\backslash\text{longdiv}x^3+0x^2-6x-10$ $-(x^3-3x^2)$	Focus on the first terms again: how many x’s are there in x^3? Since $\frac{x^3}{x}=x^2$, put x^2 in the quotient.
x^2 $x-3\backslash\text{longdiv}x^3+0x^2-6x-10$ $-x^3+3x^2$ $3x^2$	Multiply $x^2(x-3)=x^3-3x^2$ and write this underneath the dividend and prepare to subtract.
x^2 $x-3\backslash\text{longdiv}x^3+0x^2-6x-10$ $-x^3+3x^2$ $3x^2$	Rewrite the subtraction using the opposite of the expression x^3-3x^2. Then add.
x^2 $x-3\backslash\text{longdiv}x^3+0x^2-6x-10$ $-x^3+3x^2$ $3x^2-6x-10$	Bring down the rest of the expression in the dividend. It’s helpful to bring down *all* of the remaining terms.
x^2+3x $x-3\backslash\text{longdiv}x^3+0x^2-6x-10$ $-x^3+3x^2$ $3x^2-6x-10$ $-(3x^2-9x)$	Now, repeat the process with the remaining expression as the dividend: $3x^2-6x-10$.
x^2+3x $x-3\backslash\text{longdiv}x^3+0x^2-6x-10$ $-x^3+3x^2$ $3x^2-6x-10$ $-3x^2+9x$ $3x-10$	Remember to watch the signs!
x^2+3x+3 $x-3\backslash\text{longdiv}x^3+0x^2-6x-10$ $-x^3+3x^2$ $3x^2-6x-10$ $-3x^2+9x$ $3x-10$ $-(3x-9)$	How many x’s are there in $3x$? Since there are 3, multiply $3(x-3)=3x-9$, write this underneath the dividend and prepare to subtract.
x^2+3x+3 $x-3\backslash\text{longdiv}x^3+0x^2-6x-10$ $-x^3+3x^2$ $3x^2-6x-10$ $-3x^2+9x$ $3x-10$ $-3x-9$ -1	Continue until the **degree** of the remainder is *less* than the degree of the divisor. In this case the degree of the remainder, -1, is 0, which is less than the degree of $x-3$, which is 1. Also notice that you have brought down all the terms in the dividend, and that the quotient extends to the right edge of the dividend. These are other ways to check whether you have completed the problem.
$(x^3-6x-10)\div(x-3)=x^2+3x+3,\mathrm{R}-1$ $x^2+3x+3+\frac{-1}{x-3}$, or $x^2+3x+3-\frac{1}{x-3}$	You can write the remainder using the symbol \R, or as a fraction added to the rest of the quotient with the remainder in the numerator and the divisor in the denominator. In this case, since the remainder is negative, you can also subtract the opposite.

Check the result:

$$\begin{aligned}(x-3)\left(x^2+3x+3\right)&=x\left(x^2+3x+3\right)-3\left(x^2+3x+3\right)\\&=x^3+3x^2+3x-3x^2-9x-9\\&=x^3-6x-9\\x^3-6x-9+(-1)&=x^3-6x-10\end{aligned}$$

? Exercise

Divide: $\left(x^2-8x+3\right)\div(x-5)$

A. $x-3-\frac{12}{x-5}$
B. $x-13+\frac{68}{x-5}$
C. $x-13-\frac{62}{x-5}$
D. $x-15$

Answer

A. Correct.

$$\begin{array}{r}x-\quad 3\\ x-5\backslash\text{longdiv}x^2-8x+\quad 3\\ -(x^2-5x)\quad\quad\\ \hline -3x+\quad 3\\ -(-3x+15)\\ \hline -\quad 12\end{array}$$

The correct answer is $x-3-\frac{12}{x-5}$.

B. Incorrect. Remember that you are subtracting whole expressions, not just the first term.

$$\begin{array}{r}x-\quad 3\\ x-5\backslash\text{longdiv}x^2-8x+\quad 3\\ -(x^2-5x)\quad\quad\\ \hline -3x+\quad 3\\ -(-3x+15)\\ \hline -\quad 12\end{array}$$

The correct answer is $x-3-\frac{12}{x-5}$.

C. Incorrect. Remember that you are subtracting whole expressions, not just the first term.

$$\begin{array}{r}x-\quad 3\\ x-5\backslash\text{longdiv}x^2-8x+\quad 3\\ -(x^2-5x)\quad\quad\\ \hline -3x+\quad 3\\ -(-3x+15)\\ \hline -\quad 12\end{array}$$

The correct answer is $x-3-\frac{12}{x-5}$.

D. Incorrect. The remainder is not just added to the quotient. It becomes the numerator of a fraction that is added to the quotient.

$$\begin{array}{r}x-\quad 3\\ x-5\backslash\text{longdiv}x^2-8x+\quad 3\\ -(x^2-5x)\quad\quad\\ \hline -3x+\quad 3\\ -(-3x+15)\\ \hline -\quad 12\end{array}$$

The correct answer is $x-3-\frac{12}{x-5}$.

Dividing Polynomials by Polynomials

The process above works for dividing any polynomials, no matter how many terms are in the divisor or the dividend. The main things to remember are:

- When subtracting, be sure to subtract the whole expression, not just the first term. This is very easy to forget, so be careful!
- Stop when the degree of the remainder is less than the degree of the dividend, or when you have brought down all the terms in the dividend, and that the quotient extends to the right edge of the dividend.

✓ Example

Divide: $\left(3x^3+2x^2-3x+4\right) \div \left(x^2-3x+5\right)$

Solution

$$\begin{array}{r}
3x+11 \\
x^2-3x+5\backslash\text{longdiv}3x^3+2x^2-\;\;3x+\;\;4 \\
-\left(3x^3-9x^2+15x\right) \\
\hline
11x^2-18x+\;\;4 \\
-\left(11x^2-33x+55\right) \\
\hline
15x-51
\end{array}$$

Focus on the first terms: What is $\frac{3x^3}{x^2}$?
Since $\frac{3x^3}{x^2}=3x$, start by putting $3x$ in the quotient.
Follow the process as above.
Watch the signs!
The degree of the remainder is 1, which is less than the degree of the divisor, 2.
You can stop.

$$\begin{array}{rcl}
\left(3x^3+2x^2-3x+4\right) \div \left(x^2-3x+5\right) & = & 3x+11\,\mathrm{R}\,51x-51 \\
& \text{or} & 3x+11+\frac{15x-51}{x^2-3x+5}
\end{array}$$

Check:

$$\begin{aligned}
\left(x^2-3x+5\right)(3x+11) &= \left(x^2-3x+5\right)(3x)+\left(x^2-3x+5\right)(11) \\
&= 3x^3-9x^2+15x+11x^2-33x+55 \\
&= 3x^3+2x^2-18x+55 \\
3x^3+2x^2-18x+55+(15x-51) &= 3x^3+2x^2-3x+4
\end{aligned}$$

? Exercise

Divide: $\left(x^3-2x^2+3x+7\right) \div \left(x^2+2x-1\right)$

A. $x+\frac{2x+7}{x^2+2x-1}$
B. $x+8+\frac{23}{x^2+2x-1}$
C. $x-4+\frac{12x+3}{x^2+2x-1}$
D. $13x-1$

Answer

A. Incorrect. Remember that you are subtracting whole expressions, not just the first term.

$$\begin{array}{r}
x-4 \\
x^2+2x-1\backslash\text{longdiv}x^3-2x^2+3x+7 \\
-\left(x^3+2x^2-x\right) \\
\hline
-4x^2+4x+7 \\
-\left(-4x^2-8x+4\right) \\
\hline
12x+\;\;3
\end{array}$$

The correct answer is $x-4+\frac{12x+3}{x^2+2x-1}$.

B. Incorrect. Check your division again, especially the subtraction of terms.

$$\begin{array}{r} x-4 \\ x^2+2x-1\backslash\text{longdiv}x^3-2x^2+3x+7 \\ -(x^3+2x^2-x) \\ \hline -4x^2+4x+7 \\ -(-4x^2-8x+4) \\ \hline 12x+\ \ 3 \end{array}$$

The correct answer is $x-4+\frac{12x+3}{x^2+2x-1}$.

C. Correct.

$$\begin{array}{r} x-4 \\ x^2+2x-1\backslash\text{longdiv}x^3-2x^2+3x+7 \\ -(x^3+2x^2-x) \\ \hline -4x^2+4x+7 \\ -(-4x^2-8x+4) \\ \hline 12x+\ \ 3 \end{array}$$

The correct answer is $x-4+\frac{12x+3}{x^2+2x-1}$.

D. Incorrect. The remainder is not just added to the quotient. It becomes the numerator of a fraction that is added to the quotient.

$$\begin{array}{r} x-4 \\ x^2+2x-1\backslash\text{longdiv}x^3-2x^2+3x+7 \\ -(x^3+2x^2-x) \\ \hline -4x^2+4x+7 \\ -(-4x^2-8x+4) \\ \hline 12x+\ \ 3 \end{array}$$

The correct answer is $x-4+\frac{12x+3}{x^2+2x-1}$.

Summary

Dividing polynomials by polynomials of more than one term can be done using a process very much like long division of whole numbers. You must be careful to subtract entire expressions, not just the first term. Stop when the degree of the remainder is less than the degree of the divisor. The remainder can be written using R notation, or as a fraction added to the quotient with the remainder in the numerator and the divisor in the denominator.

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SECTION OVERVIEW

11.3: Polynomials with Several Variables

Topic hierarchy

11.3.1: Simplifying and Evaluating Polynomials with More than One Term

11.3.2: Operations with Polynomials

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11.3.1: Simplifying and Evaluating Polynomials with More than One Term

Learning Objectives

- Evaluate a polynomial for given values of each variable.
- Simplify polynomials by collecting like terms.

Introduction

You have studied **polynomials** consisting of constants and/or variables combined by addition or subtraction. The variables may include exponents. The examples so far have been limited to expressions such as $5x^4+3x^3-6x^2+2x$ containing one variable, but polynomials can also contain multiple variables. An example of a polynomial with two variables is $4x^2y-2xy^2+x-7$.

Many formulas are polynomials with more than one variable, such as the formula for the surface area of a rectangular prism: $2ab+2bc+2ac$, where a, b, and c are the lengths of the three sides. By substituting in the values of the lengths, you can determine the value of the surface area. By applying the same principles for polynomials with one variable, you can evaluate or combine like terms in polynomials with more than one variable.

Evaluating Polynomials for Given Values of Each Variable

When you evaluate an expression for a given value, you substitute that given value in the expression, and find its numerical value. In the following example, $x=-2$, you replace all of the $x's$ with a value of -2 and simplify the expression following the order of operations.

Example

Evaluate $7x^2-3x+2$ for $x=-2$.

Solution

$7(-2)^2-3(-2)+2$	Substitute (-2) for each x in the polynomial.
$7(4)-3(-2)+2$	Following the order of operations, evaluate exponents first.
$28+6+2$	Perform the multiplication next.
$34+2$	Combine terms beginning from the left.
36	Find the sum.

You can follow the same procedure when there are two variables in an expression. Let's review an example.

Example

Evaluate $8c-7b$ for $b=4$ and $c=5$.

Solution

$8(5)-7(4)$	Substitute 5 for each c in the polynomial and 4 for each b.
$40-28$	Multiply.
12	Find the difference.

12

As with polynomials with one variable, you must pay attention to the rules of exponents and the order of operations so that you correctly evaluate an expression with two or more variables.

✓ Example

Evaluate x^2+3y^3 for $x=7$ and $y=-2$.

Solution

$(7)^2+3(-2)^3$	Substitute the given values for x and y.
$49+3(-8)$	Evaluate the exponents first.
$49+(-24)$	Multiply.
25	Add.

25

✓ Example

Evaluate $4x^2y-2xy^2+x-7$ for $x=3$ and $y=-1$.

Solution

$4\left(3^2\right)(-1)-2(3)(-1)^2+3-7$	Substitute the given values for x and y.
$4(9)(-1)-2(3)(1)+3-7$	Evaluate the exponents first.
$-36-6+3-7$	Perform multiplication next.
$-42+3-7$ $-39-7$	Perform addition and subtraction from left to right.
-46	Find the difference.

The next example shows how to evaluate a polynomial with two variables. This polynomial is the formula for perimeter of a rectangle.

✓ Example

The formula for the perimeter, P, of a rectangle is $2a+2b$ in which a and b are the lengths of the sides of the rectangle.

Evaluate the formula for $a=6$ inches and $b=10$ inches.

Solution

$2(6)+2(10)$	Substitute the given values for a and b.
$12+20$	Multiply.
32	Add.

? Exercise

Evaluate: $2x^3-xy^2+6$ for $x=-2$ and $y=5$

A. -158
B. -60
C. 14
D. 40

Answer

A. Incorrect. Unless there are parentheses, an exponent only goes with the variable immediately to its left. So, the exponent 3 does not apply to the 2 in that term, only to the variable x. This is true in the next term also; x is not squared. The correct answer is 40.
B. Incorrect. Be more careful with your sign work. The middle term is positive 50, not negative 50. When you substitute in the values of the variables, you get $2(-2)^3-(-2)(5)^2+6$, which is simplified to $2(-8)-(-2)(25)+6$, which equals $-16+50+6$. The correct answer is 40.
C. Incorrect. An exponent indicates the number of times a value is used as a factor. x^3 means x times x times x, not x times 3. $2(-2)(-2)(-2)-(-2)(5)(5)+6=-16+50+6$. The correct answer is 40.
D. Correct. $2(-2)^3-(-2)(5)^2+6=-16+50+6=34+6=40$

Identifying the Degree of a Polynomial with Two or More Variables

Mathematicians use conventions for writing and describing polynomials. A polynomial with one variable can be described by the number of terms it has and the degree of the term with the greatest exponent. Polynomials are commonly written with their terms in descending order of degree. Let's start by looking at an example of a polynomial with one variable: $t^3-10t^2-5t-32$. This polynomial has been written in descending order of degree, starting with the **term** with an exponent of 3 and ending with the term whose degree is 0 because it has no variable. This polynomial is called a third-degree polynomial because its term with the highest degree is the **monomial** t^3**.** (Note that the **degree of a monomial**, t^3, is also 3, because the variable t has an exponent of 3.)

When a polynomial has more than one variable, you can still describe it according to its degree and the degree of its terms. It's a little more complicated. Let's look at a polynomial with two variables: $7x^2y-3xy^3+2x$. This polynomial has three terms and therefore can be called a **trinomial**. To determine the degree of a term, you find the sum of the exponents of all the variables in the term. Here are some examples:

Terms	Sum of the Exponents	Degree of the Term
$7x^2y$	$2+1=3$	3
$-3xy^3$	$1+3=4$	4
$2x$	$1=1$	1

The **degree of a polynomial** is the same as the degree of the term with the highest degree. In this case, $7x^2y-3xy^3+2x$ is a fourth-degree polynomial.

? Exercise

What description below best matches the expression: $2x^4y-5x^3-10xy^3$?

A. A twelfth-degree trinomial
B. A fifth-degree trinomial
C. A third-degree polynomial
D. A fourth-degree polynomial

Answer

A. Incorrect. The degree of a polynomial is the same as the degree of the term whose exponents have the greatest sum, not by the sum of all the exponents in the polynomial. The correct answer is a *fifth degree trinomial*.
B. Correct. $2x^4y$ has a degree of 5 because the exponents of its variables are 4 and 1, and $4+1=5$.
C. Incorrect. The degree of a polynomial is not determined by the number of terms it has. The degree is the same as the degree of the term whose exponents have the greatest sum. The correct answer is a *fifth degree trinomial*.
D. Incorrect. When there are multiple variables, find the degree of a polynomial by determining which term's exponents have the greatest sum. This is the degree of the polynomial. The $2x^4y$ term is the term with the highest degree, the sum of the exponents in the term $2x^4y$ is $4+1=5$. Since there are 3 terms, this polynomial is called a trinomial. The correct answer is a *fifth degree trinomial*.

Simplifying Polynomials with More than One Variable by Combining Like Terms

If a polynomial has **like terms**, the polynomial can be simplified by combining the like terms.

You'll recall that like terms contain the same variables raised to the same power. If there is more than one variable, the same is true: same exact variable(s) each raised to the same exact power.

The polynomial $3xy^3z^2+5xy^3z^2+6x^2y^3z$ has like terms that can be combined. $3xy^3z^2$ and $5xy^3z^2$ are like terms because they have the same variables, x, y, and z, raised to the same powers, x, y^3, and z^2. They can be collected, or combined, to give a result of $8xy^3z^2$. Notice that while $6x^2y^3z$ has the same variables, x, y, and z, the exponents in this term are different, x^2 instead of x, and z instead of z^2. So, $6x^2y^3z$ cannot be combined with the other terms. Instead, the simplified polynomial is written with two terms: $8xy^3z^2+6x^2y^3z$.

✓ Example

Simplify $2xy^2-8x-3xy^2+3x$

Solution

$2xy^2-8x-3xy^2+3x$	Identify any like terms.
$2xy^2-3xy^2-8x+3x$ $(2-3)xy^2+(-8+3)x$ $-1xy^2+(-5)x$	Combine like terms using the distributive property.
$-1xy^2-5x$	Rewrite using subtraction, and check to make sure all like terms have been combined.

$2xy^2-8x-3xy^2+3x=-xy^2-5x$

As with polynomials with one variable, you can combine like terms in polynomials with more than one variable by combining the coefficients of those like terms and keeping the variable part the same. That step is written in the example below. But to save time, you can also just perform the computation in your head.

✓ Example

Simplify $5ba^2+3a^2+a^2b-4a^2-2ab^2$.

Solution

$5ba^2+3a^2+a^2b-4a^2-2ab^2$	Identify the like terms in the polynomial. Since $5ba^2$ can also be written $5a^2b$, it is a like term to a^2b.
$(5+1)a^2b+(3-4)a^2-2ab^2$ $6a^2b-a^2-2ab^2$	Combine like terms using the distributive property and check to make sure all like terms have been combined.

$5ba^2+3a^2+a^2b-4a^2-2ab^2=6a^2b-a^2-2ab^2$

? Exercise

Simplify by collecting like terms: $4\left(x^2y+7y\right)-5y\left(3x^2-y\right)-10y$

A. $-11x^2y+5y^2+18y$
B. $4x^2y+11y-8yx^2-16y$
C. $4x^2y+18y-15yx^2+5y^2$
D. $4x^2y-5y\left(3x^2-y\right)-3y$

Answer

A. Correct.

$4\left(x^2y+7y\right)-5y\left(3x^2-y\right)-10y$ equals
$4x^2y+28y-15yx^2+5y^2-10y$. This equals
$4x^2y+18y-15yx^2+5y^2$, which equals
$-11x^2y+5y^2+18y$

B. Incorrect. The 4 is distributed to both terms in the parentheses by multiplying each term by 4 resulting in $28y$, not $11y$. The $-5y$ is similarly distributed to each term in the parentheses, resulting in $-15yx^2+5y^2$. The correct answer is $-11x^2y+5y^2+18y$.

C. Incorrect. This polynomial can be further simplified by combining the like terms $4x^2y$ and $-15yx^2$. The order of the variables in a term does not matter. The correct answer is $-11x^2y+5y^2+18y$.

D. Incorrect. Before combining like terms, you must distribute to clear the parentheses. The correct answer is $-11x^2y+5y^2+18y$.

Summary

Polynomials can contain more than one variable and can be evaluated in the same way as polynomials with one variable. To evaluate any polynomial, you substitute the given values for the variable and perform the computation to simplify the polynomial to a numerical value. The order of operations and integer operations must be properly applied to correctly evaluate a polynomial.

Polynomials with more than one variable can be simplified by combining like terms, as you can do with polynomials with one variable. Like terms must contain the same exact variables raised to the same exact power. In terms with more than one variable, the order in which the variables are written does not matter.

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11.3.2: Operations with Polynomials

Learning Objectives

- Add polynomials with more than one variable.
- Subtract polynomials with more than one variable.
- Multiply polynomials with more than one variable.
- Divide polynomials with more than one variable.

Introduction

Just as you can perform the four operations on **polynomials** with one variable, you can add, subtract, multiply, and divide polynomials with more than one variable. The process is exactly the same, but you have more variables to keep track of. When you are adding and subtracting polynomials with more than one variable, you have to pay particular attention to combining like terms only. When you multiply and divide, you also need to pay particular attention to the multiple variables and terms. You can multiply and divide terms that aren't like, but to add and subtract terms they must be like terms.

Adding Polynomials with More Than One Variable

To add polynomials, you first need to identify the like terms in the polynomials and then combine them according to the correct integer operations. Since like terms must have the same variables raised to the same exact power, identifying them in polynomials with more than one variable takes a careful eye. Sometimes parentheses are used to distinguish between the addition of two polynomials and the addition of a collection of monomials. With addition, you can simply remove the parentheses and perform the addition.

Example

Add. $(4x^2-12xy+9y^2)+(25x^2+4xy-32y^2)$

Solution

$4x^2+(-12xy)+9y^2+25x^2+4xy+(-32y^2)$	Remove the parentheses grouping the polynomial and rewrite any subtraction as addition of the opposite.
$(4x^2+25x^2)+[(-12xy)+4xy]+[9y^2+(-32y^2)]$	Group like terms using commutative and associative properties.
$29x^2+(-8xy)+(-23y^2)$	Combine like terms. Then rewrite as subtraction for the answer.

The sum is $29x^2-8xy-23y^2$

Some people find that writing the polynomial addition in a vertical form makes it easy to combine like terms. The process of adding the polynomials is the same, but the arrangement of the terms is different. The example below shows this "vertical" method of adding polynomials:

Example

Add. $(3x+2y-4z)+(45x-y+75z)$

Solution

$\begin{array}{r} 3x+2y-4z \\ +45x-y+75z \\ \hline 48x+y+71z \end{array}$	Write one polynomial below the other, making sure to line up like terms. Combine like terms, paying close attention to the signs.

The sum is $48x+y+71z$.

When there isn't a matching like term for every term in each polynomial, there will be empty places in the vertical arrangement of the polynomials. This layout makes it easy to check that you are combining like terms only.

✓ Example

Add. $(10ab+15ac-25bc+5)+(4ab-8bc-12)$

Solution

$\begin{array}{rrrrr} & 10ab & +15ac & -25bc & +\ 5 \\ + & 4ab & & -\ 8bc & -12 \\ \hline & 14ab & +15ac & -33bc & -7 \end{array}$	Write one polynomial below the other, making sure to line up like terms. Combine like terms, paying close attention to the signs.

The sum is $14ab+15ac-33bc-7$.

? Exercise

Add. $(8a^3b^2+6a^2b-4b^2+5)+(10a^2b-4a^3b^2+6a^2-7)$

A. $18a^3b^2+2a^2b+2b^2-2$
B. $4a^3b^2+16a^2b+6a^2-4b^2-2$
C. $18a^3b^2+2a^2b+6a^2-4b^2-2$
D. $4a^6b^4+16a^4b^2+6a^2-4b^2-2$

Answer

A. Incorrect. Be sure to combine like terms only. The sequence of terms in the polynomials is not the same. Also, the terms $-4b^2$ and $6a^2$ are not like terms and cannot be combined. The correct answer is $4a^3b^2+16a^2b+6a^2-4b^2-2$.
B. Correct. $(8a^3b^2+6a^2b-4b^2+5)+(10a^2b-4a^3b^2+6a^2-7)$ equals $8a^3b^2-4a^3b^2+6a^2b+10a^2b-4b^2+6a^2-7+5$ which equals $4a^3b^2+16a^2b+6a^2-4b^2-2$.
C. Incorrect. Be sure to combine like terms only. The sequence of terms in the polynomials is not the same. The correct answer is $4a^3b^2+16a^2b+6a^2-4b^2-2$.
D. Incorrect. When you add like terms, do not add the exponents, only the coefficients. The correct answer is $4a^3b^2+16a^2b+6a^2-4b^2-2$.

Subtracting Polynomials with More Than One Variable

You can apply the same process used to subtract polynomials with one variable to subtract polynomials with more than one variable. In order to remove the parentheses following a subtraction sign, you must multiply each term by -1.

✓ Example

Subtract. $(14x^3y^2-5xy+14y)-(7x^3y^2-8xy+10y)$

Solution

$14x^3y^2-5xy+14y-7x^3y^2+8xy-10y$	Remove the parentheses. Notice the signs!
$14x^3y^2-7x^3y^2-5xy+8xy+14y-10y$	Regroup to put like terms together. When you regroup or rearrange terms involving subtraction, think of the subtraction as “adding the opposite” and move the negative sign along with the term.
$7x^3y^2+3xy+4y$	Combine like terms.

The difference is $7x^3y^2+3xy+4y$

An alternative to the approach shown above is the vertical method for arranging the subtraction problem. This method is shown below for a different problem. Both methods are effective for subtracting polynomials. The idea is to identify and organize like terms in order to compute with them accurately.

Example

Subtract. $(10a^3+5b^2-5c+10)-(15+5c-15b^2+10a^3)$

Solution

$\begin{array}{r} 10a^3+\quad 5b^2-5c+10 \\ \underline{-(10a^3-15b^2+5c+15)} \end{array}$	Organize the like terms using the vertical approach.
$0+20b^2-10c-5$	Combine like terms. Pay attention to the signs when subtracting.

The difference is $20b^2-10c-5$.

The examples that follow illustrate the left-to-right and vertical methods for the same polynomial subtraction problem. Think about which method you find easier.

Example

Subtract. $(3x^4y^3+5x^3y^2-2x^2y^2)-(-2x^4y^3+4x^3y^2-2x^2y^3-1)$

Solution

$3x^4y^3+5x^3y^2-2x^2y^2+2x^4y^3-4x^3y^2+2x^2y^3+1$	Remove the parentheses. The first polynomial stays the same. The signs change in the second polynomial.
$3x^4y^3+2x^4y^3+5x^3y^2-4x^3y^2-2x^2y^2+2x^2y^3+1$	Regroup using commutative and associative properties.
$5x^4y^3+x^3y^2-2x^2y^2+2x^2y^3+1$	Combine like terms.

The difference is $5x^4y^3+x^3y^2-2x^2y^2+2x^2y^3+1$.

Example

Subtract. $(3x^4y^3+5x^3y^2-2x^2y^2)-(-2x^4y^3+4x^3y^2-2x^2y^3-1)$

Solution

$\begin{array}{r} 3x^4y^3+5x^3y^2-2x^2y^2\qquad\qquad\quad \\ \underline{-(-2x^4y^3+4x^3y^2\qquad -2x^2y^3-1)} \\ 5x^4y^3+x^3y^2-2x^2y^2+2x^2y^3+1 \end{array}$	Write one polynomial below the other, making sure to line up like terms. Combine like terms, paying close attention to the signs.

The difference is $5x^4y^3+x^3y^2-2x^2y^2+2x^2y^3+1$.

Multiplying Polynomials with More Than One Variable

Polynomials with more than one variable can also be multiplied by one another. You use the same techniques you used when you multiplied polynomials with only one variable. Consider the following example.

$$(4x^2y^3)(5x^4y^2)$$

This is an example of multiplication of two polynomials, specifically monomials, with two variables. To do this multiplication, you multiply the coefficients and use the rules of exponents to find the exponent for each variable in order to find the product. Let's look.

$$(4x^2y^3)(5x^4y^2)=(4\cdot 5)(x^{2+4})(y^{3+2})=20x^6y^5$$

To multiply a monomial by a binomial, you use the distributive property in the same way as multiplying polynomials with one variable.

✓ Example

Multiply. $5x^2y^2\left(2x^2+5xy-10\right)$

Solution

$5x^2y^2\left(2x^2+5xy-10\right)$ $5x^2y^2\left(2x^2\right)+5x^2y^2(5xy)-5x^2y^2(10)$ $5(2)x^2\left(x^2\right)y^2+5(5)x^2(x)y^2(y)-5(10)x^2y^2$ $10x^4y^2+25x^3y^3-50x^2y^2$	Multiply, using the distributive property.

The product is $10x^4y^2+25x^3y^3-50x^2y^2$.

To multiply two binomials containing more than one variable, you can still use the FOIL (First, Outer, Inner, Last) method that works for binomials with one variable. After all, FOIL is simply a shortcut for using the distributive property to multiply each term in one binomial by each term in the other binomial. This process works for multiplying any two binomials. Two examples follow.

✓ Example

Multiply. $(4x-7xy)(2y+3x)$

Solution

First: $4x\cdot 2y=8xy$ Outer: $4x\cdot 3x=12x^2$ Inner: $-7xy\cdot 2y=-14xy^2$ Last: $-7xy\cdot 3x=-21x^2y$	Be careful to include the negative sign with $-7xy$, since this term is being subtracted.
$8xy+12x^2-14xy^2-21x^2y$	Combine terms into one expression.

The product is $8xy+12x^2-14xy^2-21x^2y$.

The next example shows the product of a binomial and a trinomial, each with two variables. Since FOIL can only be used with the product of two binomials, you need to systematically multiply each term in the binomial by each term in the trinomial.

✓ Example

Multiply. $(9b-ab)\left(5a^2b+7ab-b\right)$

Solution

$9b\left(5a^2b+7ab-b\right)$ $45a^2b^2+63ab^2-9b^2$	Multiply $9b$ by each term in the trinomial, paying attention to the signs.
$-ab\left(5a^2b+7ab-b\right)$ $-5a^3b^2-7a^2b^2+ab^2$	Multiply $-ab$ by each term in the trinomial, paying attention to the signs.
$45a^2b^2+63ab^2-9b^2-5a^3b^2-7a^2b^2+ab^2$	Combine the products.
$45a^2b^2+63ab^2-9b^2-5a^3b^2-7a^2b^2+ab^2$ $38a^2b^2+64ab^2-9b^2-5a^3b^2$	Combine like terms.

The product is $38a^2b^2+64ab^2-9b^2-5a^3b^2$.

When multiplying multivariable polynomials like this, some people prefer to set up the multiplication in a vertical fashion, as you would do if you were multiplying $45\cdot 189$. The example below shows $(9b-ab)\left(5a^2b+7ab-b\right)$ set up in vertical fashion.

Example

Multiply. $(9b - ab)\left(5a^2b + 7ab - b\right)$

Solution

$\begin{array}{rr} & 9b - ab \\ \times & 5a^2b + 7ab - b \\ \hline & -9b^2 + ab^2 \end{array}$	Set up the problem in vertical form, and begin by multiplying $9b - ab$ by $-b$. Make sure to pay attention to the signs! Place the products underneath, as shown.
$\begin{array}{rrr} & 9b - & ab \\ \times & 5a^2b + 7ab - & b \\ \hline & -9b^2 + & ab^2 \\ -7a^2b^2 & & +63ab^2 \end{array}$	Now multiply $9b - ab$ by $+7ab$. Notice that $(9b)(7ab) = 63ab^2$; since this term is like ab^2, place it directly beneath it.
$\begin{array}{rrr} & 9b - & ab \\ \times & 5a^2b + 7ab - & b \\ \hline & -9b^2 + & ab^2 \\ & -7a^2b^2 & +63ab^2 \\ -5a^3b^2 + 45a^2b^2 & & \\ \hline \end{array}$	Finally, multiply $9b - ab$ by $5a^2b$.
$\begin{array}{rrr} & 9b - & ab \\ \times & 5a^2b + 7ab - & b \\ \hline & -9b^2 + & ab^2 \\ & -7a^2b^2 & +63ab^2 \\ -5a^3b^2 + 45a^2b^2 & & \\ \hline & -5a^3b^2 + 38a^2b^2 - 9b^2 + 64ab^2 & \end{array}$	Now add like terms.

$-5a^3b^2 + 38a^2b^2 - 9b^2 + 64ab^2$

Notice that the products of the two examples are the same, although the order of the individual terms is different due to the different solution methods.

Exercise

Find the product. $-4pt^2\left(5pt^3 + 3pt^2 - t\right)$

A. $-20p^2t^5 - 12p^2t^4 + 4pt^3$
B. $-20t^5 + 12p^2t^4 - 4pt^3$
C. $-20pt^6 - 12pt^4 + 4pt^2$
D. $-20p^2t^5 + 3pt^2 - t$

Answer

A. Correct. Rewriting the subtraction as adding the opposite gives $-4pt^2\left(5pt^3 + 3pt^2 + (-t)\right)$. Distributing the monomial $-4pt^2$ gives $-4pt^2 \cdot 5pt^3 + \left(-4pt^2 \cdot 3pt^2\right) + \left(-4pt^2 \cdot -t\right)$, which is $-20p^2t^5 - 12p^2t^4 + 4pt^3$.

B. Incorrect. The negative must be distributed to all terms along with the $4pt^2$. This changes the sign of the middle and last terms. The correct answer is $-20p^2t^5 - 12p^2t^4 + 4pt^3$.

C. Incorrect. By the laws of exponents, you add (not multiply) exponents when multiplying: $-4pt^2 \cdot 5pt^3 + \left(-4pt^2 \cdot 3pt^2\right) + \left(-4pt^2 \cdot -t\right)$ is $-20p^2t^5 - 12p^2t^4 + 4pt^3$. The correct answer is $-20p^2t^5 - 12p^2t^4 + 4pt^3$.

D. Incorrect. You must distribute the monomial to all three terms in the polynomial, not just the first one: $-4pt^2 \cdot 5pt^3 + \left(-4pt^2 \cdot 3pt^2\right) + \left(-4pt^2 \cdot -t\right)$. The correct answer is $-20p^2t^5 - 12p^2t^4 + 4pt^3$.

Dividing Polynomials with More Than One Variable

The fourth arithmetic operation is division. Polynomials with more than one variable can also be divided. When dividing monomials with more than one variable, you divide the coefficients and then divide variables. When there are exponents with the same base, the law of exponents says you divide by subtracting the exponents. Consider this example.

Example

Divide. $\frac{14x^3y}{28x^6y^4}$

Solution

$\left(\frac{14}{28}\right)\left(\frac{x^3}{x^6}\right)\left(\frac{y}{y^4}\right)$	To make it easier, you can break up the coefficients and variables into numeric and variable factors.
$\frac{1}{2}x^{-3}y^{-3}$	Divide the coefficients, and divide the variables by subtracting the exponents with like bases.
$\frac{1}{2x^3y^3}$	Rewrite with positive exponents.

The quotient is $\frac{1}{2x^3y^3}$.

Now let’s look at an example of dividing a trinomial with more than one variable by a monomial with more than one variable. This follows the same procedure as when you have one variable, but you need to pay attention to distinguishing between the variables.

Example

Divide. $\frac{4x^4y^5-2x^8y^3+6x^3y^2}{2x^2y}$

Solution

$\frac{4x^4y^5-2x^8y^3+6x^3y^2}{2x^2y}$ $\left(\frac{4x^4y^5}{2x^2y}\right)-\left(\frac{2x^8y^3}{2x^2y}\right)+\left(\frac{6x^3y^2}{2x^2y}\right)$	To make it easier, you can break the division out by the terms in the polynomial since each term is being divided by $2x^2y$.
$2x^2y^4-x^6y^2+3xy$	Perform the division of each term by dividing the coefficients, and by dividing the variables by subtracting the exponents of variables with like bases.

The quotient is $2x^2y^4-x^6y^2+3xy$.

Exercise

Divide. $\frac{25s^2t^4-10s^3t^3+5st^2}{5st^2}$

A. $5st^2-2s^2t+1$
B. $5st^2-10s^3t^3+5st^2$
C. $20st^2-5s^2t$
D. $\frac{25s^2t^4-10s^3t^3+5st^2}{5st^2}$

Answer

A. Correct. Divide each term in the polynomial by the monomial: $\frac{25s^2t^4}{5st^2}-\frac{10s^3t^3}{5st^2}+\frac{5st^2}{5st^2}$.
B. Incorrect. You only divided the first term. Divide each term in the polynomial by the monomial: $\frac{25s^2t^4}{5st^2}-\frac{10s^3t^3}{5st^2}+\frac{5st^2}{5st^2}$. The correct answer is $5st^2-2s^2t+1$.
C. Incorrect. Divide, don’t subtract, the coefficients. The correct answer is $5st^2-2s^2t+1$.
D. Incorrect. Perform the division to simplify. It’s okay that they are not like terms, you can still divide. You have to have like terms when you add or subtract terms, not when you multiply and divide terms. Divide each term in the polynomial by the monomial: $\frac{25s^2t^4}{5st^2}-\frac{10s^3t^3}{5st^2}+\frac{5st^2}{5st^2}$. The correct answer is $5st^2-2s^2t+1$.

Summary

Performing addition, subtraction, multiplication, and division of polynomials with more than one variable follows the same steps as operating on polynomials in one variable. The key things to pay attention to are combining only like terms and applying the laws of exponents, integer operations, and the order of operations accurately.

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CHAPTER OVERVIEW

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SECTION OVERVIEW

12.1: Introduction to Factoring

Topic hierarchy

12.1.1: Greatest Common Factor

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12.1.1: Greatest Common Factor

Learning Objectives

- Find the greatest common factor (GCF) of monomials.
- Factor polynomials by factoring out the greatest common factor (GCF).
- Factor expressions with four terms by grouping.

Introduction

Factors are the building blocks of multiplication. They are the numbers that you can multiply together to produce another number: 2 and 10 are factors of 20, as are 4 and 5 and 1 and 20. To factor a number is to rewrite it as a product. $20=4\cdot 5$.

Likewise to factor a **polynomial**, you rewrite it as a product. Just as any integer can be written as the product of factors, so too can any **monomial** or polynomial be expressed as a product of factors. **Factoring** is very helpful in simplifying and solving equations using polynomials.

A **prime factor** is similar to a **prime number**: it has only itself and 1 as factors. The process of breaking a number down into its prime factors is called **prime factorization**.

Greatest Common Factor

Let's first find the **greatest common factor (GCF)** of two whole numbers. The GCF of two numbers is the greatest number that is a factor of *both* of the numbers. Take the numbers 50 and 30.

$$50=10\cdot 5$$
$$30=10\cdot 3$$

Their greatest common factor is 10, since 10 is the greatest factor that both numbers have in common.

To find the GCF of greater numbers, you can factor each number to find their prime factors, identify the prime factors they have in common, and then multiply those together.

Example

Find the greatest common factor of 210 and 168.

Solution

$$210=\mathbf{2}\cdot\mathbf{3}\cdot 5\cdot\mathbf{7}$$
$$168=\mathbf{2}\cdot 2\cdot 2\cdot\mathbf{3}\cdot\mathbf{7}$$
$$\text{GCF}=\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{7}$$

$$\text{GCF}=42$$

Because the GCF is the product of the prime factors that these numbers have in common, you know that it is a factor of both numbers. (If you want to test this, go ahead and divide both 210 and 168 by 42: they are both evenly divisible by this number!)

Finding the greatest common factor in a set of monomials is not very different from finding the GCF of two whole numbers. The method remains the same: factor each monomial independently, look for common factors, and then multiply them to get the GCF.

Example

Find the greatest common factor of $25b^3$ and $10b^2$.

Solution

$$25b^3=\mathbf{5}\cdot 5\cdot\mathbf{b}\cdot\mathbf{b}\cdot b$$
$$10b^2=\mathbf{5}\cdot 2\cdot\mathbf{b}\cdot\mathbf{b}$$
$$\text{GCF}=\mathbf{5}\cdot\mathbf{b}\cdot\mathbf{b}$$

$$\text{GCF}=5b^2$$

The monomials have the factors 5, b, and b in common, which means their greatest common factor is $5 \cdot b \cdot b$, or simply $5b^2$.

✓ Example

Find the greatest common factor of $81c^3d$ and $45c^2d^2$.

Solution

$$81c^3d = \mathbf{3} \cdot \mathbf{3} \cdot 3 \cdot 3 \cdot \mathbf{c} \cdot \mathbf{c} \cdot c \cdot d$$
$$45c^2d^2 = \mathbf{3} \cdot \mathbf{3} \cdot 5 \cdot \mathbf{c} \cdot \mathbf{c} \cdot \mathbf{d} \cdot d$$
$$\text{GCF} = \mathbf{3} \cdot \mathbf{3} \cdot \mathbf{c} \cdot \mathbf{c} \cdot \mathbf{d}$$

$$\text{GCF} = 9c^2d$$

? Exercise

Find the greatest common factor of $56xy$ and $16y^3$.

A. 8
B. $8y$
C. $16y$
D. $8xy^3$

Answer

A. Incorrect. 8 is a common factor, but you need to account for the variable terms that the two monomials have in common as well. The correct answer is $8y$.
B. Correct. The expression $56xy$ can be factored as $2 \cdot 2 \cdot 2 \cdot 7 \cdot x \cdot y$, and $16y^3$ can be factored as $2 \cdot 2 \cdot 2 \cdot 2 \cdot y \cdot y \cdot y$. They have the factors $2 \cdot 2 \cdot 2$ and y in common. Multiplying them together will give you the GCF: $8y$.
C. Incorrect. y is a common factor, but 16 is not: 56 is not evenly divisible by 16. Think of numbers that are factors of both 56 and 16. The correct answer is $8y$.
D. Incorrect. 8 is a common factor, but you need to account for the variable terms that the two monomials have in common as well. The entire term xy^3 is not a factor of either monomial. The correct answer is $8y$.

Using the GCF to Factor Polynomials

When two or more monomials are combined (either added or subtracted), the resulting expression is called a polynomial. If you can find common factors for each term of the polynomial, then you can factor the polynomial.

As you look at the examples of simple polynomials below, try to identify factors that the terms of the polynomial have in common.

Polynomial	**Terms**	**Common Factors**
$6x+9$	$6x$ and 9	3 is a factor of $6x$ and 9
a^2-2a	a^2 and $2a$	a is a factor of a^2 and $2a$
$4c^3+4c$	$4c^3$ and $4c$	4 and c are factors of $4c^3$ and $4c$

To factor a polynomial, first identify the greatest common factor of the terms. You can then use the distributive property to rewrite the polynomial in a factored form. Recall that the **distributive property of multiplication over addition** states that a product of a number and a sum is the same as the sum of the products.

Product of a number and a sum: $a(b+c) = a \cdot b + a \cdot c$. You can say that " a is being distributed over $b+c$.")

Sum of the products: $a \cdot b + a \cdot c = a(b+c)$. Here you can say that " a is being factored out."

In both cases, it is the distributive property that is being used.

Example

Factor $25b^3 + 10b^2$**.**

Solution

$25b^3 = \mathbf{5} \cdot 5 \cdot \mathbf{b} \cdot \mathbf{b} \cdot b$ $10b^2 = \mathbf{5} \cdot 2 \cdot \mathbf{b} \cdot \mathbf{b}$	Find the GCF. From a previous example, you found the GCF of $25b^3$ and $10b^2$ to be $5b^2$.
$\text{GCF} = \mathbf{5} \cdot \mathbf{b} \cdot \mathbf{b} = 5b^2$ $25b^3 = 5b^2 \cdot 5b$ $10b^2 = 5b^2 \cdot 2$	Rewrite each term with the GCF as one factor.
$5b^2 \cdot 5b + 5b^2 \cdot 2$	Rewrite the polynomial using the factored terms in place of the original terms.
$5b^2(5b + 2)$	Factor out the $5b^2$.

$5b^2(5b + 2)$

The factored form of the polynomial $25b^3 + 10b^2$ is $5b^2(5b + 2)$. You can check this by doing the multiplication. $5b^2(5b + 2) = 25b^3 + 10b^2$.

Note that if you do not factor the greatest common factor at first, you can continue factoring, rather than start all over.

For example:

Factor out 5: $25b^3 + 10b^2 = 5\left(5b^3 + 2b^2\right)$

Then factor out b^2: $5b^2(5b + 2)$

Notice that you arrive at the same simplified form whether you factor out the GCF immediately or if you pull out factors individually.

Example

Factor $81c^3d + 45c^2d^2$**.**

Solution

$\mathbf{3} \cdot \mathbf{3} \cdot 9 \cdot \mathbf{c} \cdot \mathbf{c} \cdot c \cdot \mathbf{d}$	Factor $81c^3d$.
$\mathbf{3} \cdot \mathbf{3} \cdot 5 \cdot \mathbf{c} \cdot \mathbf{c} \cdot \mathbf{d} \cdot d$	Factor $45c^2d^2$.
$\mathbf{3} \cdot \mathbf{3} \cdot \mathbf{c} \cdot \mathbf{c} \cdot \mathbf{d} = 9c^2d$	Find the GCF.
$81c^3d = 9c^2d \cdot 9c$ $45c^2d^2 = 9c^2d \cdot 5d$	Rewrite each term as the product of the GCF and the remaining terms.
$9c^2d \cdot 9c + 9c^2d \cdot 5d$	Rewrite the polynomial expression using the factored terms in place of the original terms.
$9c^2d(9c + 5d)$	Factor out $9c^2d$.

$9c^2d(9c + 5d)$

? Exercise

Factor $8a^6 - 11a^5$.

A. $88\left(a^6 - a^5\right)$
B. $8a\left(a^5 - 3\right)$
C. $a^5(a - 1)$
D. $a^5(8a - 11)$

Answer

A. Incorrect. 88 is the least common multiple, not the greatest common factor, of 11 and 8. If $88\left(a^6-a^5\right)$ were expanded it would become $88a^6-88a^5$, not $8a^6-11a^5$. The correct answer is $a^5(8a-11)$.
B. Incorrect. 8 is not a common factor of 8 and 11. If $8a\left(a^5-3\right)$ were expanded, it would become $8a^6-24a$, not $8a^6-11a^5$. The correct answer is $a^5(8a-11)$.
C. Incorrect. a^5 is a common factor, but the values 8 and 11 have been left out of this factorization. If $a^5(a-1)$ were expanded it would become a^6-a^5, not $8a^6-11a^5$. The correct answer is $a^5(8a-11)$.
D. Correct. The values 8 and 11 share no common factors, but the GCF of a^6 and a^5 is a^5. So you can factor out a^5 and rewrite the polynomial as $a^5(8a-11)$.

Factoring by Grouping

The distributive property allows you to factor out common factors. However, what do you do if the terms within the polynomial do not share any common factors?

If there is no common factor for all of the terms in the polynomial, another technique needs to be used to see if the polynomial can be factored. It involves organizing the polynomial in groups.

✓ Example

Factor $4ab+12a+3b+9$

Solution

$(4ab+12a)+(3b+9)$	Group terms into pairs.
$4ab=\mathbf{2}\cdot\mathbf{2}\cdot\mathbf{a}\cdot b$ $12a=3\cdot\mathbf{2}\cdot\mathbf{2}\cdot\mathbf{a}$ $\text{GCF}\ =4a$	Find the GCF of the first pair of terms.
$(4a\cdot b+4a\cdot 3)+(3b+9)$ $4a(b+3)+(3b+9)$	Factor the GCF, $4a$, out of the first group.
$3b=\mathbf{3}\cdot b$ $9=\mathbf{3}\cdot 3$ $\text{GCF}\ =3$	Find the GCF of the second pair of terms.
$4a(b+3)+(3\cdot b+3\cdot 3)$ $4a(b+3)+3(b+3)$	Factor 3 out of the second group.
$4a(b+3)+3(b+3)$	Notice that the two terms have a common factor, $(b+3)$.
$(b+3)(4a+3)$	Factor out the common factor $(b+3)$ from the two terms.

$(b+3)(4a+3)$

Notice that when you factor two terms, the result is a monomial times a polynomial. But the factored form of a four-term polynomial is the product of two binomials.

This process is called the *grouping technique*. Broken down into individual steps, here's how to do it (you can also follow this process in the example below).

- Group the terms of the polynomial into pairs.
- Factor each pair of terms (find the greatest common factor and then use the distributive property to pull out the GCF).
- Look for common factors between the factored forms of the paired terms.
- Factor the common polynomial out of the groups.

Let’s try factoring a few more four-term polynomials. Notice that in the example below, the first term is x^2, and x is the only variable present.

✓ Example

Factor $x^2+2x+5x+10$**.**

Solution

$(x^2+2x)+(5x+10)$	Group terms of the polynomial into pairs.
$x(x+2)+(5x+10)$	Factor out the like factor, x, from the first group.
$x(x+2)+5(x+2)$	Factor out the like factor, 5, from the second group.
$(x+2)(x+5)$	Look for common factors between the factored forms of the paired terms. Here, the common factor is $(x+2)$. Factor out the common factor, $(x+2)$, from both terms. The polynomial is now factored.

$(x+2)(x+5)$

This method of factoring only works in some cases. Notice that both factors here contain the term.

✓ Example

Factor $2x^2-3x+8x-12$**.**

Solution

$(2x^2-3x)+(8x-12)$	Group terms into pairs.
$x(2x-3)+4(2x-3)$	Factor the common factor, x, out of the first group and the common factor, 4, out of the second group.
$(x+4)(2x-3)$	Factor out the common factor, $(2x-3)$, from both terms.

$(x+4)(2x-3)$

✓ Example

Factor $3x^2+3x-2x-2$**.**

Solution

$(3x^2+3x)+(-2x-2)$	Group terms into pairs. Since subtraction is the same as addition of the opposite, you can write $-2x-2 \text{ as } +(-2x-2)$
$3x(x+1)+(-2x-2)$	Factor the common factor $3x$ out of first group.
$3x(x+1)-2(x+1)$	Factor the common factor of -2 out of the second group. Notice what happens to the signs within the parentheses once -2 is factored out.
$(x+1)(3x-2)$	Factor out the common factor, $(x+1)$, from both terms.

$(x+1)(3x-2)$

? Exercise

Factor $10ab+5b+8a+4$.

A. $(2a+1)(5b+4)$
B. $(5b+2a)(4+1)$
C. $5(2ab+b+8a+4)$

D. $(4+2a)(5b+1)$

Answer

A. Correct. The polynomial $10ab+5b+8a+4$ can be grouped as $(10ab+5b)+(8a+4)$. Pulling out common factors, you find: $5b(2a+1)+4(2a+1)$. Since $(2a+1)$ is a common factor, the factored form is $(2a+1)(5b+4)$.
B. Incorrect. When factoring $5b$ out of $10ab$ and $5b$, the remaining $2a$ and 1 must still be added and multiplied by the common factor $5b$: $5b(2a+1)$. Similarly, factoring out the 4 from $8a+4$ leaves $4(2a+1)$. Then you can factor $(2a+1)$ from the sum of those expressions to get the correct factorization, $(2a+1)(5b+4)$.
C. Incorrect. The 5 is a common factor only for $10ab+5b$, giving $5b(2a+1)$. The other pair, $8a+4$, has a common factor of 4. Factoring them gives $4(2a+1)$. Since both expressions have a common factor of $2a+1$, you can factor again to get $(2a+1)(5b+4)$.
D. Incorrect. You correctly identified $5b$ as a factor of one pair, leaving $2a$ and 1, and 4 as the factor of the other pair, also leaving $2a$ and 1. However, this gives $5b(2a+1)+4(2a+1)$. If you had $5bx+4x$, you could factor out the x to get $x(5b+4)$, so factoring out the $(2a+1)$ gives $(2a+1)(5b+4)$.

Sometimes, you will encounter polynomials that, despite your best efforts, cannot be factored into the product of two binomials.

✓ Example

Factor $7x^2-21x+5x-5$.

Solution

$(7x^2-21x)+(5x-5)$	Group terms into pairs.
$7x(x-3)+(5x-5)$	Factor the common factor $7x$ out of the first group.
$7x(x-3)+5(x-1)$	Factor the common factor 5 out of the second group.
$7x(x-3)+5(x-1)$	The two groups $7x(x-3)$ and $5(x-1)$ do not have any common factors, so this polynomial cannot be factored any further.

Cannot be factored

In the example above, each pair can be factored, but then there is no common factor between the pairs!

Summary

A whole number, monomial, or polynomial can be expressed as a product of factors. You can use some of the same logic that you apply to factoring integers to factoring polynomials. To factor a polynomial, first identify the greatest common factor of the terms, and then apply the distributive property to rewrite the expression. Once a polynomial in $a\cdot b+a\cdot c$ form has been rewritten as $a(b+c)$, where a is the GCF, the polynomial is in factored form.

When factoring a four-term polynomial using grouping, find the common factor of pairs of terms rather than the whole polynomial. Use the distributive property to rewrite the grouped terms as the common factor times a binomial. Finally, pull any common binomials out of the factored groups. The fully factored polynomial will be the product of two binomials.

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SECTION OVERVIEW

12.2: Factoring Polynomials

Topic hierarchy

12.2.1: Factoring Trinomials

12.2.2: Factoring- Special Cases

12.2.3: Special Cases- Cubes

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12.2.1: Factoring Trinomials

Note

- Factor trinomials with a leading coefficient of 1.
- Factor trinomials with a common factor.
- Factor trinomials with a leading coefficient other than 1.

Introduction

A **polynomial** with three terms is called a **trinomial**. Trinomials often (but not always!) have the form x^2+bx+c. At first glance, it may seem difficult to factor trinomials, but you can take advantage of some interesting mathematical patterns to factor even the most difficult-looking trinomials.

So, how do you get from $6x^2+2x-20$ to $(2x+4)(3x-5)$? Let’s take a look.

Factoring Trinomials: x^2+bx+c

Trinomials in the form x^2+bx+c can often be factored as the product of two **binomials**. Remember that a binomial is simply a two-term polynomial. Let’s start by reviewing what happens when two binomials, such as $(x+2)$ and $(x+5)$, are multiplied.

Example

Multiply $(x+2)(x+5)$**.**

Solution

$(x+2)(x+5)$	Use the FOIL method to multiply binomials.
$x^2+5x+2x+10$	Then combine the like terms $2x$ and $5x$.

$x^2+7x+10$

Factoring is the reverse of multiplying. So let’s go in reverse and factor the trinomial $x^2+7x+10$. The individual terms x^2, $7x$, and 10 share no common factors. So look at rewriting $x^2+7x+10$ as $x^2+5x+2x+10$.

And, you can group pairs of factors: $(x^2+5x)+(2x+10)$

Factor each pair: $x(x+5)+2(x+5)$

Then factor out the common factor $x+5$: $(x+5)(x+2)$

Here is the same problem done in the form of an example:

Example

Factor $x^2+7x+10$**.**

Solution

$x^2+5x+2x+10$	Rewrite the middle term $7x$ as $5x+2x$.
$x(\mathbf{x+5})+2(\mathbf{x+5})$	Group the pairs and factor out the common factor x from the first pair and 2 from the second pair.
$(\mathbf{x+5})(x+2)$	Factor out the common factor $(x+5)$.

$(x+5)(x+2)$

How do you know how to rewrite the middle term? Unfortunately, you can’t rewrite it just any way. If you rewrite $7x$ as $6x+x$, this method won’t work. Fortunately, there's a rule for that.

Factoring Trinomials in the form x^2+bx+c

To factor a trinomial in the form x^2+bx+c, find two integers, r and s, whose product is and whose sum is b.

Rewrite the trinomial as $x^2+rx+sx+c$ and then use grouping and the distributive property to factor the polynomial. The resulting factors will be $(x+r)$ and $(x+s)$.

For example, to factor $x^2+7x+10$, you are looking for two numbers whose sum is 7 (the coefficient of the middle term) and whose product is 10 (the last term).

Look at factor pairs of 10: 1 and 10, 2 and 5. Do either of these pairs have a sum of 7? Yes, 2 and 5. So you can rewrite $7x$ as $2x+5x$, and continue factoring as in the example above. Note that you can also rewrite $7x$ as $5x+2x$. Both will work.

Let’s factor the trinomial x^2+5x+6. In this polynomial, the b part of the middle term is 5 and the c term is 6. A chart will help us organize possibilities. On the left, list all possible factors of the c term, 6; on the right you'll find the sums.

Factors whose product is 6	**Sum of the factors**
$1\cdot 6=6$	$1+6=7$
$2\cdot 3=6$	$2+3=5$

There are only two possible factor combinations, 1 and 6, and 2 and 3. You can see that $2+3=5$. So $2x+3x=5x$, giving us the correct middle term.

Example

Factor x^2+5x+6

Solution

$x^2+2x+3x+6$	Use values from the chart above. Replace $5x$ with $2x+3x$.
$(x^2+2x)+(3x+6)$	Group the pairs of terms.
$x(x+2)+(3x+6)$	Factor x out of the first pair of terms.
$x(x+2)+3(x+2)$	Factor 3 out of the second pair of terms.
$(x+2)(x+3)$	Factor out $(x+2)$.

$(x+2)(x+3)$

Note that if you wrote x^2+5x+6 as $x^2+3x+2x+6$ and grouped the pairs as $(x^2+3x)+(2x+6)$, then factored, $x(x+3)+2(x+3)$, and factored out $x+3$, the answer would be $(x+3)(x+2)$. Since multiplication is commutative, the order of the factors does not matter. So this answer is correct as well; they are equivalent answers.

Finally, let’s take a look at the trinomial x^2+x-12. In this trinomial, the c term is -12. So look at all of the combinations of factors whose product is -12. Then see which of these combinations will give you the correct middle term, where b is 1.

Factors whose product is -12	**Sum of the factors**
$1\cdot -12=-12$	$1+-12=-11$
$2\cdot -6=-12$	$2+-6=-4$
$3\cdot -4=-12$	$3+-4=-1$
$4\cdot -3=-12$	$4+-3=1$
$6\cdot -2=-12$	$6+-2=4$
$12\cdot -1=-12$	$12+-1=11$

There is only one combination where the product is -12 and the sum is 1, and that is when $r=4$, and $s=-3$. Let’s use these to factor our original trinomial.

✓ Example

Factor x^2+x-12.

Solution

$x^2+4x+-3x-12$	Rewrite the trinomial using the values from the chart above. Use values $r=4$ and $s=-3$.
$(x^2+4x)+(-3x-12)$	Group pairs of terms.
$x(x+4)+(-3x-12)$	Factor x out of the first group.
$x(x+4)-3(x+4)$	Factor -3 out of the second group.
$(x+4)(x-3)$	Factor out $(x+4)$.

$(x+4)(x-3)$

In the above example, you could also rewrite x^2+x-12 as $x^2-3x+4x-12$ first. Then factor $x(x-3)+4(x-3)$, and factor out $(x-3)$ getting $(x-3)(x+4)$. Since multiplication is commutative, this is the same answer.

Factoring Tips

Factoring trinomials is a matter of practice and patience. Sometimes, the appropriate number combinations will just pop out and seem so obvious! Other times, despite trying many possibilities, the correct combinations are hard to find. And, there are times when the trinomial cannot be factored.

While there is no foolproof way to find the right combination on the first guess, there are some tips that can ease the way.

Tips for Finding Values that Work

When factoring a trinomial in the form x^2+bx+c, consider the following tips.

Look at the c term first.

- If the c term is a positive number, then the factors of c will both be positive or both be negative. In other words, r and s will have the same sign.
- If the c term is a negative number, then one factor of c will be positive, and one factor of c will be negative. Either r or s will be negative, but not both.

Look at the b term second.

- If the c term is positive and the term is positive, then both r and s are positive.
- If the c term is positive and the b term is negative, then both r and s are negative.
- If the c term is negative and the b term is positive, then the factor that is positive will have the greater absolute value. That is, if $|r|>|s|$, then r is positive and s is negative.
- If the c term is negative and the b term is negative, then the factor that is negative will have the greater absolute value. That is, if $|r|>|s|$, then r is negative and s is positive.

After you have factored a number of trinomials in the form x^2+bx+c, you may notice that the numbers you identify for r and s end up being included in the factored form of the trinomial. Have a look at the following chart, which reviews the three problems you have seen so far.

Trinomial	$x^2+7x+10$	x^2+5x+6	x^2+x-12
r and s values	$r=+5, s=+2$	$r=+2, s=+3$	$r=+4, s=-3$
Factored form	$(x+5)(x+2)$	$(x+2)(x+3)$	$(x+4)(x-3)$

Notice that in each of these examples, the r and s values are repeated in the factored form of the trinomial.

So what does this mean? It means that in trinomials of the form x^2+bx+c (where the coefficient in front of x^2 is 1), if you can identify the correct r and s values, you can effectively skip the grouping steps and go right to the factored form. You may want to stick with the grouping method until you are comfortable factoring, but this is a neat shortcut to know about!

? Exercise

Jess is trying to use the grouping method to factor the trinomial $v^2-10v+21$. How should she rewrite the central b term, $-10v$?

A. $+7v+3v$
B. $-7v-3v$
C. $-7v+3v$
D. $+7v-3v$

Answer

A. Incorrect. Because the c term is positive and the b term is negative, both terms should be negative. (Notice that using the integers 7 and 3, $7+3=+10$, so this would provide the term $10v$ instead of $-10v$.) The correct answer is $-7v-3v$.
B. Correct. Because the c term is positive and the b term is negative, both terms should be negative. Check: using the integers -7 and -3, $-7+-3=-10$ and $-7\cdot-3=21$, so this provides both terms $-10v$ and 21 correctly.
C. Incorrect. Because the c term is positive and the b term is negative, both terms should be negative. (Notice that using the integers -7 and 3, $-7+3=-4$ and $-7\cdot3=-21$, so this would provide $-4v$ instead of $-10v$ and -21 instead of 21.) The correct answer is $-7v-3v$.
D. Incorrect. Because the c term is positive and the b term is negative, both terms should be negative. (Notice that using the integers 7 and -3, $7+-3=4$ and $7\cdot-3=-21$, so this would provide the term $4v$ instead of $-10v$ and -21 instead of 21.) The correct answer is $-7v-3v$.

Identifying Common Factors

Not all trinomials look like x^2+5x+6, where the coefficient in front of the x^2 term is 1. In these cases, your first step should be to look for common factors for the three terms.

Trinomial	Factor out Common Factor	Factored
$2x^2+10x+12$	$2\left(x^2+5x+6\right)$	$2(x+2)(x+3)$
$-5a^2-15a-10$	$-5\left(a^2+3a+2\right)$	$-5(a+2)(a+1)$
c^3-8c^2+15c	$c\left(c^2-8c+15\right)$	$c(c-5)(c-3)$
$y^4-9y^3-10y^2$	$y^2\left(y^2-9y-10\right)$	$y^2(y-10)(y+1)$

Notice that once you have identified and pulled out the common factor, you can factor the remaining trinomial as usual. This process is shown below.

✓ Example

Factor $3x^3-3x^2-90x$**.**

Solution

$3\left(x^3-x^2-30x\right)$	Since 3 is a common factor for the three terms, factor out the 3.
$3x\left(x^2-x-30\right)$	x is also a common factor. Factor out x. Now you can factor the trinomial x^2-x-30. To find r and s, identify two numbers whose product is -30 and whose sum is -1.

$3x\left(x^2-6x+5x-30\right)$	The pair of factors is -6 and 5. So replace $-x$ with $-6x+5x$.
$3x\left[\left(x^2-6x\right)+(5x-30)\right]$	Use grouping to consider the terms in pairs.
$3x[(x(x-6)+5(x-6)]$	Factor x out of the first group and factor 5 out of the second group.
$3x(x-6)(x+5)$	Then factor out $x-6$.

$3x(x-6)(x+5)$

Factoring Trinomials: ax^2+bx+c

The general form of trinomials with a leading coefficient of a is ax^2+bx+c. Sometimes the factor of a can be factored as you saw above; this happens when a can be factored out of all three terms. The remaining trinomial that still needs factoring will then be simpler, with the leading term only being an x^2 term, instead of an ax^2 term.

However, if the coefficients of all three terms of a trinomial don't have a common factor, then you will need to factor the trinomial with a coefficient of something other than 1.

Factoring Trinomials in the form ax^2+bx+c

To factor a trinomial in the form ax^2+bx+c, find two integers, r and s, whose sum is b and whose product is ac. Rewrite the trinomial as $ax^2+rx+sx+c$ and then use grouping and the distributive property to factor the polynomial.

This is almost the same as factoring trinomials in the form x^2+bx+c, as in this form $a=1$. Now you are looking for two factors whose product is $a\cdot c$, and whose sum is b.

Let's see how this strategy works by factoring $6z^2+11z+4$.

In this trinomial, $a=6$, $b=11$, and $c=4$. According to the strategy, you need to find two factors, r and s, whose sum is $b(11)$ and whose product is $ac(\text{ or } 6\cdot 4=24)$. You can make a chart to organize the possible factor combinations. (Notice that this chart only has positive numbers. Since ac is positive and b is positive, you can be certain that the two factors you're looking for are also positive numbers.)

Factors whose product is 24	***Sum of the factors***
$1\cdot 24=24$	$1+24=25$
$2\cdot 12=24$	$2+12=14$
$3\cdot 8=24$	$3+8=11$
$4\cdot 6=24$	$4+6=10$

There is only one combination where the product is 24 and the sum is 11, and that is when $r=3$ and $s=8$. Let's use these values to factor the original trinomial.

Example

Factor $6z^2+11z+4$**.**

Solution

$6z^2+3z+8z+4$	Rewrite the middle term, $11z$, as $3z+8z$ (from the chart above.)
$\left(6z^2+3z\right)+(8z+4)$	Group pairs. Use grouping to consider the terms in pairs.
$3z(2z+1)+4(2z+1)$	Factor $3z$ out of the first group and 4 out of the second group.
$(2z+1)(3z+4)$	Factor out $(2z+1)$.

$(2z+1)(3z+4)$

Before going any further, it is worth mentioning that not all trinomials can be factored using integer pairs. Take the trinomial $2z^2+35z+7$, for instance. Can you think of two integers whose sum is $b(35)$ and whose product is $ac(2\cdot 7=14)$? There are none! This type of trinomial, which cannot be factored using integers, is called a prime trinomial.

? Exercise

Factor $3x^2+x-2$.

A. $(3x+2)(x-1)$
B. $(3x-2)(x+1)$
C. $(3x+1)(x-2)$
D. $(3x-1)(x+2)$

Answer

A. Incorrect. $(3x+2)(x-1)$ has a product of $3x^2-x-2$; look for two numbers whose product is -6 and whose sum is +1. Then use those numbers to factor by grouping. The correct answer is $(3x-2)(x+1)$.
B. Correct. The product of $(3x-2)(x+1)$ is $3x^2+x-2$.
C. Incorrect. The product of $(3x+1)(x-2)$ is $3x^2-5x-2$; look for two numbers whose product is -6 and whose sum is +1. Then use those numbers to factor by grouping. The correct answer is $(3x-2)(x+1)$.
D. Incorrect. The product of $(3x-1)(x+2)$ is $3x^2+5x-2$; look for two numbers whose product is -6 and whose sum is +1. Then use those numbers to factor by grouping. The correct answer is $(3x-2)(x+1)$.

Negative Terms

In some situations, a is negative, as in $-4h^2+11h+3$. It often makes sense to factor out -1 as the first step in factoring, as doing so will change the sign of ax^2 from negative to positive, making the remaining trinomial easier to factor.

✓ Example

Factor $-4h^2+11h+3$

Solution

<table>
<tr><td>$-1\left(4h^2-11h-3\right)$</td><td>Factor -1 out of the trinomial. Notice that the signs of all three terms have changed.
To factor the trinomial, you need to figure out how to rewrite $-11h$. The product of $rs=4\cdot -3=-12$ and the sum of $rs=-11$.</td></tr>
<tr><td>$-1\left(4h^2-12h+1h-3\right)$</td><td><table>
<tr><td>$r\cdot s=-12$</td><td>$r+s=-11$</td></tr>
<tr><td>$-12\cdot 1=-12$</td><td>$-12+1=-11$</td></tr>
<tr><td>$-6\cdot 2=-12$</td><td>$-6+2=-4$</td></tr>
<tr><td>$-4\cdot 3=-12$</td><td>$-4+3=-1$</td></tr>
</table>
Rewrite the middle term $-11h$ as $-12h+1h$.</td></tr>
<tr><td>$-1\left[\left(4h^2-12h\right)+(1h-3)\right]$</td><td>Group terms.</td></tr>
<tr><td>$-1[4h(h-3)+1(h-3)]$</td><td>Factor out $4h$ from the first pair. The second group cannot be factored further, but you can write it as $+1(h-3)$ since $+1(h-3)=(h-3)$. This helps with factoring in the next step.</td></tr>
</table>

$-1[(h-3)(4h+1)]$	Factor out a common factor of $(h-3)$. Notice you are left with $(h-3)(4h+1)$; the $+1$ comes from the term $+1(h-3)$ in the previous step.

$-1(h-3)(4h+1)$

Note that the answer above can also be written as $(-h+3)(4h+1)$ or $(h-3)(-4h-1)$ if you multiply -1 times one of the other factors.

Summary

Trinomials in the form x^2+bx+c can be factored by finding two integers, r and s, whose sum is b and whose product is c. Rewrite the trinomial as $x^2+rx+sx+c$ and then use grouping and the distributive property to factor the polynomial.

When a trinomial is in the form of ax^2+bx+c, where a is a coefficient other than 1, look first for common factors for all three terms. Factor out the common factor first, then factor the remaining simpler trinomial. If the remaining trinomial is still of the form ax^2+bx+c, find two integers, r and s, whose sum is b and whose product is ac. Then rewrite the trinomial as $ax^2+rx+sx+c$ and use grouping and the distributive property to factor the polynomial.

When ax^2 is negative, you can factor -1 out of the whole trinomial before continuing.

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12.2.2: Factoring- Special Cases

Learning Objectives

- Factor trinomials that are perfect squares.
- Factor binomials in the form of the difference of squares.

Introduction

One of the keys to factoring is finding patterns between the **trinomial** and the factors of the trinomial. Learning to recognize a few common polynomial types will lessen the amount of time it takes to factor them. Knowing the characteristic patterns of special products, trinomials that come from squaring **binomials**, for example, provides a shortcut to finding their factors.

Perfect Squares

Perfect squares are numbers that are the result of a whole number multiplied by itself or squared. For example 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100 are all perfect squares; they come from squaring each of the numbers from 1 to 10. Notice that these perfect squares can also come from squaring the negative numbers from -1 to -10, as $(-1)(-1)=1$, $(-2)(-2)=4$, $(-3)(-3)=9$, and so on.

A **perfect square trinomial** is a trinomial that is the result of a binomial multiplied by itself or squared. For example, $(x+3)^2=(x+3)(x+3)=x^2+6x+9$. The trinomial x^2+6x+9 is a perfect square trinomial. Let's factor this trinomial using the methods you have already seen.

Example

Factor x^2+6x+9.

Solution

$x^2+3x+3x+9$	Rewrite $6x$ as $3x+3x$, as $3\cdot 3=9$, the last term, and $3+3=6$, the middle term.
$(x^2+3x)+(3x+9)$	Group pairs of terms.
$x(x+3)+3(x+3)$	Factor x out of the first pair, and factor 3 out of the second pair.
$(x+3)(x+3)$ or $(x+3)^2$	Factor out $x+3$. $(x+3)(x+3)$ can also be written as $(x+3)^2$.

$(x+3)(x+3)$ or $(x+3)^2$

Notice that in the trinomial x^2+6x+9, the a and c terms are each a perfect square, as $x^2=x\cdot x$, and $9=3\cdot 3$. Also the middle term is twice the product of the x and 3 terms, $2(3)x=6x$.

Let's consider a slightly different example next. The above example shows how $(x+3)^2=x^2+6x+9$. What do you suppose $(x-3)^2$ equals? Using what you know about multiplying binomials, you see the following.

$$(x-3)^2$$
$$(x-3)(x-3)$$
$$x^2-3x-3x+9$$
$$x^2-6x+9$$

Notice that $(x+3)^2=x^2+6x+9$ and $(x-3)^2=x^2-6x+9$! Here, 9 can be written as $(-3)^2$, so the middle term is $2(-3)x=-6x$. So when the sign of the middle term is negative, the trinomial may be factored as $(a-b)^2$.

Let's try one more example: $9x^2-24x+16$. Notice that $9x^2$ is a perfect square, as $(3x)^2=9x^2$ and that 16 is a perfect square, as $4^2=16$. However, the middle term, $-24x$ is negative, so try $16=(-4)^2$. In this case, the middle term is $2(3x)(-4)=-24x$. So the trinomial $9x^2-24x+16$ is a perfect square and factors as $(3x-4)^2$.

You can also continue to factor using grouping as shown below.

✓ Example

Factor $9x^2-24x+16$**.**

Solution

$9x^2-12x-12x+16$	Rewrite $-24x$ as $-12x-12x$
$(9x^2-12x)+(-12x+16)$	Group pairs of terms. (Keep the negative sign with the 12.)
$3x(3x-4)-4(3x-4)$	Factor $3x$ out of the first group, and factor out -4 from the second group.
$(3x-4)(3x-4)$ or $(3x-4)^2$	Factor out $(3x-4)$. $(3x-4)(3x-4)$ can also be written as $(3x-4)^2$.

$(3x-4)^2$

Notice that if you had factored out 4 rather than -4, the $3x-4$ factor would have been $-3x+4$, which is the opposite of $3x-4$. By factoring out the -4, the factors from the grouping come out the same, both as $3x-4$. We need that to happen if we are going to pull a common grouping factor out for our next step.

The pattern for factoring perfect square trinomials lead to this general rule.

Perfect Square Trinomials

A trinomial in the form $a^2+2ab+b^2$ can be factored as $(a+b)^2$.

A trinomial in the form $a^2-2ab+b^2$ can be factored as $(a-b)^2$.

Examples:

The factored form of $4x^2+20x+25$ is $(2x+5)^2$.

The factored form of $x^2-10x+25$ is $(x-5)^2$.

Let's factor a trinomial using the rule above. Once you have determined that the trinomial is indeed a perfect square, the rest is easy. Notice that the c term is always positive in a perfect trinomial square.

✓ Example

Factor $x^2-14x+49$**.**

Solution

$x^2-14x+49$	Determine if this is a perfect square trinomial. The first term is a square, as $x^2=x\cdot x$. The last term is a square as $7\cdot 7=49$. Also $-7\cdot -7=49$.
$-14x=-7x+-7x$	So, $a=x$ and $b=7$ or -7. The middle term is $-2ab$ if we use $b=7$, because $-2x\cdot 7=-14x$. It is a perfect square trinomial.
$(x-7)^2$	Factor as $(a-b)^2$.

$(x-7)^2$

You can, and should, always multiply to check the answer. $(x-7)^2=(x-7)(x-7)=x^2-7x-7x+49=x^2-14x+49$.

? Exercise

Factor $x^2-12x+36$.

A. $(x-4)(x-9)$

B. $(x+6)^2$
C. $(x-6)^2$
D. $(x+6)(x-6)$

Answer

A. Incorrect. While $-4 \cdot (-9)$ gives the constant term 36, the middle term would be $-13x$ rather than $-12x$. The correct answer is $(x-6)^2$.
B. Incorrect. While 6^2 is 36, the middle term in the original polynomial is negative, so you must subtract in the binomial you square. The correct answer is $(x-6)^2$.
C. Correct. This is a perfect square trinomial $a^2-2ab+b^2$ where $a=x$ and $b=6$. The factored form is $(a-b)^2$, or $(x-6)^2$.
D. Incorrect. This is a perfect square trinomial $a^2-2ab+b^2$ where $a=x$ and $b=6$, so the factored form is $(a-b)^2$, or $(x-6)^2$. Notice that if you expand $(x+6)(x-6)$, you get $x^2+6x-6x-36$. The 36 is subtracted rather than added, and the $6x-6x$ gives a middle term of 0 (that is, no middle term at all).

Factoring a Difference of Squares

The difference of two squares, a^2-b^2, is also a special product that factors into the product of two binomials.

Let's factor $9x^2-4$ by writing it as a trinomial, $9x^2+0x-4$. Now you can factor this trinomial just as you have been doing.

$9x^2+0x-4$ fits the standard form of a trinomial, ax^2+bx+c. Let's factor this trinomial the same way you would any other trinomial. Find the factors of $ac(9 \cdot -4=-36)$ whose sum is b, in this case, 0.

Factors of -36	***Sum of the factors***
$1 \cdot -36=-36$	$1+(-36)-35$
$2 \cdot -18=-36$	$2+(-18)=-16$
$3 \cdot -12=-36$	$3+(-12)=-9$
$4 \cdot -9=-36$	$4+(-9)=-5$
$6 \cdot -6=-36$	$6+(-6)=0$
$9 \cdot -4=-36$	$9+(-4)=5$
...	...

There are more factors, but you have found the pair that has a sum of 0, 6 and -6. You can use these to factor $9x^2-4$.

✓ Example

Factor $9x^2-4$.

Solution

$9x^2+0x-4$ $9x^2-6x+6x-4$	Rewrite $0x$ as $-6x+6x$.
$(9x^2-6x)+(6x-4)$	Group pairs.
$3x(3x-2)+2(3x-2)$	Factor $3x$ out of the first group. Factor 2 out of the second group.
$(3x-2)(3x+2)$	Factor out $(3x-2)$.

$(3x-2)(3x+2)$

Since multiplication is commutative, the answer can also be written as $(3x+2)(3x-2)$.

You can check the answer by multiplying $(3x-2)(3x+2)=9x^2+6x-6x-4=9x^2-4$.

Factoring a Difference of Squares

A binomial in the form a^2-b^2 can be factored as $(a+b)(a-b)$.

Examples

The factored form of x^2-100 is $(x+10)(x-10)$.

The factored form of $49y^2-25$ is $(7y+5)(7y-5)$.

Let's factor the difference of two squares using the above rule. Once you have determined that you have the difference of two squares, you just follow the pattern.

Example

Factor $4x^2-36$**.**

Solution

$4x^2-36$	$4x^2=(2x)^2$, so $a=2x$ $36=6^2$, so $b=6$ And $4x^2-36$ is the difference of two squares.
$(2x+6)(2x-6)$	Factor as $(a+b)(a-b)$.

$(2x+6)(2x-6)$

Check the answer by multiplying: $(2x+6)(2x-6)=4x^2-12x+12x-36=4x^2-36$.

Exercise

Factor $4b^2-25$.

A. $(2b-25)(2b+1)$
B. $(2b+5)^2$
C. $(2b-5)^2$
D. $(2b+5)(2b-5)$

Answer

A. Incorrect. $(2b-25)(2b+1)=4b^2+2b-50b-25=4b^2-48b-25$. The middle term should be $0b$, not $-48b$. The correct answer is $(2b+5)(2b-5)$.
B. Incorrect. $(2b+5)^2=(2b+5)(2b+5)=4b^2+10b+10b+25=4b^2+20b+25$. The middle term should be $0b$, not $20b$. The correct answer is $(2b+5)(2b-5)$.
C. Incorrect. $(2b-5)^2=(2b-5)(2b-5)=4b^2-10b-10b+25=4b^2-20b+25$. The middle term should be $0b$, not $-20b$. The correct answer is $(2b+5)(2b-5)$.
D. Correct. $4b^2-25$ is a special case. It is the difference of two squares. $(2b+5)(2b-5)=4b^2-10b+10b-25=(2b+5)(2b-5)$, which is correct.

Notice that you **cannot** factor the sum of two squares, a^2+b^2. You might be tempted to factor this as $(a+b)^2$, but check it by multiplying: $(a+b)^2=(a+b)(a+b)=a^2+ab+ab+b^2=a^2+2ab+b^2$, NOT a^2+b^2.

Summary

Learning to identify certain patterns in polynomials helps you factor some "special cases" of polynomials quickly. The special cases are:

- trinomials that are perfect squares, $a^2+2ab+b^2$ and $a^2-2ab+b^2$, which factor as $(a+b)^2$ and $(a-b)^2$, respectively;

- binomials that are the difference of two squares, a^2-b^2, which factors as $(a+b)(a-b)$.

For some polynomials, you may need to combine techniques (looking for common factors, grouping, and using special products) to factor the polynomial completely.

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12.2.3: Special Cases- Cubes

Learning Objectives

- Factor the sum of cubes.
- Factor the difference of cubes.

Introduction

In many ways, **factoring** is about patterns: if you recognize the patterns that numbers make when they are multiplied together, you can use those patterns to separate these numbers into their individual factors.

Some interesting patterns arise when you are working with cubed quantities within polynomials. Specifically, there are two more special cases to consider: a^3+b^3 and a^3-b^3.

Let’s take a look at how to factor sums and differences of cubes.

Sum of Cubes

The term “cubed” is used to describe a number raised to the third power. In geometry, a cube is a six-sided shape with equal width, length, and height; since all these measures are equal, the volume of a cube with width x can be represented by x^3.

(Notice the exponent!)

Cubed numbers get large very quickly. $1^3=1, 2^3=8, 3^3=27, 4^3=64$, and $5^3=125$.

Before looking at factoring a sum of two cubes, let’s look at the possible factors.

It turns out that a^3+b^3 can actually be factored as $(a+b)\left(a^2-ab+b^2\right)$. Let’s check these factors by multiplying.

Does $(a+b)\left(a^2-ab+b^2\right)=a^3+b^3$?	
$(\mathrm{a})\left(a^2-ab+b^2\right)+(b)\left(a^2-ab+b^2\right)$	Apply the distributive property.
$\left(a^3-a^2b+ab^2\right)+(b)\left(a^2-ab+b^2\right)$	Multiply by a.
$\left(a^3-a^2b+ab^2\right)+\left(a^2b-ab^2+b^3\right)$	Multiply by b.
$a^3-a^2b+a^2b+ab^2-ab^2+b^3$	Rearrange terms in order to combine the like terms.
a^3+b^3	Simplify.

Did you see that? Four of the terms cancelled out, leaving us with the (seemingly) simple binomial a^3+b^3. So, the factors are correct.

You can use this pattern to factor binomials in the form a^3+b^3, otherwise known as “the sum of cubes.”

The Sum of Cubes

A binomial in the form a^3+b^3 can be factored as $(a+b)\left(a^2-ab+b^2\right)$.

Examples:

The factored form of x^3+64 is $(x+4)\left(x^2-4x+16\right)$.

The factored form of $8x^3+y^3$ is $(2x+y)\left(4x^2-2xy+y^2\right)$.

Example

Factor x^3+8y^3.

Solution

x^3+8y^3	Identify that this binomial fits the sum of cubes pattern: a^3+b^3 $\cdot\, a=x$ and $b=2y$ (since $2y\cdot 2y\cdot 2y=8y^3$).
$(x+2y)\left(x^2-x(2y)+(2y)^2\right)$	Factor the binomial as $(a+b)\left(a^2-ab+b^2\right)$, substituting $a=x$ and $b=2y$ into the expression.
$(x+2y)\left(x^2-x(2y)+4y^2\right)$	Square $(2y)^2=4y^2$
$(x+2y)\left(x^2-2xy+4y^2\right)$	Multiply $-x\cdot 2y=-2xy$ (writing the coefficient first).

And that's it. The binomial x^3+8y^3 can be factored as $(x+2y)\left(x^2-2xy+4y^2\right)$. Let's try another one.

You should always look for a common factor before you follow any of the patterns for factoring.

✓ Example

Factor $16m^3+54n^3$**.**

Solution

$16m^3+54n^3$	Factor out the common factor, 2.
$2\left(8m^3+27n^3\right)$	$8m^3$ and $27n^3$ are cubes, so you can factor $8m^3+27n^3$ as the sum of two cubes: $a=2m$ and $b=3n$.
$2(2m+3n)\left[(2m)^2-(2m)(3n)+(3n)^2\right]$	Factor the binomial $8m^3+27n^3$ substituting $a=2m$ and $b=3n$ into the expression $(a+b)\left(a^2-ab+b^2\right)$.
$2(2m+3n)\left[\mathbf{4m^2}-(2m)(3n)+\mathbf{9n^2}\right]$	Square: $(2m)^2=4m^2$ and $(3n)^2=9n^2$.
$2(2m+3n)\left(4m^2-6mn+9n^2\right)$	Multiply $-(2m)(3n)=-6mn$.

? Exercise

Factor $125x^3+64$.

A. $(5x+64)\left(25x^2-125x+16\right)$
B. $(5x+4)\left(25x^2-20x+16\right)$
C. $(x+4)\left(x^2-2x+16\right)$
D. $(5x+4)\left(25x^2+20x-64\right)$

Answer

A. Incorrect. Check your values for a and b here. $b^3=64$, so what is b? The correct answer is $(5x+4)\left(25x^2-20x+16\right)$.
B. Correct. $5x$ is the cube root of $125x^3$, and 4 is the cube root of 64. Substituting these values for a and b, you find $(5x+4)\left(25x^2-20x+16\right)$.
C. Incorrect. Check your values for a and b here. $a^3=125x^3$, so what is a? The correct answer is $(5x+4)\left(25x^2-20x+16\right)$.
D. Incorrect. Check the mathematical signs; the b^2 term is positive, not negative, when factoring a sum of cubes. The correct answer is $(5x+4)\left(25x^2-20x+16\right)$.

Difference of Cubes

Having seen how binomials in the form a^3+b^3 can be factored, it should not come as a surprise that binomials in the form a^3-b^3 can be factored in a similar way.

The Difference of Cubes

A binomial in the form $a^3 - b^3$ can be factored as $(a - b)\left(a^2 + ab + b^2\right)$.

Examples:

The factored form of $x^3 - 64$ is $(x - 4)\left(x^2 + 4x + 16\right)$.

The factored form of $27x^3 - 8y^3$ is $(3x - 2y)\left(9x^2 + 6xy + 4y^2\right)$.

Notice that the basic construction of the factorization is the same as it is for the sum of cubes; the difference is in the + and - signs. Take a moment to compare the factored form of $a^3 + b^3$ with the factored form of $a^3 - b^3$.

Factored form of $a^3 + b^3$: $(a + b)\left(a^2 - ab + b^2\right)$

Factored form of $a^3 - b^3$: $(a - b)\left(a^2 + ab + b^2\right)$

This can be tricky to remember because of the different signs. The factored form of $a^3 + b^3$ contains a negative, and the factored form of $a^3 - b^3$ contains a positive! Some people remember the different forms like this:

“Remember one sequence of variables: $a^3 b^3 = (ab)\left(a^2 \ ab \ b^2\right)$. There are 4 missing signs. Whatever the first sign is, it is also the second sign. The third sign is the opposite, and the fourth sign is always +.”

Try this for yourself. If the first sign is +, as in $a^3 + b^3$, according to this strategy how do you fill in the rest: $(ab)\left(a^2 \ ab \ b^2\right)$?

Does this method help you remember the factored form of $a^3 + b^3$ and $a^3 - b^3$?

Let’s go ahead and look at a couple of examples. Remember to factor out all common factors first.

Example

Factor $8x^3 - 1,000$.

Solution

$8\left(x^3 - 125\right)$	Factor out 8.
$8\left(x^3 - 125\right)$	Identify that the binomial fits the pattern $a^3 - b^3$: $a = x$ and $b = 5$ (since $5^3 = 125$).
$8(x - 5)\left[x^2 + (x)(5) + 5^2\right]$	Factor $x^3 - 125$ as $(a - b)\left(a^2 + ab + b^2\right)$, substituting $a = x$ and $b = 5$ into the expression.
$8(x - 5)\left(x^2 + 5x + 25\right)$	Square the first and last terms, and rewrite $(x)(5)$ as $5x$.

$8(x - 5)\left(x^2 + 5x + 25\right)$

Let’s see what happens if you don’t factor out the common factor first. In this example, it can still be factored as the difference of two cubes. However, the factored form still has common factors, which need to be factored out.

Example

Factor $8x^3 - 1,000$.

Solution

$8x^3 - 1,000$	Identify that this binomial fits the pattern $a^3 - b^3$: $a = 2x$ and $b = 10$ (since $10^3 = 1,000$).
$(2x - 10)\left[(2x)^2 + 2x \cdot 10 + 10^2\right]$	Factor as $(a - b)\left(a^2 + ab + b^2\right)$, substituting $a = 2x$ and $b = 10$ into the expression.
$(2x - 10)\left(4x^2 + 20x + 100\right)$	Square and multiply: $(2x)^2 = 4x^2$, $(2x)(10) = 20x$, and $10^2 = 100$.

$2(x-5)(4)\left(x^2+5x+25\right)$	Factor out remaining common factors in each factor. Factor out 2 from the first factor, factor out 4 from the second factor.
$(2\cdot 4)(x-5)\left(x^2+5x+25\right)$	Multiply the numerical factors.

$8(x-5)\left(x^2+5x+25\right)$

As you can see, this last example still worked, but required a couple of extra steps. It is always a good idea to factor out all common factors first. In some cases, the only efficient way to factor the binomial is to factor out the common factors first.

Here is one more example. Note that $r^9=\left(r^3\right)^3$ and that $8s^6=\left(2s^2\right)^3$.

✓ Example

Factor r^9-8s^6**.**

Solution

r^9-8s^6	Identify this binomial as the difference of two cubes. As shown above, it is. Using the laws of exponents, rewrite r^9 as $\left(r^3\right)^3$.
$\left(r^3\right)^3-\left(2s^2\right)^3$	Rewrite r^9 as $\left(r^3\right)^3$ and rewrite $8s^6$ as $\left(2s^2\right)^3$.
$\left(r^3-2s^2\right)\left[\left(r^3\right)^2+\left(r^3\right)\left(2s^2\right)+\left(2s^2\right)^2\right]$	Now the binomial is written in terms of cubed quantities. Thinking of a^3-b^3, $a=r^3$ and $b=2s^2$. Factor the binomial as $(a-b)\left(a^2+ab+b^2\right)$, substituting $a=r^3$ and $b=2s^2$ into the expression.
$\left(r^3-2s^2\right)\left(r^6+2r^3s^2+4s^4\right)$	Multiply and square the terms.

$\left(r^3-2s^2\right)\left(r^6+2r^3s^2+4s^4\right)$

? Exercise

Using the difference of cubes, identify the product of $3(x-3y)\left(x^2+3xy+9y^2\right)$.

A. x^3-y^3
B. $3x-81y$
C. $3x^3+81y^3$
D. $3x^3-81y^3$

Answer

A. Incorrect. If this were true, the expression shown above would be $(x-y)\left(x^2+xy+y^2\right)$. The correct answer is $3x^3-81y^3$.
B. Incorrect. Neither of the terms in this binomial is a cubed number! The correct answer is $3x^3-81y^3$.
C. Incorrect. Check your signs. If this expression falls into the *difference* of cubes category, the symbol between $3x^3$ and $81y^3$ should be -. The correct answer is $3x^3-81y^3$.
D. Correct. Recognizing that this expression is in the form $(a-b)\left(a^2+ab+b^2\right)$, you find $a=x$ and $b=3y$. This means the resulting a^3-b^3 monomial is x^3-27y^3. It $3x^3-81y^3$.

Summary

You encounter some interesting patterns when factoring. Two special cases, the sum of cubes and the difference of cubes, can help you factor some binomials that have a degree of three (or higher, in some cases). The special cases are:

- A binomial in the form a^3+b^3 can be factored as $(a+b)\left(a^2-ab+b^2\right)$
- A binomial in the form a^3-b^3 can be factored as $(a-b)\left(a^2+ab+b^2\right)$

Always remember to factor out any common factors first.

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SECTION OVERVIEW

12.3: Solving Quadratic Equations

Topic hierarchy

12.3.1: Solve Quadratic Equations by Factoring

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12.3.1: Solve Quadratic Equations by Factoring

Learning Objectives

- Solve equations in factored form by using the Principle of Zero Products.
- Solve quadratic equations by factoring and then using the Principle of Zero Products.
- Solve application problems involving quadratic equations.

Introduction

When a polynomial is set equal to a value (whether an integer or another polynomial), the result is an equation. An equation that can be written in the form $ax^2+bx+c=0$ is called a **quadratic equation**. You can solve a quadratic equation using the rules of algebra, applying factoring techniques where necessary, and by using the **Principle of Zero Products**.

The Principle of Zero Products

The Principle of Zero Products states that if the product of two numbers is 0, then at least one of the factors is 0. (This is not really new.)

Principle of Zero Products

If $ab=0$, then either $a=0$ or $b=0$, or both a and b are 0.

This property may seem fairly obvious, but it has big implications for solving quadratic equations. If you have a factored polynomial that is equal to 0, you know that at least one of the factors or both factors equal 0.

You can use this method to solve quadratic equations. Let's start with one that is already factored.

Example

Solve $(x+4)(x-3)=0$ for x.

Solution

$(x+4)(x-3)=0$	Applying the Principle of Zero Products, you know that if the product is 0, then one or both of the factors has to be 0.
$\begin{array}{ccc} x+4=0 & \text{or} & x-3=0 \\ x+4-4=0-4 & & x-3+3=0+3 \\ x=-4 & \text{or} & x=3 \end{array}$	Set each factor equal to 0. Solve each equation.

$x=-4$ OR $x=3$

You can check these solutions by substituting each one at a time into the original equation, $(x+4)(x-3)=0$. You can also try another number to see what happens.

$\textbf{Checking } x=-4$
$(x+4)(x-3)=0$
$(-4+4)(-4-3)=0$
$(0)(-7)=0$
$0=0$

$\textbf{Checking } x=3$
$(x+4)(x-3)=0$
$(3+4)(3-3)=0$
$(7)(0)=0$
$0=0$

Trying $x = 5$
$(x+4)(x-3)=0$
$(5+4)(5-2)=0$
$(9)(3)=0$
$27 \neq 0$

The two values that we found via factoring, $x=-4$ and $x=3$, lead to true statements: $0=0$. So, the solutions are correct. But $x = 5$, the value not found by factoring, creates an untrue statement: 27 does not equal 0!

? Exercise

Solve for x.

$(x-5)(2x+7)=0$

A. $x=5$ or $\frac{-7}{2}$
B. $x=5$ or -7
C. $x=0$ or $\frac{-7}{2}$
D. $x=0$

Answer

A. Correct. To find the roots of this equation, apply the Principle of Zero Products and set each factor, $(x-5)$ and $(2x+7)$, equal to 0. $x-5=0$, so $x=5$; you also find that $2x+7=0$, so $2x=-7$, and $x=\frac{-7}{2}$. Both answers, $x=5$ and $\frac{-7}{2}$, are solutions.
B. Incorrect. While $x=5$ does make the equation true, the Principle of Zero Products states if $(x-5)(2x+7)=0$, then either $x-5=0$ or $2x+7=0$. This happens when $x=5$ or $\frac{-7}{2}$.
C. Incorrect. While $x=\frac{-7}{2}$ does make the equation true, the Principle of Zero Products states if $(x-5)(2x+7)=0$, then either $x-5=0$ or $2x+7=0$. This happens when $x=5$ or $\frac{-7}{2}$.
D. Incorrect. A value of $x=0$ does not make the equation true: $(0-5)[2(0)+7]=(-5)(7)=-35$, not 0. The Principle of Zero Products states if $(x-5)(2x+7)=0$, then either $x-5=0$ or $2x+7=0$. This happens when $x=5$ or $\frac{-7}{2}$.

Solving Quadratics

Let's try solving an equation that looks a bit different: $5a^2+15a=0$.

✓ Example

Solve for a**:** $5a^2+15a=0$**.**

Solution

$5a^2+15a=0$	Begin by factoring the left side of the equation.
$5a(a+3)=0$	Factor out $5a$, which is a common factor of $5a^2$ and $15a$.
$5a=0 \quad \text{or} \quad a+3=0$	Set each factor equal to zero.
$\frac{5a}{5}=\frac{0}{5} \quad \text{or} \quad a+3-3=0-3$ $a=0 \qquad\qquad a=-3$	Solve each equation.

$a=0 \quad \text{OR} \quad a=-3$

To check your answers, you can substitute both values directly into the original equation and see if you get a true sentence for each.

Checking $a=0$
$5a^2+15a=0$
$5(0)^2+15(0)=0$
$5(0)+0=0$
$0+0=0$
$0=0$

Checking $a=-3$
$5a^2+15a=0$
$5(-3)^2+(15)(-3)=0$
$5(9)-45=0$
$45-45=0$
$0=0$

Both solutions check.

You can use the Principle of Zero Products to solve quadratic equations in the form $ax^2+bx+c=0$. First, factor the expression and set each factor equal to 0.

✓ Example

Solve for r: $r^2-5r+6=0$.

Solution

$r^2-3r+-2r+6=0$	Rewrite $-5r$ as $-3r-2r$, as $(-3)(-2)=6$, and $-3+-2=-5$.
$(r^2-3r)+(-2r+6)=0$	Group pairs.
$r(r-3)-2(r-3)=0$	Factor out r from the first pair and factor out -2 from the second pair.
$(r-3)(r-2)=0$	Factor out $(r-3)$.
$r-3=0$ or $r-2=0$	Use the Principle of Zero Products and set each of the factors equal to 0.
$r=3$ or $r=2$	Solve each equation.
$r=3$ or $r=2$	The roots of the original equation are 3 or 2.

Note in the example above, if the common factor of 2 had been factored out, the resulting factor would be $(-r+3)$, which is the negative of $(r-3)$. So factoring out -2 will result in the common factor of $(r-3)$. If we had gotten $(-r+3)$ as a factor, then when setting that factor equal to zero and solving for r we would have gotten:

$(-r+3)=0$	Principle of Zero Products
$(-1)(-r+3)=(-1)0$	Multiplying both sides by -1.
$r-3=0$	Multiplying.
$r=3$	Adding 3 to both sides.

More work, but the same result as before, $r=3$ or $r=2$.

? Exercise

Solve for h.

$h(2h+5)=0$

A. $h=0$
B. $h=2$ or 5
C. $h=0$ or $\frac{5}{2}$
D. $h=0$ or $-\frac{5}{2}$

Answer

A. Incorrect. While $h = 0$ does make the equation true (since the first factor is h), there is another solution when $2h+5 = 0$. The correct answer is $h = 0$ or $-\frac{5}{2}$.
B. Incorrect. The Principle of Zero Products says if $h(2h+5) = 0$ then either $h = 0$ or $2h+5 = 0$. This happens when $h = 0$ or $-\frac{5}{2}$.
C. Incorrect. While $h = 0$ does make the equation true (since the first factor is h), the second factor is 0 when $h = -\frac{5}{2}$, not $\frac{5}{2}$. The correct answer is $h = 0$ or $-\frac{5}{2}$.
D. Correct. To find the roots of this equation, apply the Principle of Zero Products and set each factor, h and $(2h+5)$, equal to 0. Then solve those equations for h. Both answers are possible solutions.

Applying Quadratic Equations

There are many applications for quadratic equations. When you use the Principle of Zero Products to solve a quadratic equation, you need to make sure that the equation is equal to zero. For example, $12x^2 + 11x + 2 = 7$ must first be changed to $12x^2 + 11x + -5 = 0$ by subtracting 7 from both sides.

✓ Example

The area of a rectangular garden is 30 square feet. If the length is 7 feet longer than the width, find the dimensions.

Solution

$A = l \cdot w$	The formula for the area of a rectangle is $A = l \cdot w$.
$30 = (w+7)(w)$	$\text{width} = w$ $\text{length} = w+7$ $\text{area} = 30$
$30 = w^2 + 7w$	Multiply.
$w^2 + 7w - 30 = 0$	Subtract 30 from both sides to set the equation equal to 0.
$w^2 + 10w - 3w - 30 = 0$	Find two numbers whose product is -30 and whose sum is 7, and write the middle term as $10w - 3w$.
$w(w+10) - 3(w+10) = 0$	Factor w out of the first pair and -3 out of the second pair.
$(w-3)(w+10) = 0$	Factor out $w+10$.
$w - 3 = 0 \quad \text{or} \quad w + 10 = 0$ $w = 3 \quad \text{or} \quad w = -10$	Use the Zero Product Property to solve for w.
The width = 3 feet The length is 3 + 7 = 10 feet	The solution $w = -10$ does not work for this application, as the width cannot be a negative number, so we discard the -10. Therefore, the width is 3 feet. Substitute $w = 3$ into the expression $w + 7$ to find the length: $3 + 7 = 10$.

The width of the garden is 3 feet and the length is 10 feet.

The example below shows another quadratic equation where neither side is originally equal to zero. (Note that the factoring sequence has been shortened.)

✓ Example

Solve $5b^2 + 4 = -12b$ for b.

Solution

$5b^2 + 4 + 12b = -12b + 12b$	The original equation has $-12b$ on the right. To make this side equal to 0, add $12b$ to both sides.
$5b^2 + 12b + 4 = 0$	Combine like terms.

$5b^2+10b+2b+4=0$	Rewrite $12b$ as $10b+2b$.
$5b(b+2)+2(b+2)=0$	Factor out $5b$ from the first pair and 2 from the second pair.
$(5b+2)(b+2)=0$	Factor out $b+2$.
$5b+2=0$ or $b+2=0$	Apply the Zero Product Property.
$b=-\frac{2}{5}$ or $b=-2$	Solve each equation.

$b=-\frac{2}{5}$ or $b=-2$

If you factor out a constant, the constant will never equal 0. So it can essentially be ignored when solving. See the following example.

✓ Example

A small toy rocket is launched from a 4-foot pedestal. The height (h, in feet) of the rocket t seconds after taking off is given by the formula $h=-2t^2+7t+4$. How long will it take the rocket to hit the ground?

Solution

$h=-2t^2+7t+4$ $0=-2t^2+7t+4$	The rocket will be on the ground when the height is 0. So, substitute 0 for h in the formula.
$0=-2t^2+8t-t+4$	Factor the trinomial by grouping.
$0=-2t(t-4)-1(t-4)$ $0=(-2t-1)(t-4)$ $0=-1(2t+1)(t-4)$	Factor.
$2t+1=0$ or $t-4=0$	Use the Zero Product Property. There is no need to set the constant factor -1 to zero, because -1 will never equal zero.
$t=-\frac{1}{2}$ or $t=4$	Solve each equation.
$t=4$	Interpret the answer. Since t represents time, it cannot be a negative number; only $t=4$ makes sense in this context.

The rocket will hit the ground 4 seconds after being launched.

? Exercise

Solve for m: $2m^2+10m=48$.

A. $m=-8$ or 3
B. $m=-3$ or 8
C. $m=0$ or -5
D. $m=0$ or 5

Answer

A. Correct. The original equation has 48 on the right. To make this side equal to 0, subtract 48 from both sides: $2m^2+10m-48=0$. Then factor out the common factor of 2, $2\left(m^2+5m-24\right)=0$. Then set the trinomial to 0 and solve for m. You find that $2(m+8)(m-3)=0$, so $m=-8$ or 3.
B. Incorrect. You probably either factored the quadratic incorrectly or you solved the individual equations incorrectly. The correct answer is $m=-8$ or 3.
C. Incorrect. You probably factored $2m^2+10m$ as $2m(m+5)$ and then set the factors equal to 0. However, the original equation is not equal to 0, it's equal to 48. To use the Zero Product Property, one side *must* be 0. The correct answer is $m=-8$ or 3.

D. Incorrect. You probably factored $2m^2+10m$ as $2m(m+5)$ and then set the factors equal to 0, as well as making a sign mistake when solving $m+5=0$. However, the original equation is not equal to 0, it's equal to 48. The correct answer is $m=-8$ or 3.

Summary

You can find the solutions, or roots, of quadratic equations by setting one side equal to zero, factoring the polynomial, and then applying the Zero Product Property. The Principle of Zero Products states that if $ab=0$, then either $a=0$ or $b=0$, or both a and b are 0. Once the polynomial is factored, set each factor equal to zero and solve them separately. The answers will be the set of solutions for the original equation.

Not all solutions are appropriate for some applications. In many real-world situations, negative solutions are not appropriate and must be discarded.

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CHAPTER OVERVIEW

13: Graphing

13.1: Graphs and Applications

  13.1.1: The Coordinate Plane

  13.1.2: Graphing Linear Equations

13.2: Slope and Writing the Equation of a Line

  13.2.1: Finding the Slope of a Line

  13.2.2: Writing the Equation of a Line

  13.2.3: Parallel and Perpendicular Lines

  13.2.4: Graphing Linear Inequalities

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SECTION OVERVIEW

13.1: Graphs and Applications

Topic hierarchy

13.1.1: The Coordinate Plane

13.1.2: Graphing Linear Equations

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13.1.1: The Coordinate Plane

Learning Objectives

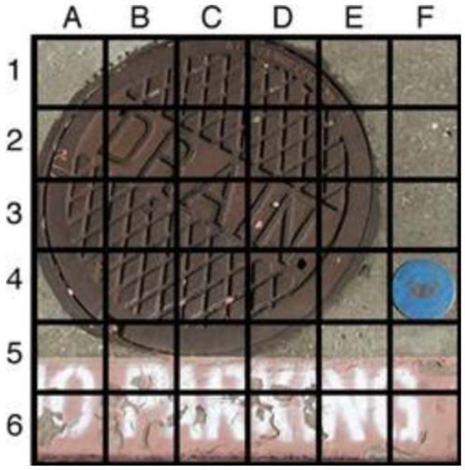
- Plot ordered pairs on a coordinate plane.
- Given an ordered pair, determine its quadrant.

Introduction

The **coordinate plane** was developed centuries ago and refined by the French mathematician René Descartes. In his honor, the system is sometimes called the Cartesian coordinate system. The coordinate plane can be used to plot points and graph lines. This system allows us to describe algebraic relationships in a visual sense, and also helps us create and interpret algebraic concepts.

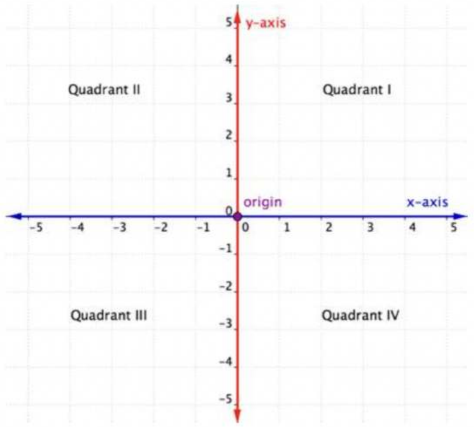
Getting to Know the Coordinate Plane

You have likely used a coordinate plane before. For example, have you ever used a gridded overlay to map the position of an object? (This is often done with road maps, too.)



This “map” uses a horizontal and vertical grid to convey information about an object’s location. Notice that the letters A through F are listed along the top, and the numbers 1 through 6 are listed along the left edge. The general location of any item on this map can be found by using the letter and number of its grid square. For example, you can find the item that exists at square 4 F by moving your finger along the horizontal to letter F and then straight down so you are in line with the 4. You’ll find a blue disc is at this location on the map.

The coordinate plane has similar elements to the grid shown above. It consists of a horizontal **axis** and a vertical axis, number lines that intersect at right angles. (They are perpendicular to each other.)



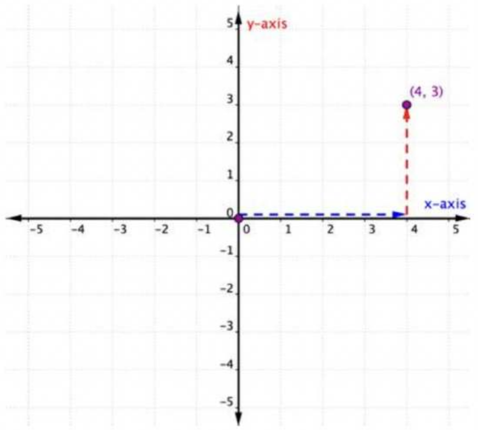
The horizontal axis in the coordinate plane is called the **x-axis**. The vertical axis is called the **y-axis**. The point at which the two axes intersect is called the **origin**. The origin is at 0 on the x-axis and 0 on the y-axis.

The intersecting x- and y-axes divide the coordinate plane into four sections. These four sections are called **quadrants**. Quadrants are named using the Roman numerals I, II, III, and IV beginning with the top right quadrant and moving counter clockwise.

Locations on the coordinate plane are described as **ordered pairs**. An ordered pair tells you the location of a point by relating the point's location along the x-axis (the first value of the ordered pair) and along the y-axis (the second value of the ordered pair).

In an ordered pair, such as (x, y), the first value is called the **x-coordinate** and the second value is the **y-coordinate**. Note that the x-coordinate is listed before the y-coordinate. Since the origin has an x-coordinate of 0 and a y-coordinate of 0, its ordered pair is written (0, 0).

Consider the point below.



To identify the location of this point, start at the origin (0, 0) and move right along the x-axis until you are under the point. Look at the label on the x-axis. The 4 indicates that, from the origin, you have traveled four units to the right along the x-axis. This is the x-coordinate, the first number in the ordered pair.

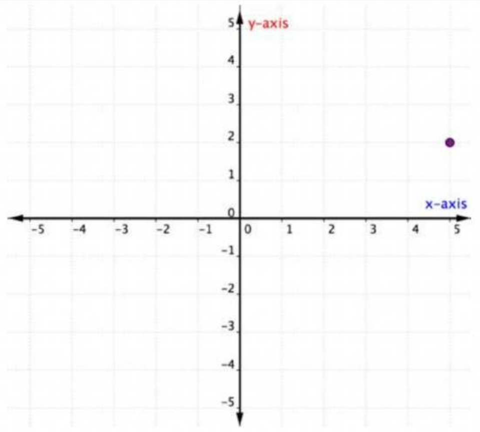
From 4 on the x-axis move up to the point and notice the number with which it aligns on the y-axis. The 3 indicates that, after leaving the x-axis, you traveled 3 units up in the vertical direction, the direction of the y-axis. This number is the y-coordinate, the second number in the ordered pair. With an x-coordinate of 4 and a y-coordinate of 3, you have the ordered pair (4, 3).

Let's look at another example.

✓ Example

Describe the point shown as an ordered pair.

Solution



$(5, y)$	Begin at the origin and move along the x-axis. This is the x-coordinate and is written first in the ordered pair.
$(5, 2)$	Move from 5 up to the ordered pair and read the number on the y-axis. This is the y-coordinate and is written second in the ordered pair.

The point shown as an ordered pair is (5, 2).

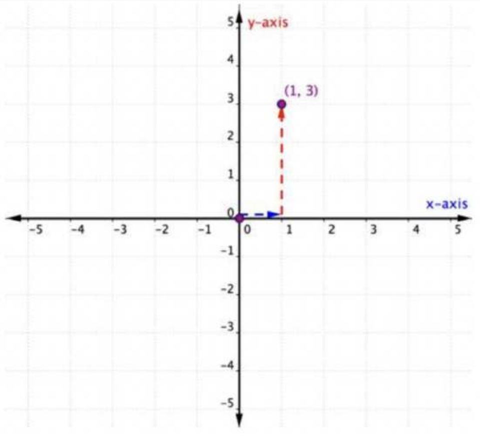
Plotting Points in the Coordinate Plane

Now that you know how to use the x- and y--axes, you can plot an ordered pair as well. Just remember, both processes start at the origin—the beginning! The example that follows shows how to graph the ordered pair (1, 3).

✓ Example

Plot the point (1, 3).

Solution



The x-coordinate is 1 because it comes first in the ordered pair. Start at the origin and move a distance of 1 unit in a positive direction (to the right) from the origin along the x-axis.	The y-coordinate is 3 because it comes second in the ordered pair. From here move directly 3 units in a positive direction (up). If you look over to the y-axis, you should be lined up with 3 on that axis.

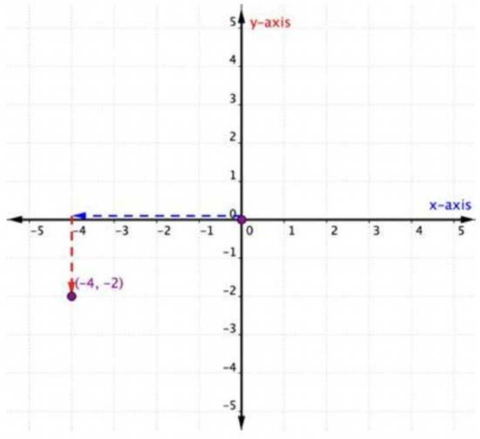
Draw a point at this location and label the point (1, 3).

In the previous example, both the x- and y-coordinates were positive. When one (or both) of the coordinates of an ordered pair is negative, you will need to move in the negative direction along one or both axes. Consider the example below in which both coordinates are negative.

✓ Example

Plot the point $(-4, -2)$**.**

Solution



The x-coordinate is -4 because it comes first in the ordered pair. Start at the origin and move 4 units in a negative direction (left) along the x-axis.	The y-coordinate is -2 because it comes second in the ordered pair. Now move 2 units in a negative direction (down). If you look over to the y-axis, you should be lined up with -2 on that axis.

Draw a point at this location and label the point $(-4, -2)$.

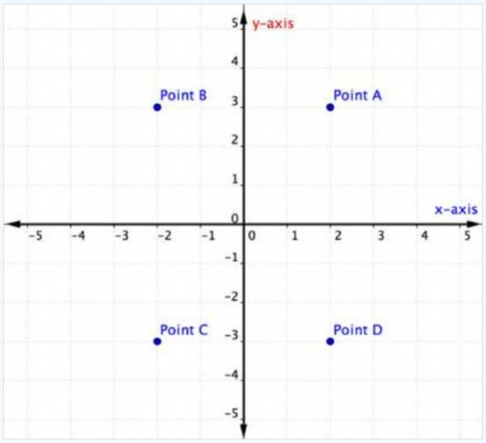
The steps for plotting a point are summarized below.

Steps for Plotting an Ordered Pair (x, y) in the Coordinate Plane

- Determine the x-coordinate. Beginning at the origin, move horizontally, the direction of the x-axis, the distance given by the x-coordinate. If the x-coordinate is positive, move to the right; if the x-coordinate is negative, move to the left.
- Determine the y-coordinate. Beginning at the x-coordinate, move vertically, the direction of the y-axis, the distance given by the y-coordinate. If the y-coordinate is positive, move up; if the y-coordinate is negative, move down.
- Draw a point at the ending location. Label the point with the ordered pair.

Exercise

Which point represents the ordered pair (-2, 3)?

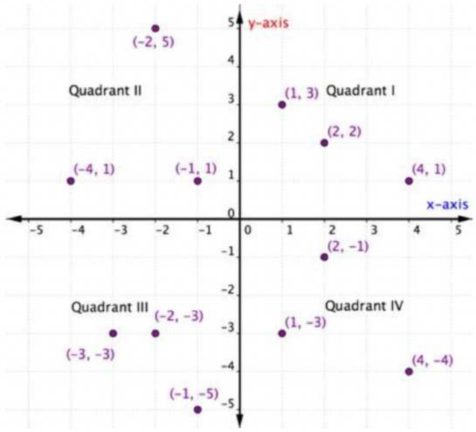


Answer

Point B is correct. Begin at the origin and move 2 units in the negative direction (left) along the x-axis. Then, move 3 units in a positive direction along the y-axis (3 units up). Plot a point there.

The Four Quadrants

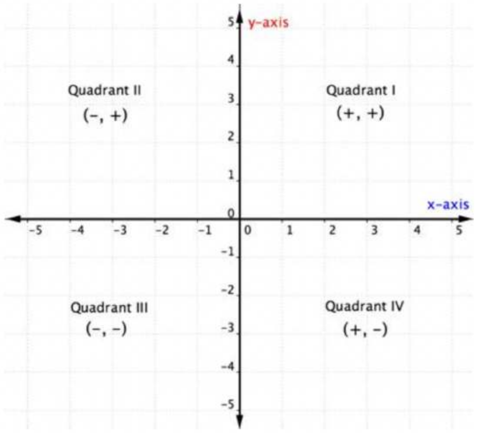
Ordered pairs within any particular quadrant share certain characteristics. Look at each quadrant in the graph below. What do you notice about the signs of the x- and y-coordinates of the points within each quadrant?



Within each quadrant, the signs of the x-coordinates and y-coordinates of each ordered pair are the same. They also follow a pattern, which is outlined in the table below.

Quadrant	**General Form of Point in this Quadrant**	**Example**	**Description**
I	$(+,+)$	$(5,4)$	Starting from the origin, go along the x-axis in a positive direction (right) and along the y-axis in a positive direction (up).
II	$(-,+)$	$(-5,4)$	Starting from the origin, go along the x-axis in a negative direction (left) and along the y-axis in a positive direction (up).
III	$(-,-)$	$(-5,-4)$	Starting from the origin, go along the x-axis in a negative direction (left) and along the y-axis in a negative direction (down).
IV	$(+,-)$	$(5,-4)$	Starting from the origin, go along the x-axis in a positive direction (right) and along the y-axis in a negative direction (down).

Once you know about the quadrants in the coordinate plane, you can determine the quadrant of an ordered pair without even graphing it by looking at the chart above. Here’s another way to think about it.



The example below details how to determine the quadrant location of a point just by thinking about the signs of its coordinates. Thinking about the quadrant location before plotting a point can help you prevent a mistake. It is also useful knowledge for checking that you have plotted a point correctly.

✓ Example

In which quadrant is the point (-7, 10) located?

Solution

$(-7, 10)$	Look at the signs of the x- and y-coordinates. For this ordered pair, the signs are $(-,+)$.
Points with the pattern $(-,+)$ are in Quadrant II.	Using the table or grid above, locate the pattern $(-,+)$.

The point (-7, 10) is in Quadrant II.

✓ Example

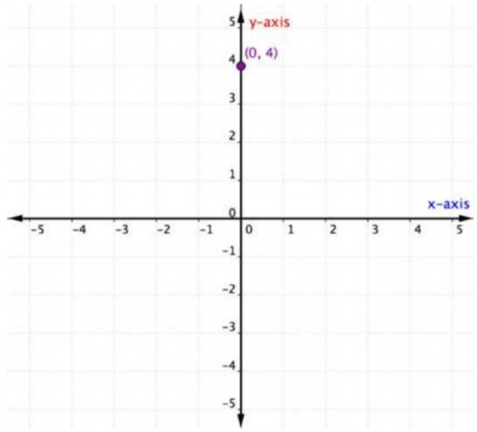
In which quadrant is the point (-10, -5) located?

Solution

$(-10, -5)$	Look at the signs of the x- and y-coordinates. For this ordered pair, the signs are $(-,-)$.
Points with the pattern $(-,-)$ are in Quadrant III.	Using the table or grid above, locate the pattern $(-,-)$.

The point (-10, -5) is in Quadrant III.

What happens if an ordered pair has an x- or y-coordinate of zero? The example below shows the graph of the ordered pair (0, 4).



A point located on one of the axes is not considered to be in a quadrant. It is simply on one of the axes. Whenever the x-coordinate is 0, the point is located on the y-axis. Similarly, any point that has a y-coordinate of 0 will be located on the x-axis.

Exercise

Which of the descriptions below best describes the location of the point (8, 0)?

A. Quadrant I
B. It is on the x-axis
C. It is on the y-axis
D. The coordinate plane

Answer

A. Incorrect. The axes are not considered part of the quadrant. Rather, they are the boundaries that define the quadrant. The correct answer is the x-axis.
B. Correct. To plot this point, you would begin at the origin and move 8 units in the positive direction along the x-axis. Since the y-coordinate is 0, you do not move off the location on the x-axis.
C. Incorrect. Since the y-coordinate is 0, this point is located on the x-axis. To plot this point, you would begin at the origin and move 8 units in the positive direction along the x-axis. Since the y-coordinate is 0, you do not move off the location on the x-axis. The correct answer is the x-axis.
D. Incorrect. Even though this point is located in the coordinate plane, you can provide a better description of its location. The correct answer is the x-axis.

Summary

The coordinate plane is a system for graphing and describing points and lines. The coordinate plane is comprised of a horizontal x-axis and a vertical y-axis. The intersection of these lines creates the origin, which is the point (0, 0). The coordinate plane is split into four quadrants. Together, these features of the coordinate system allow for the graphical representation and communication about points, lines, and other algebraic concepts.

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13.1.2: Graphing Linear Equations

Learning Objectives

- Use coordinate pairs to graph linear relationships.
- Graph a linear equation using x- and y-intercepts.
- Determine whether an ordered pair is a solution of an equation.
- Solve application problems involving graphs of linear equations.

Introduction

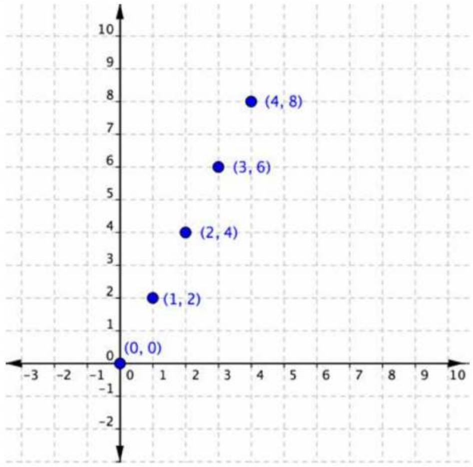
Graphing ordered pairs is only the beginning of the story. Once you know how to place points on a grid, you can use them to make sense of all kinds of mathematical relationships.

You can use a **coordinate plane** to plot points and to map various relationships, such as the relationship between an object's distance and the elapsed time. Many mathematical relationships are **linear relationships**. Let's look at what a linear relationship is.

Linear Relationships

A linear relationship is a relationship between variables such that when plotted on a coordinate plane, the points lie on a line. Let's start by looking at a series of points in Quadrant I on the coordinate plane.

Look at the five **ordered pairs** (and their x- and y-coordinates) below. Do you see any pattern to the location of the points? If this pattern continued, what other points could be on the line?



You probably identified that if this pattern continued, the next ordered pair would be at (5, 10). This makes sense because the point. (5, 10) "lines up" with the other points in the series—it is literally on the same line as the others. Applying the same logic, you may identify that the ordered pairs (6, 12) and (7, 14) would also belong if this coordinate plane were larger; they, too, will line up with the other points.

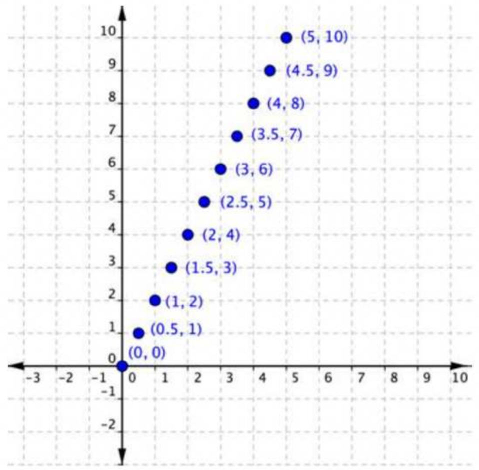
These series of points can also be represented in a table. In the table below, the x- and y-coordinates of each ordered pair on the graph are recorded.

x-coordinate	y-coordinate
0	0
1	2
2	4
3	6
4	8

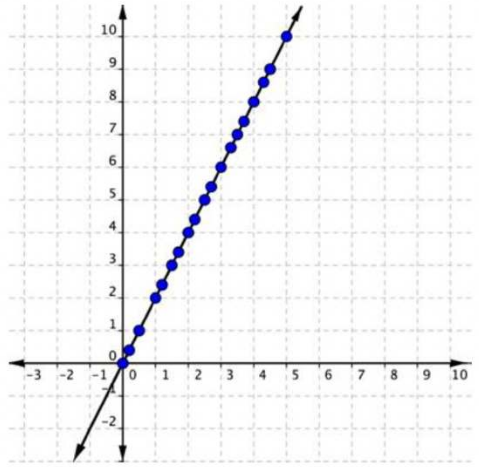
5	10
6	12
7	14

Notice that each y-coordinate is twice the corresponding x-value. All of these x- and y-values follow the same pattern, and, when placed on a coordinate plane, they all line up.

Once you know the pattern that relates the x- and y-values, you can find a y-value for any x-value that lies on the line. So if the rule of this pattern is that each y-coordinate is *twice* the corresponding x-value, then the ordered pairs (1.5, 3), (2.5, 5), and (3.5, 7) should all appear on the line too, correct? Look to see what happens.



If you were to keep adding ordered pairs (x, y) where the y-value was twice the x-value, you would end up with a graph like this.



Look at how all of the points blend together to create a line. You can think of a line, then, as a collection of an infinite number of individual points that share the same mathematical relationship. In this case, the relationship is that the y-value is twice the x-value.

There are multiple ways to represent a linear relationship—a table, a linear graph, and there is also a **linear equation**. A linear equation is an equation with two variables whose ordered pairs graph as a straight line.

There are several ways to create a graph from a linear equation. One way is to create a table of values for x and y, and then plot these ordered pairs on the coordinate plane. Two points are enough to determine a line. However, it’s always a good idea to plot more than two points to avoid possible errors.

Then you draw a line through the points to show all of the points that are on the line. The arrows at each end of the graph indicate that the line continues endlessly in both directions. Every point on this line is a solution to the linear equation.

✓ Example

Graph the linear equation $y = -1.5x$.

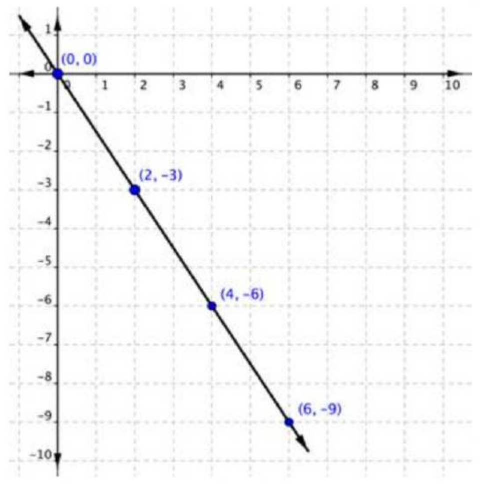
Solution

x values	$-1.5x$	y values
0	-1.5(0)	0
2	-1.5(2)	-3
4	-1.5(4)	-6
6	-1.5(6)	-9

Evaluate $y = -1.5x$ for different values of x, and create a table of corresponding x and y values.
Since the coefficient of x is -1.5, it is convenient to choose multiples of 2 for x. This ensures that y is an integer, and makes the line easier to graph.
Convert the table to ordered pairs. Then plot the ordered pairs.

$(0, 0)$
$(2, -3)$
$(4, -6)$
$(6, -9)$

Draw a line through the points to indicate all of the points on the line.



✓ Example

Graph the linear equation $y = 2x + 3$.

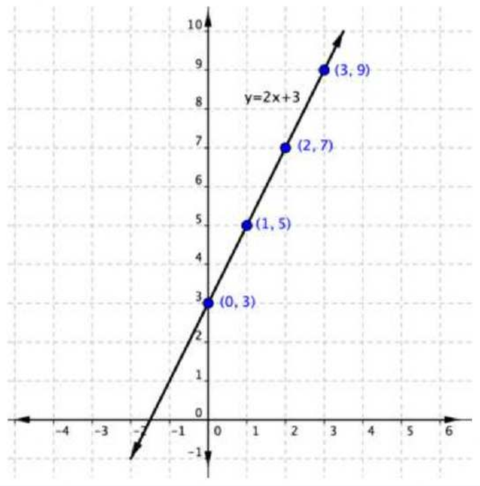
Solution

x values	$2x + 3$	y values
0	2(0)+3	3
1	2(1)+3	5
2	2(2)+3	7
3	2(3)+3	9

Evaluate $y = 2x + 3$ for different values of x, and create a table of corresponding x and y values.

$(0, 3)$
$(1, 5)$
$(2, 7)$
$(3, 9)$

Convert the table to ordered pairs.
Plot the ordered pairs (shown below).
Draw a line through the points to indicate all of the points on the line.



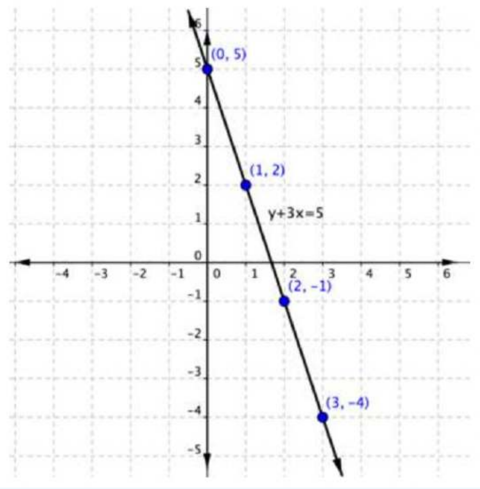
The linear equations graphed above were solved for y. If the equation is not stated in terms of y, it is best to first solve the equation for y. If there is no y in the equation, then solve the equation for x.

✓ Example

Graph the linear equation $y+3x=5$.

Solution

<table>
<tr><td>Solve $y+3x=5$ for y
$y+3x-3x=5-3x$
$y=5-3x$</td><td></td></tr>
<tr><td><table>
<tr><th>x values</th><th>$5-3x$</th><th>y values</th></tr>
<tr><td>0</td><td>5-3(0)</td><td>5</td></tr>
<tr><td>1</td><td>5-3(1)</td><td>2</td></tr>
<tr><td>2</td><td>5-3(2)</td><td>-1</td></tr>
<tr><td>3</td><td>5-3(3)</td><td>-4</td></tr>
</table></td><td>Evaluate $y=5-3x$ for different values of x, and create a table of corresponding x and y values.</td></tr>
<tr><td>$(0,5)$
$(1,2)$
$(2,-1)$
$(3,-4)$</td><td>Plot the ordered pairs (shown below).
Draw a line through the points to indicate all of the points on the line.</td></tr>
</table>



The linear equations $x=2$ and $y=-3$ only have one variable in each of them. However, because these are linear equations, then they will graph on a coordinate plane just as the linear equations above do. Just think of the equation $x=2$ as $x=0y+2$ and think of $y=-3$ as $y=0x-3$.

✓ Example

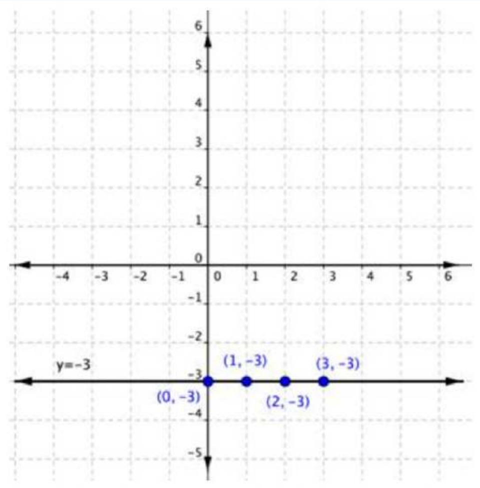
Graph $y=-3$.

Solution

x values	$0x-3$	y values
0	0(0)-3	-3
1	0(1)-3	-3
2	0(2)-3	-3
3	0(3)-3	-3

Write $y=-3$ as $y=0x-3$, and evaluate y when x has several values. Or just realize that $y=-3$ means every y value will be -3, no matter what x is.

$(0,-3)$ $(1,-3)$ $(2,-3)$ $(3,-3)$	Plot the ordered pairs (shown below). Draw a line through the points to indicate all of the points on the line.



Notice that $y=-3$ graphs as a horizontal line.

Exercise

Which table of values could be generated by the equation $2y-5x=10$?

A.

x	y
5	0
7.5	1
10	2

B.

x	y
1	5
2	6
3	7

C.

x	y
1	15
2	20
3	25

D.

x	y
0	5
1	7.5
2	10

Answer

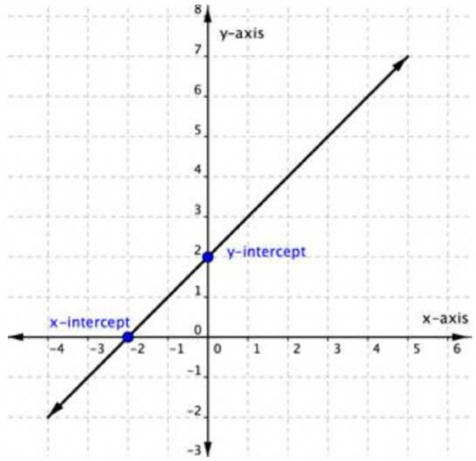
The correct answer is D. To generate a table of values for this equation, first solve the equation for y. $y=\frac{5}{2}x+5$

Now pick values for x and then evaluate the equation for y. When $x=0$, $y=5$, when $x=1$, $y=7.5$, when $x=2$, $y=10$.

x- and y- Intercepts

The intercepts of a line are the points where the line intercepts, or crosses, the horizontal and vertical axes. To help you remember what “intercept” means, think about the word “intersect.” The two words sound alike and in this case mean the same thing.

The straight line on the graph below intercepts the two coordinate axes. The point where the line crosses the x-axis is called the **x-intercept**. The **y-intercept** is the point where the line crosses the y-axis.



The x-intercept above is the point (-2, 0). The y-intercept above is the point (0, 2).

Notice that the y-intercept always occurs where $x=0$, and the x-intercept always occurs where $y=0$.

To find the x- and y-intercepts of a linear equation, you can substitute 0 for y and for x respectively.

For example, the linear equation $3y+2x=6$ has an x intercept when $y=0$, so $3(0)+2x=6$.

$$2x=6$$
$$x=3$$

The x-intercept is (3, 0).

Likewise, the y-intercept occurs when $x=0$.

$$3y+2(0)=6$$
$$3y=6$$
$$y=2$$

The y-intercept is (0, 2).

? Exercise

What is the y-intercept of a line with the equation $y=5x-4$?

A. $\left(\frac{4}{5},0\right)$
B. $(-4,0)$
C. $(0,-4)$
D. $(5,-4)$

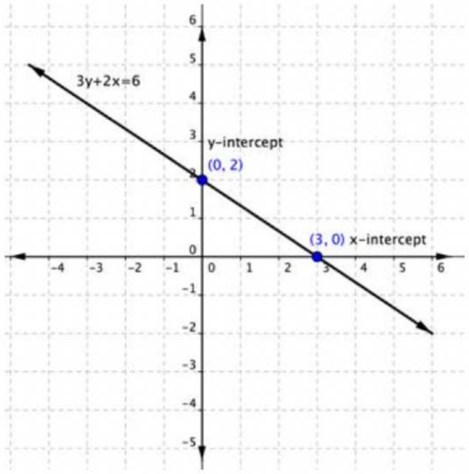
Answer

A. Incorrect. $\left(\frac{4}{5},0\right)$ is the x-intercept. At the y-intercept, $x=0$. When 0 is substituted for x in the equation, $y=-4$. The correct answer is (0, -4).
B. Incorrect. This answer switches the values of x and y. Coordinates are given in the order (x,y). At the y-intercept, $x=0$. When 0 is substituted for x in the equation, $y=-4$. The correct answer is $(0,-4)$.
C. Correct. At the y-intercept, $x=0$. When 0 is substituted for x in the equation, $y=-4$.
D. Incorrect. This is the coefficient of x and the constant, not the y-intercept. At the y-intercept, $x=0$. When 0 is substituted for x in the equation, $y=-4$. The correct answer is $(0,-4)$.

Using Intercepts to Graph Linear Equations

You can use intercepts to graph linear equations. Once you have found the two intercepts, draw a line through them.

Let’s do it with the equation $3y+2x=6$. You figured out that the intercepts of the line this equation represents are $(0,2)$ and $(3,0)$. That’s all you need to know.

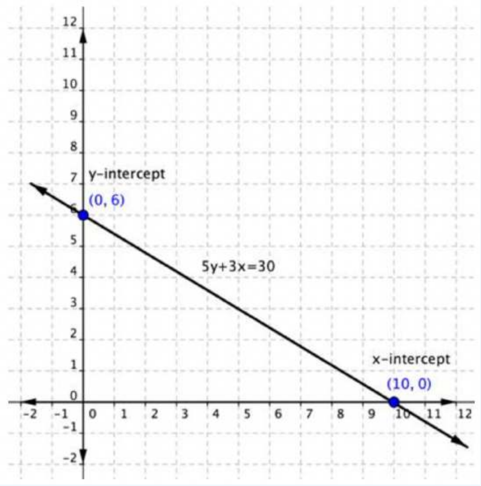


Example

Graph $5y+3x=30$**.**

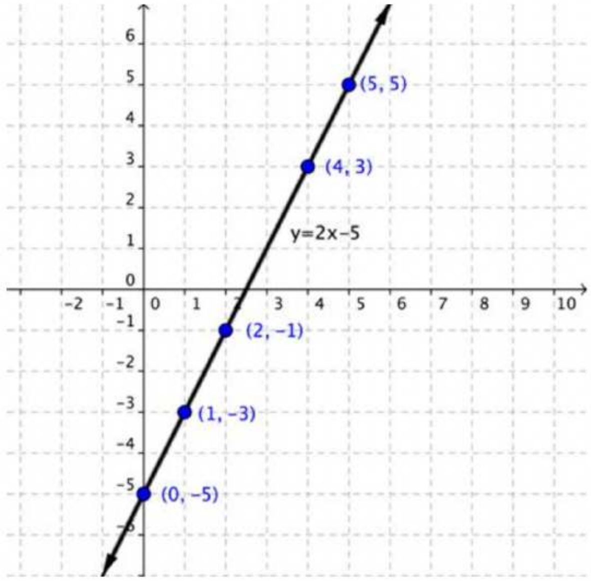
Solution

$5y+3x=30$	When an equation is in $Ax+By=C$ form, you can easily find the x- and y-intercepts and then graph.
$\begin{aligned} 5y+3x&=30 \\ 5y+3(0)&=30 \\ 5y+0&=30 \\ 5y&=30 \\ y&=6 \\ y\text{ -intercept: }&(0,6) \end{aligned}$	To find the y-intercept, set $x=0$ and solve for y.
$\begin{aligned} 5y+3x&=30 \\ 5(0)+3x&=30 \\ 0+3x&=30 \\ 3x&=30 \\ x&=10 \\ x\text{ -intercept: }&(10,0) \end{aligned}$	To find the x-intercept, set $y=0$ and solve for x.



Ordered Pairs as Solutions

So far, you have considered the following ideas about lines: a line is a visual representation of a linear equation, and the line itself is made up of an infinite number of points (or ordered pairs). The picture below shows the line of the linear equation $y = 2x - 5$ with some of the specific points on the line.



Every point on the line is a solution to the equation $y = 2x - 5$. You can try any of the points that are labeled like the ordered pair, $(1, -3)$.

$$\begin{aligned}&y = 2x - 5\\&-3 = 2(1) - 5\\&-3 = 2 - 5\\&-3 = -3 \text{ This is true.}\end{aligned}$$

You can also try ANY of the other points on the line. Every point on the line is a solution to the equation $y = 2x - 5$. All this means is that determining whether an ordered pair is a solution of an equation is pretty straightforward. If the ordered pair is on the line created by the linear equation, then it is a solution to the equation. But if the ordered pair is not on the line—no matter how close it may look—then it is not a solution to the equation.

Identifying Solutions

To find out whether an ordered pair is a solution of a linear equation, you can do the following:

- Graph the linear equation, and graph the ordered pair. If the ordered pair appears to be on the graph of a line, then it is a possible solution of the linear equation. If the ordered pair does not lie on the graph of a line, then it is not a solution.
- Substitute the (x, y) values into the equation. If the equation yields a true statement, then the ordered pair is a solution of the linear equation. If the ordered pair does not yield a true statement, then it is not a solution.

Example

Determine whether (-2, 4) is a solution to the equation $4y + 5x = 3$.

Solution

$4y + 5x = 3$ $4(4) + 5(-2) = 3$	For this problem, you will use the substitution method. Substitute $x = -2$ and $y = 4$ into the equation.
$16 + (-10) = 3$ $6 = 3$	Evaluate. The statement is not true, so $(-2, 4)$ is not a solution to the equation $4y + 5x = 3$

$(-2,4)$ is not a solution to the equation $4y+5x=3$.

Application Problems

Linear equations can be used to model a number of real-life problems, like how much money you make over time, or the distance that a bicyclist will travel given a steady rate of pedaling. Graphing these relationships on a coordinate plane can often help you think about (and find solutions to) the problem.

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SECTION OVERVIEW

13.2: Slope and Writing the Equation of a Line

Topic hierarchy

13.2.1: Finding the Slope of a Line

13.2.2: Writing the Equation of a Line

13.2.3: Parallel and Perpendicular Lines

13.2.4: Graphing Linear Inequalities

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13.2.1: Finding the Slope of a Line

Learning Objectives

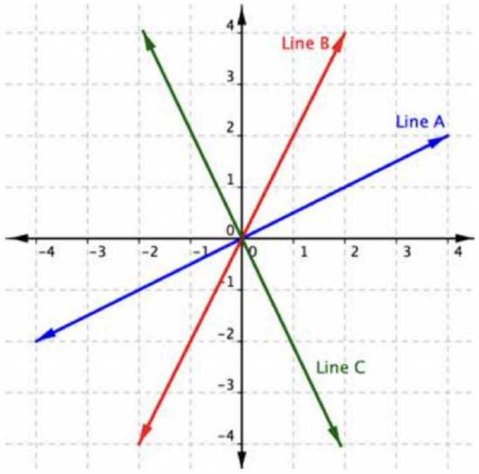
- Find the slope of a line from a graph.
- Find the slope of a line given two points.
- Find the slope of the lines $x = a$ and $y = b$.

Introduction

The idea of slope is something you encounter often in everyday life. Think about rolling a cart down a ramp or climbing a set of stairs. Both the ramp and the stairs have a slope. You can describe the slope, or steepness, of the ramp and stairs by considering horizontal and vertical movement along them. In conversation, you use words like “gradual” or “steep” to describe slope. Along a gradual slope, most of the movement is horizontal. Along a steep slope, the vertical movement is greater.

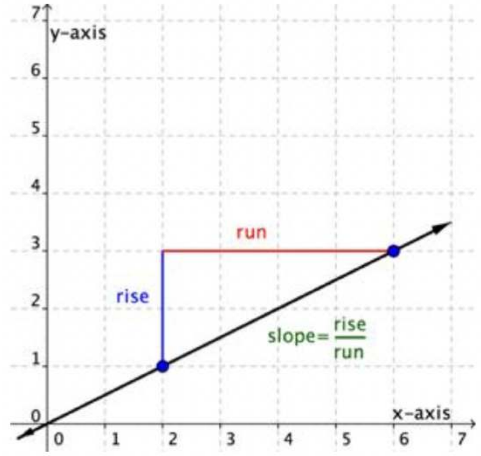
Defining Slope

The mathematical definition of **slope** is very similar to our everyday one. In math, slope is used to describe the steepness and direction of lines. By just looking at the graph of a line, you can learn some things about its slope, especially relative to other lines graphed on the same coordinate plane. Consider the graphs of the three lines shown below:



First, let’s look at lines A and B. If you imagined these lines to be hills, you would say that line B is steeper than line A. Line B has a greater slope than line A.

Next, notice that lines A and B slant up as you move from left to right. We say these two lines have a positive slope. Line C slants down from left to right. Line C has a negative slope. Using two of the points on the line, you can find the slope of the line by finding the rise and the run. The vertical change between two points is called the **rise**, and the horizontal change is called the **run**. The slope equals the rise divided by the run: $\text{Slope } = \frac{\text{rise}}{\text{run}}$.

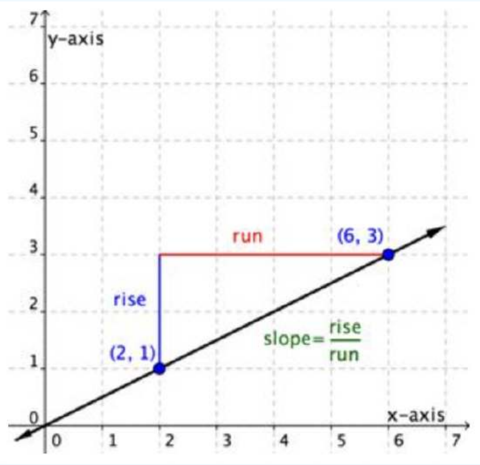


Finding the Slope of a Line from a Graph

You can determine the slope of a line from its graph by looking at the rise and run. One characteristic of a line is that its slope is constant all the way along it. So, you can choose any 2 points along the graph of the line to figure out the slope. Let’s look at an example.

✓ Example

Use the graph to find the slope of the line.



Solution

$\text{rise } = 2$	Start from a point on the line, such as (2, 1) and move vertically until in line with another point on the line, such as (6, 3). The rise is 2 units. It is positive as you moved up.
$\text{run } = 4$	Next, move horizontally to the point (6, 3). Count the number of units. The run is 4 units. It is positive as you moved to the right.
$\text{Slope } = \frac{2}{4} = \frac{1}{2}$	$\text{Slope } = \frac{\text{rise}}{\text{run}}$

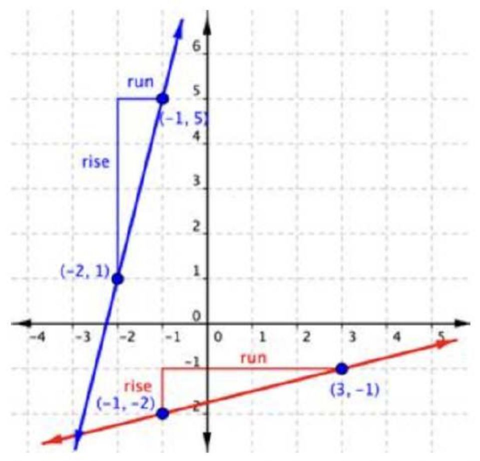
The slope is $\frac{1}{2}$.

This line will have a slope of $\frac{1}{2}$ no matter which two points you pick on the line. Try measuring the slope from the origin, (0, 0), to the point (6, 3). You will find that the $\text{rise } = 3$ and the $\text{run } = 6$. The slope is $\frac{rise}{run} = \frac{3}{6} = \frac{1}{2}$. It is the same!

Let’s look at another example.

Example

Use the graph to find the slope of the two lines.



Solution

	Notice that both of these lines have positive slopes, so you expect your answers to be positive.
$\text{rise} = 4$	Blue line Start with the blue line, going from point (-2, 1) to point (-1, 5). This line has a rise of 4 units up, so it is positive.
$\text{run} = 1$	Run is 1 unit to the right, so it is positive.
$\text{Slope} = \frac{4}{1} = 4$	Substitute the values for the rise and run in the formula $\text{Slope} = \frac{\text{rise}}{\text{run}}$.
$\text{rise} = 1$	Red line The red line, going from point (-1, -2) to point (3, -1) has a rise of 1 unit.
$\text{run} = 4$	The red line has a run of 4 units.
$\text{Slope} = \frac{1}{4}$	Substitute the values for the rise and run into the formula $\text{Slope} = \frac{\text{rise}}{\text{run}}$.

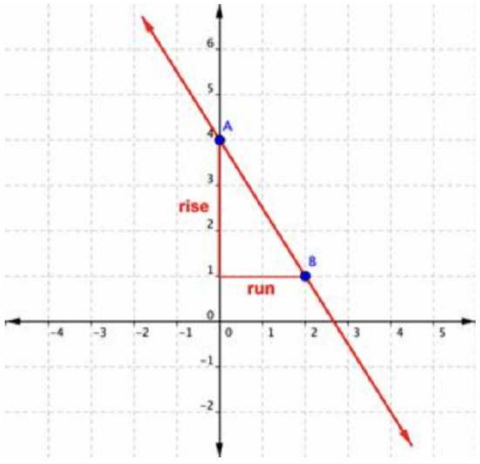
The slope of the blue line is 4 and the slope of the red line is $\frac{1}{4}$.

When you look at the two lines, you can see that the blue line is steeper than the red line. It makes sense the value of the slope of the blue line, 4, is greater than the value of the slope of the red line, $\frac{1}{4}$. The greater the slope, the steeper the line.

The next example shows a line with a negative slope.

Example

Find the slope of the line graphed below.



Solution

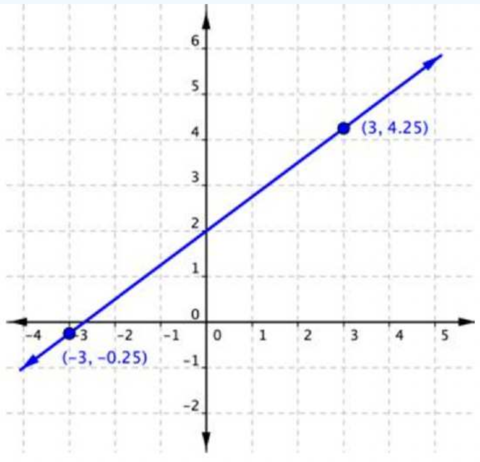
rise $= -3$	Start at Point A, (0, 4) and rise -3. This means moving 3 units in a negative direction.
run $= 2$	From there, run 2 units in a positive direction to Point B (2, 1).
Slope $= \frac{-3}{2}$	Slope $= \frac{\text{rise}}{\text{run}}$

The slope of the line is $-\frac{3}{2}$.

Direction is important when it comes to determining slope. It's important to pay attention to whether you are moving up, down, left, or right; that is, if you are moving in a positive or negative direction. If you go up to get to your second point, the rise is positive. If you go down to get to your second point, the rise is negative. If you go right to get to your second point, the run is positive. If you go left to get to your second point, the run is negative. In the example above, you could have found the slope by starting at point B, running -2, and then rising +3 to arrive at point A. The result is still a slope of $\frac{rise}{run} = \frac{+3}{-2} = -\frac{3}{2}$.

✓ Advanced Example

Find the slope of the line graphed below.



Solution

rise $= 4.5$	Start at (-3, -0.25) and rise 4.5. This means moving 4.5 units in a positive direction.
run $= 6$	From there, run 6 units in a positive direction to (3, 4.25).

Slope $= \frac{4.5}{6} = 0.75$	Slope $= \frac{\text{rise}}{\text{run}}$

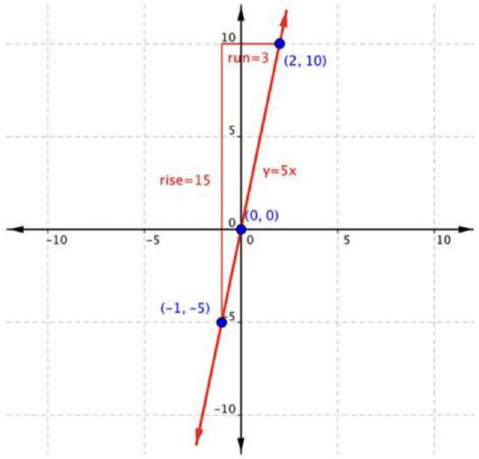
The slope of the line is 0.75.

Looking at Equations

The slope of a line can sometimes be quickly determined from its equation. Let’s consider the line whose equation is $y = 5x$. You can create a table of values to find three points on the line.

x	y
-1	-5
0	0
2	10

Plotting these points, create the graph of the line and determine the slope.



As you move from the point (-1, -5) to the point (2, 10), the line has a rise of 15 and a run of 3, so the slope of the line is $\frac{15}{3} = 5$. Notice that the number 5 also appears in the equation: $y = 5x$.

Whenever the equation of a line is written in the form $y = mx + b$, it is called the slope-intercept form of the equation. The m is the slope of the line. And b is the b in the point that is the y-intercept $(0, b)$.

For example, for the equation $y = 3x - 7$, the slope is 3, and the y-intercept is (0, -7).

What if the equation is written as $2y = 5x + 1$? Then you must rewrite the equation in the form $y = mx + b$. Solve for y.

$2y = 5x + 1$

$y = \frac{5}{2}x + \frac{1}{2}$ divide both sides of the equation by 2.

The slope is $\frac{5}{2}$, and the y-intercept is $\left(0, \frac{1}{2}\right)$.

? Exercise

What is the slope of the line whose equation is $y = -2x + 7$?

A. 7
B. 2
C. -2
D. $-\frac{7}{2}$

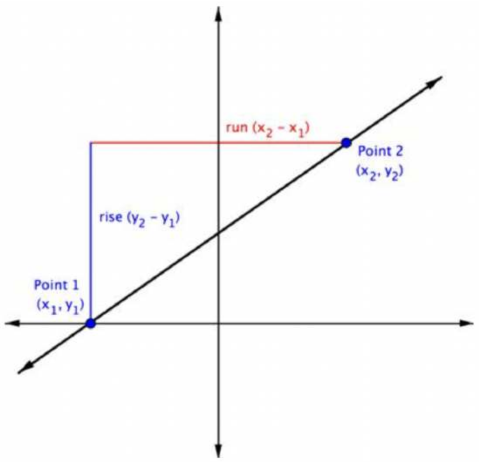
Answer

A. Incorrect. The slope for a line written in $y = mx + b$ is given by the coefficient of x. The correct answer is -2.
B. Incorrect. The slope for a line written in $y = mx + b$ is given by the coefficient of x. The coefficient is -2. The correct answer is -2.
C. Correct. The slope for a line written in $y = mx + b$ is given by the coefficient of x. For this line, the coefficient, or m, the slope, is -2.
D. Incorrect. The slope for a line written in $y = mx + b$ is given by the coefficient of x. The coefficient is -2. The correct answer is -2.

Finding the Slope of a Line Given Two Points

You've seen that you can find the slope of a line on a graph by measuring the rise and the run. You can also find the slope of a straight line without its graph if you know the coordinates of any two points on that line. Every point has a set of coordinates: an x-value and a y-value, written as an ordered pair (x, y). The x value tells you where a point is horizontally. The y value tells you where the point is vertically.

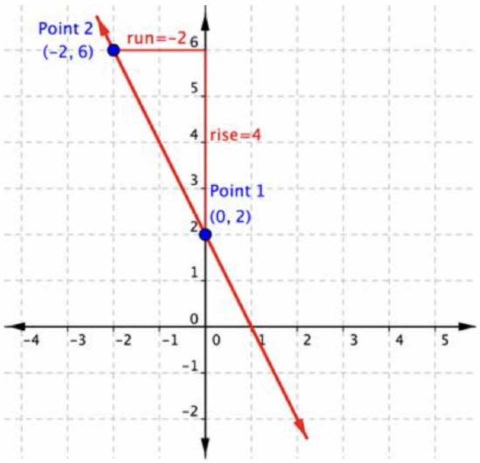
Consider two points on a line—Point 1 and Point 2. Point 1 has coordinates (x_1, y_1) and Point 2 has coordinates (x_2, y_2).



The rise is the vertical distance between the two points, which is the difference between their y-coordinates. That makes the rise $y_2 - y_1$. The run between these two points is the difference in the x-coordinates, or $x_2 = x_1$.

So, $\text{Slope} = \frac{\text{rise}}{\text{run}}$ or $m = \frac{y_2 - y_1}{x_2 - x_1}$

In the example below, you'll see that the line has two points each indicated as an ordered pair. The point (0, 2) is indicated as Point 1, and (-2, 6) as Point 2. So you are going to move from Point 1 to Point 2. A triangle is drawn in above the line to help illustrate the rise and run.



You can see from the graph that the rise going from Point 1 to Point 2 is 4, because you are moving 4 units in a positive direction (up). The run is -2, because you are then moving in a negative direction (left) 2 units. Using the slope formula, $\text{Slope} = \frac{rise}{run} = \frac{4}{-2} = -2$.

You do not need the graph to find the slope. You can just use the coordinates, keeping careful track of which is Point 1 and which is Point 2. Let's organize the information about the two points:

Name	**Ordered Pair**	**Coordinates**
Point 1	(0, 2)	$x_1 = 0$ $y_1 = 2$
Point 2	(-2, 6)	$x_2 = -2$ $y_2 = 6$

The slope, $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{-2-0} = \frac{4}{-2} = -2$. The slope of the line, m, is -2.

It doesn't matter which point is designated as Point 1 and which is Point 2. You could have called (-2, 6) Point 1, and (0, 2) Point 2. In that case, putting the coordinates into the slope formula produces the equation $m = \frac{2-6}{0-(-2)} = \frac{-4}{2} = -2$. Once again, the slope $m = -2$. That's the same slope as before. The important thing is to be consistent when you subtract: you must always subtract in the same order $y_2 - y_1$ and $x_2 - x_1$.

✓ Example

What is the slope of the line that contains the points (5, 5) and (4, 2)?

Solution

$x_1 = 4$ $y_1 = 2$	$(4, 2) = \text{ Point } 1, (x_1, y_1)$
$x_2 = 5$ $y_2 = 5$	$(5, 5) = \text{ Point } 2, (x_2, y_2)$
$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{5-2}{5-4} = \frac{3}{1}$ $m = 3$	Substitute the values into the slope formula and simplify.

The slope is 3.

The example below shows the solution when you reverse the order of the points, calling (5, 5) Point 1 and (4, 2) Point 2.

✓ Example

What is the slope of the line that contains the points (5, 5) and (4, 2)?

Solution

$x_1 = 5$ $y_1 = 5$	$(5, 5) = \text{ Point } 1, (x_1, y_1)$
$x_2 = 4$ $y_2 = 2$	$(4, 2) = \text{ Point } 2, (x_2, y_2)$
$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{2-5}{4-5} = \frac{-3}{-1} = 3$ $m = 3$	Substitute the values into the slope formula and simplify.

The slope is 3.

Notice that regardless of which ordered pair is named Point 1 and which is named Point 2, the slope is still 3.

Advanced Example

What is the slope of the line that contains the points (3, -6.25) and (-1, 8.5)?

Solution

$x_1 = 3$ $y_1 = -6.25$ $x_2 = -1$ $y_2 = 8.5$	$(3, -6.25) =$ Point 1, (x_1, y_1) $(-1, 8.5) =$ Point 2, (x_2, y_2)
$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{8.5 - (-6.25)}{-1 - 3}$ $m = \frac{14.75}{-4}$ $m = -3.6875$	Substitute the values into the slope formula and simplify.

The slope is -3.6875.

Exercise

What is the slope of a line that includes the points (-5, 1) and (-2, 3)

A. $\frac{2}{3}$
B. $-\frac{2}{7}$
C. $-\frac{2}{3}$
D. $\frac{3}{2}$

Answer

A. Correct. $m = \frac{3-1}{-2-(-5)} = \frac{2}{3}$
B. Incorrect. The denominator is $-2-(-5)$, $-2-5$. The correct answer is
C. Incorrect. Put the coordinates into the slope formula consistently: $\frac{y_2-y_1}{x_2-x_1}$. The correct answer is $m = \frac{3-1}{-2-(-5)} = \frac{2}{3}$.
D. Incorrect. You have interchanged the rise and the run. The correct answer is $m = \frac{3-1}{-2-(-5)} = \frac{2}{3}$.

Advanced Question

What is the slope of a line that includes the points $\left(5, -\frac{1}{5}\right)$ and $\left(-3, \frac{3}{5}\right)$?

A. -10
B. $\frac{8}{25}$
C. $-\frac{1}{20}$
D. $-\frac{1}{10}$

Answer

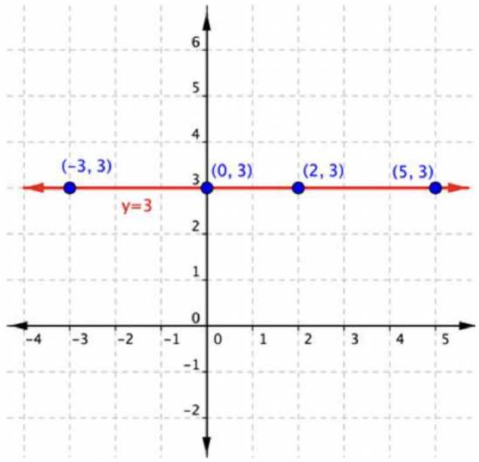
A. Incorrect. It looks like you inverted the rise and the run. Use the formula $m = \frac{y_2-y_1}{x_2-x_1}$ to find the slope. The correct answer is $-\frac{1}{10}$.
B. Incorrect. It looks like you subtracted either the y or x coordinates in the wrong order. Make sure you subtract $y_2 = y_1$, then $x_2 - x_1$, and then calculate the slope. The correct answer is $-\frac{1}{10}$.
C. Incorrect. It looks like you subtracted either the y or x coordinates in the wrong order. Make sure you subtract $y_2 = y_1$, then $x_2 - x_1$, and then calculate the slope. The correct answer is $-\frac{1}{10}$.
D. Correct. Using the formula for slope, $m = \frac{y_2-y_1}{x_2-x_1}$, you found that $m = \frac{\frac{3}{5}-\left(-\frac{1}{5}\right)}{-3-5} = \frac{\frac{4}{5}}{-8} = \frac{4}{5} \div (-8) = \frac{4}{5} \cdot \frac{1}{-8} = -\frac{4}{40} = -\frac{1}{10}$.

Finding the Slopes of Horizontal and Vertical Lines

So far, you’ve considered lines that run “uphill” or “downhill.” Their slopes may be steep or gradual, but they are always positive or negative numbers. But there are two other kinds of lines, horizontal and vertical. What is the slope of a flat line or level ground?

Of a wall or a vertical line?

Let’s consider a horizontal line on a graph. No matter which two points you choose on the line, they will always have the same y-coordinate. The equation for this line is $y = 3$. The equation can also be written as $y = (0)x + 3$.



Using the form $y = 0x + 3$, you can see that the slope is 0. You can also use the slope formula with two points on this horizontal line to calculate the slope of this horizontal line. Using $(-3, 3)$ as Point 1 and $(2, 3)$ as Point 2, you get:

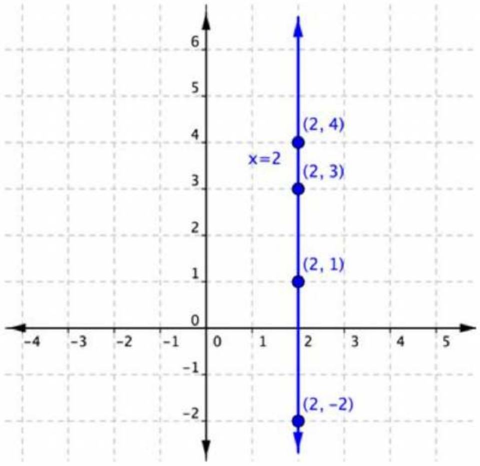
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{3-3}{2-(-3)} = \frac{0}{5} = 0$$

The slope of this horizontal line is 0.

Let’s consider any horizontal line. No matter which two points you choose on the line, they will always have the same y-coordinate. So, when you apply the slope formula, the numerator will always be 0. Zero divided by any non-zero number is 0, so the slope of any horizontal line is always 0.

The equation for the horizontal line $y = 3$ is telling you that no matter which two points you choose on this line, the y-coordinate will always be 3.

How about vertical lines? In their case, no matter which two points you choose, they will always have the same x-coordinate. The equation for this line is $x = 2$.



There is no way that this equation can be put in the slope-point form, as the coefficient of y is 0 $(x = 0y + 2)$.

So, what happens when you use the slope formula with two points on this vertical line to calculate the slope? Using $(2, 1)$ as Point 1 and $(2, 3)$ as Point 2, you get:

$$m=\frac{y_2-y_1}{x_2-x_1}$$
$$m=\frac{3-1}{2-2}=\frac{2}{0}$$

But division by zero has no meaning for the set of real numbers. Because of this fact, it is said that the slope of this vertical line is undefined. This is true for all vertical lines — they all have a slope that is undefined.

✓ Example

What is the slope of the line that contains the points (3, 2) and (-8, 2)?

Solution

$x_1=3$ $y_1=2$	$(3,2)=$ Point $1,(x_1,y_1)$
$x_2=-8$ $y_2=2$	$(-8,2)=$ Point $2,(x_2,y_2)$
$\frac{y_2-y_1}{x_2-x_1}$ $\frac{(2)-(2)}{(-8)-(3)}=\frac{0}{-11}=0$ $m=0$	Substitute the values into the slope formula and simplify.

The slope is 0, so the line is horizontal.

? Advanced Question

Which of the following points will lie on the line created by the points $(-3.75, 6.25)$ and $(-3.75, 0)$?

A. $(0,-3.75)$
B. $(-3.75,-14.55)$
C. $(3.75,-6.25)$
D. $(0,6.25)$

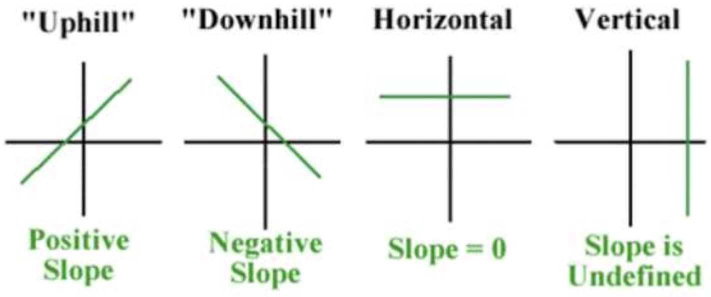
Answer

A. Incorrect. Notice that both points on the line have the same x-coordinate but different y-coordinates. That makes it a vertical line, so any other points on the line will have an x-coordinate of -3.75. The correct answer is $(-3.75,-14.55)$.
B. Correct. The points $(-3.75, 6.25)$ and $(-3.75, 0)$ form a vertical line, so any other point on that line will have to have an x-coordinate of -3.75.
C. Incorrect. Try drawing a quick sketch of the points $(-3.75, 6.25)$ and $(-3.75, 0)$. They form a vertical line, so any other points on the line will have an x-coordinate of -3.75. The correct answer is $(-3.75,-14.55)$.
D. Incorrect. Notice that both points on the line have the same x-coordinate but different y-coordinates. That makes it a vertical line, so any other points on the line will have an x-coordinate of -3.75. The correct answer is (-3.75,-14.55).

Summary

Slope describes the steepness of a line. The slope of any line remains constant along the line. The slope can also tell you information about the direction of the line on the coordinate plane. Slope can be calculated either by looking at the graph of a line or by using the coordinates of any two points on a line. There are two common formulas for slope: $\text{Slope}=\frac{\text{rise}}{\text{run}}$ and $m=\frac{y_2-y_1}{x_2-x_1}$ where $m=$ slope and (x_1,y_1) and (x_2,y_2) are two points on the line.

The images below summarize the slopes of different types of lines.


"Uphill"
"Downhill"
Horizontal
Vertical
Positive Slope
Negative Slope
Slope = 0
Slope is Undefined

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13.2.2: Writing the Equation of a Line

Learning Objectives

- Find the slope and the y-intercept, and write an equation of the line.
- Given the slope and a point on the line, write an equation of the line.
- Given two points, write the equation of a line.

Introduction

The concept of slope is important to understand because it is often used to measure the rate at which changes are taking place.

Slope-intercept Form

A linear equation can be expressed in the form $y=mx+b$. In this equation, x and y are coordinates of a point, m is the **slope**, and b is the y-coordinate of the **y-intercept**. Because this equation describes a line in terms of its slope and its y-intercept, this equation is called the **slope-intercept form**. When working with linear relationships, the slope-intercept form helps to translate between the graph of a line and the equation of a line.

Changing the value of m in a linear equation changes the slope of the line. If m is positive, the line slopes up as you move from left to right. If m is negative, the line slopes down as you move from left to right. If m is zero, the line is horizontal.

When m is positive, as m gets larger the upward slope of the line gets steeper. When the value of m gets close to zero, the upward slope flattens.

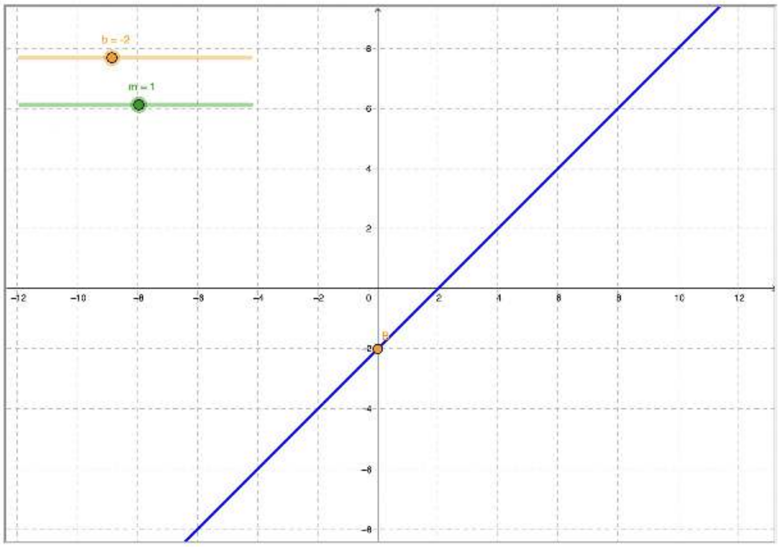
When m is negative, as the absolute value of m gets larger the downward slope of the line gets steeper. When the absolute value of m gets close to zero, the downward slope flattens.

Changing the value of b (the y-intercept) in a linear equation moves the line up and down on the coordinate plane. A positive y-intercept means the line crosses the y-axis above the origin. The larger the value of b, the higher up it crosses the y-axis. A negative y-intercept means that the line crosses below the origin. The larger the absolute value of b, the farther down it crosses the y-axis.

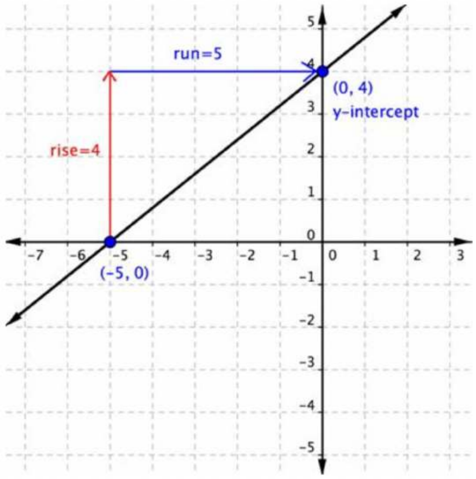
Simply by changing the values of m and b in a linear equation, you can define any line. That's how powerful and versatile the slope-intercept formula is.

Supplemental Interactive Activity

The graph below represents any line that can be written in slope-intercept form. It has two slider bars that can be manipulated. The bar labeled m lets you adjust the slope, or steepness, of the line. The bar labeled b changes the y-intercept. Try sliding each bar back and forth, and see how that affects the line.



Now that you understand the slope-intercept form, you can look at the graph of a line and write its equation just by identifying the slope and the y-intercept from the graph. Let’s try it with the line in this graph.



For this line, the slope is $\frac{4}{5}$, and the y-intercept is 4. If you put those values into the slope-intercept form, $y=mx+b$, you get the equation $y=\frac{4}{5}x+4$.

Example

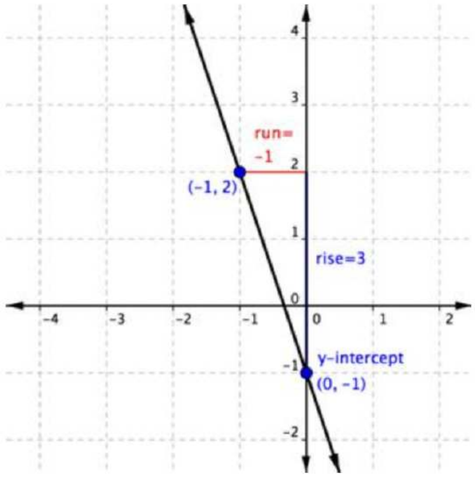
Write the equation of the line that has a slope of $\frac{1}{2}$ and a y-intercept of -5.

Solution

$y=\frac{1}{2}x+b$	Substitute the slope (m) into $y=mx+b$.
$y=\frac{1}{2}x+(-5)$ or $y=\frac{1}{2}-5$	Substitute the y-intercept (b) into the equation.

If you know the slope of a line and a point on the line, you can draw a graph. So, using an equation in the point-slope form, you can easily identify the slope and a point. Consider the equation $y=-3x-1$. You can tell from this equation that the y-intercept is at (0, -1). Start by plotting that point, (0, -1), on a graph.

You can also tell from the equation that the slope of this line is -3. So start at (0, -1) and count up 3 and over -1 (1 unit in the negative direction, left) and plot a second point. (You could also have gone down 3 and over 1.) Then draw a line through both points, and there it is, the graph of $y=-3x-1$.



? Exercise

What is the equation of a line that has a slope of -2 and goes through the point (0, 8)?

A. $y=-2x+8$
B. $y=8x-2$
C. $y=-2x+0$
D. $0=8x-2$

Answer

A. Correct. The point (0, 8) is also the y-intercept of the line, so $b=8$. The slope, m, is -2. Substituting these values into the equation $y=mx+b$, you find $y=-2x+8$.
B. Incorrect. Remember that in the slope-intercept form of an equation, $y=mx+b$, $m=$ slope and $b=$ the y-value of the y-intercept. The correct choice is $y=-2x+8$.
C. Incorrect. Remember that in the slope-intercept form of an equation, $y=mx+b$, $b=$ the y-value (not the x-value!) of the y-intercept. The correct choice is $y=-2x+8$.
D. Incorrect. Remember that in the slope-intercept form of an equation, $y=mx+b$, $m=$ slope and $b=$ the y-value of the y-intercept. The correct choice is $y=-2x+8$.

Slope and a Point on the Line

Using slope-intercept form to help write the equation of a line is possible when you know both the slope (m) and the y-intercept (b), but what if you know the slope and just any point on the line, not specifically the y-intercept? Can you still write the equation? The answer is *yes*, but you will need to put in a little more thought and work than you did previously.

Recall that a point is an (x,y) coordinate pair and that all points on the line will satisfy the linear equation. So, if you have a point on the line, it must be a solution to the equation. Although you don't know the exact equation yet, you know that you can express the line in slope-intercept form, $y=mx+b$.

You do know the slope (m), but you just don't know the value of the y-intercept (b). Since point (x,y) is a solution to the equation, you can substitute its coordinates for x and y in $y=mx+b$ and solve to find b!

This may seem a bit confusing with all the variables, but an example with an actual slope and a point will help to clarify.

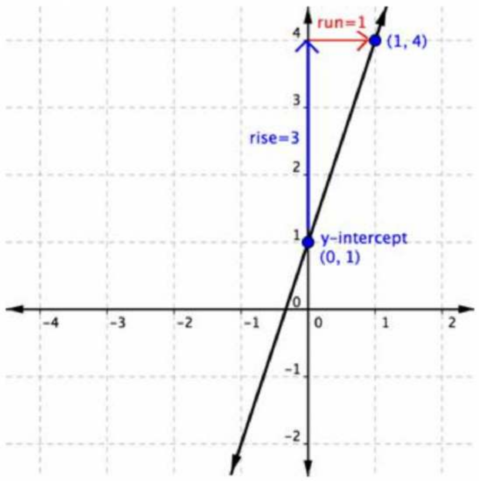
✓ Example

Write the equation of the line that has a slope of 3 and contains the point (1, 4).

Solution

$y=3x+b$	Substitute the slope (m) into $y=mx+b$.
$4=3(1)+b$	Substitute the point (1, 4) for x and y.
$4=3+b$ $1=b$	Solve for b.
$y=3x+1$	Rewrite $y=mx+b$ with $m=3$ and $b=1$.

To confirm our algebra, you can check by graphing the equation $y=3x+1$. The equation checks because when graphed, it passes through the point (1, 4).



Advanced Example

Write the equation of the line that has a slope of $-\frac{7}{8}$ and contains the point $(4,\frac{5}{4})$.

Solution

$y=mx+b$ $y=-\dfrac{7}{8}x+b$	Substitute the slope (m) into $y=mx+b$.
$\frac{5}{4}=-\frac{7}{8}(4)+b$	Substitute the point $(4,\frac{5}{4})$ for x and y.
$\frac{5}{4}=-\frac{28}{8}+b$ $\frac{5}{4}=-\frac{14}{4}+b$ $\frac{5}{4}+\frac{14}{4}=-\frac{14}{4}+\frac{14}{4}+b$ $\frac{19}{4}=b$	Solve for b.
$y=-\frac{7}{8}x+\frac{19}{4}$	Rewrite $y=mx+b$ with $m=-\frac{7}{8}$ and $b=\frac{19}{4}$.

Exercise

Write the slope-intercept form of the line with a slope of $\frac{2}{3}$ and which contains the point $(9,4)$.

A. $y=-2x+\frac{2}{3}$
B. $y=\frac{2}{3}x-2$
C. $y=\frac{2}{3}x+10$
D. $y=\frac{2}{3}x+\frac{19}{3}$

Answer

A. Incorrect. You solved correctly for the y-intercept but you have used the slope for the y-intercept and the y-intercept for the slope when plugging back into $y=mx+b$. The correct answer is $y=\frac{2}{3}x-2$.
B. Correct. The slope of this line is $\frac{2}{3}$ and the y-intercept is -2.
C. Incorrect. Check your algebra when you solved for b. To find b, substitute the known values for x, y, and m into the slope-intercept form. This gives you $4=\frac{2}{3}(9)+b$, which simplifies to $4=6+b$, so $b=-2$. The correct answer is $y=\frac{2}{3}x-2$.
D. Incorrect. When solving for b, you used the 9 as the y-coordinate and 4 as the x-coordinate. To find b, substitute the known values for x, y, and m into the slope-intercept form. This gives you $4=\frac{2}{3}(9)+b$, which simplifies to $4=6+b$, so $b=-2$. The correct answer is $y=\frac{2}{3}x-2$.

? Advanced Question

Write the slope-intercept form of the line with a slope of -0.6 and which contains the point (3.8, 7.25).

A. $y=-0.6x+3.8$
B. $y=-0.6x+4.97$
C. $y=3.8x+7.25$
D. $y=-0.6x+9.53$

Answer

A. Incorrect. The value for m is correct in this equation, but the value for b is not. Try substituting $(3.8, 7.25)$ into the equation $y=-0.6x+b$ and solve for b. The correct answer is $y=-0.6x+9.53$.
B. Incorrect. It looks like you subtracted $7.25-2.28$ to arrive at a $b\text{-value}$ of 4.97. Check the + and - signs in the problem again. The correct answer is $y=-0.6x+9.53$.
C. Incorrect. The value for m is incorrect; remember that m represents the slope. The correct answer is $y=-0.6x+9.53$.
D. Correct. Substituting $(3.8, 7.25)$ into the equation $y=-0.6x+b$, you found $b=9.53$. This means the equation of the line is $y=-0.6x+9.53$.

Two Points on the Line

Let’s suppose you don’t know either the slope or the y-intercept, but you do know the location of two points on the line. It is more challenging, but you can find the equation of the line that would pass through those two points. You will again use slope-intercept form to help you.

The slope of a linear equation is always the same, no matter which two points you use to find the slope. Since you have two points, you can use those points to find the slope (m). Now you have the slope and a point on the line! You can now substitute values for m, x, and y into the equation $y=mx+b$ and find b.

✓ Example

Write the equation of the line that passes through the points (2, 1) and (-1, -5).

Solution

$\frac{1-(-5)}{2-(-1)}=\frac{6}{3}=2$	Find the slope using the given points.
$y=2x+b$	Substitute the slope (m) into $y=mx+b$.
$1=2(2)+b$	Substitute the coordinates of either point for x and y- this example uses $(2, 1)$.
$1=4+b$ $-3=b$	Solve for b.
$y=2x+(-3)$, or $y=2x-3$	Rewrite $y=mx+b$ with $m=2$ and $b=-3$.

Notice that it doesn’t matter which point you use when you substitute and solve for b-you get the same result for b either way. In the example above, you substituted the coordinates of the point $(2, 1)$ in the equation $y=2x+b$. Let’s start with the same equation, $y=2x+b$, but substitute in $(-1, -5)$:

$$\begin{aligned} y&=2x+b\\ -5&=2(-1)+b\\ -5&=-2+b\\ -3&=b \end{aligned}$$

The final equation is the same: $y=2x-3$.

✓ Advanced Example

Write the equation of the line that passes through the points $(-4.6, 6.45)$ and $(1.15, 7.6)$.

Solution

$\frac{7.6-6.45}{1.15-(-4.6)} = \frac{1.15}{5.75} = 0.2$	Find the slope using the given points.
$y = 0.2x + b$	Substitute the slope (m) into $y = mx + b$.
$7.6 = 0.2(1.15) + b$ $7.6 = 0.23 + b$ $7.6 - 0.23 = 0.23 - 0.23 + b$ $7.37 = b$	Substitute either point for x and y. This example uses $(1.15, 7.6)$. Then solve for b.
$y = 0.2x + 7.37$	Rewrite $y = mx + b$ with $m = 0.2$ and $b = 7.37$.

The equation of the line that passes through the points $(-4.6, 6.45)$ and $(1.15, 7.6)$ is $y = 0.2x + 7.37$.

? Exercise

Write the slope-intercept form of the line that passes through $(5, 2)$ and $(-1, -10)$.

A. $y = 2x + 12$
B. $y = \frac{1}{2}x - \frac{7}{2}$
C. $y = 2x - 8$
D. $y = 2x + 1$

Answer

A. Incorrect. The slope $(m = 2)$ is correct, but check your algebra when you solved for b. Substitute $m = 2$, $x = 5$, and $y = 2$ into the equation $y = mx + b$ to find b: $2 = 2(5) + b$, so $2 = 10 + b$, so $b = -8$. The correct answer is $y = 2x - 8$.
B. Incorrect. It looks like you wrote the slope as $\frac{\text{run}}{\text{rise}}$ instead of $\frac{\text{rise}}{\text{run}}$. The correct answer is $y = 2x - 8$.
C. Correct. Find the slope $\frac{\text{rise}}{\text{run}} = \frac{-10-2}{-1-5} = \frac{-12}{-6} = 2$. Then use the slope and one of the points to find the value of b.
D. Incorrect. Make sure that you substitute the x-coordinate of the point for the x variable and the y-coordinate for the y variable of $y = mx + b$ when solving for b. The correct answer is $y = 2x - 8$.

? Advanced Question

Which of the following lines goes through the points $(\frac{3}{2}, -4)$ and $(\frac{7}{4}, -2)$?

A. $y = \frac{1}{2}x - \frac{3}{4}$
B. $y = 8x - 16$
C. $y = \frac{1}{2}x + 16$
D. $y = 8x - \frac{3}{4}$

Answer

A. Incorrect. To find the slope of the line, use the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. Then substitute the slope in for m in the equation $y = mx + b$ and solve for b. The answer is $y = 8x - 16$.
B. Correct. The slope of the line that these points lie upon is 8, and substituting either of the points into the equation $y = 8x + b$ yields $b = -16$.
C. Incorrect. Try calculating the slope again using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. When you find the slope, use it (and one of the given points) to find the value of b. The correct answer is $y = 8x - 16$.
D. Incorrect. You have found the correct slope, but the value of b is incorrect. Try substituting either $(\frac{3}{2}, -4)$ or $(\frac{7}{4}, -2)$ into the equation $y = 8x + b$ to solve for b. The correct answer is $y = 8x - 16$.

Summary

The slope-intercept form of a linear equation is written as $y=mx+b$, where m is the slope and b is the value of y at the y-intercept, which can be written as $(0,b)$. When you know the slope and the y-intercept of a line, you can use the slope-intercept form to immediately write the equation of that line. The slope-intercept form can also help you to write the equation of a line when you know the slope and a point on the line or when you know two points on the line.

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13.2.3: Parallel and Perpendicular Lines

Learning Objectives

- Find the slope of a line that is parallel or perpendicular to an existing line.
- Given a point and a line perpendicular to or parallel to the unknown line, write the equation of the unknown line.

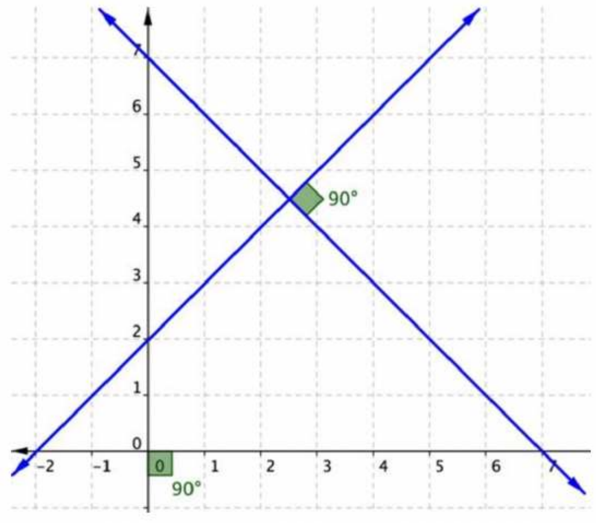
Introduction

When you graph two or more linear equations in a coordinate plane, they generally cross at a point. However, when two lines in a coordinate plane never cross, they are called **parallel lines**. You will also look at the case where two lines in a coordinate plane cross at a right angle. These are called **perpendicular lines**. The slopes of the graphs in each of these cases have a special relationship to each other.

Exploring Parallel and Perpendicular Lines

Parallel lines are two or more lines in a plane that never intersect. Examples of parallel lines are all around us, such as the opposite sides of a rectangular picture frame and the shelves of a bookcase.

Perpendicular lines are two or more lines that intersect at a 90-degree angle, like the two lines drawn on this graph. These 90-degree angles are also known as right angles.

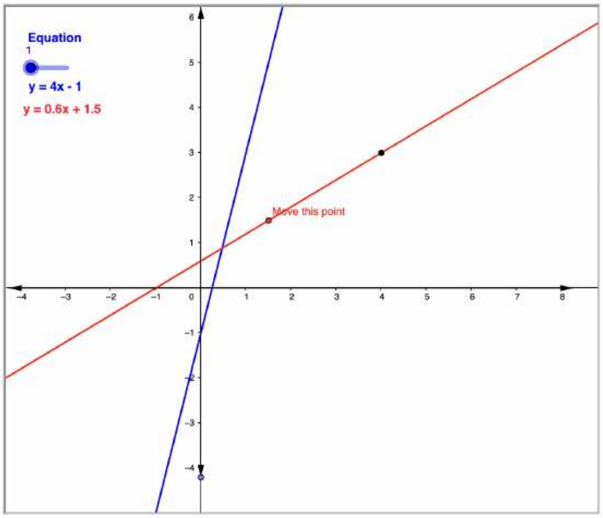


Perpendicular lines are also everywhere, not just on graph paper but also in the world around us, from the crossing pattern of roads at an intersection to the colored lines of a plaid shirt.

Supplemental Interactive Activity

Explore lines in the interactive diagram below.

- Click and drag the dot in the “Equation” slider to choose one of five example equations. The equation will be graphed in blue.
- Then, click and drag the dot on the red line to make the line parallel or perpendicular to the blue line. (Be sure to move your cursor slowly.) When the lines are parallel or perpendicular, text will appear to let you know you’ve done it!
- Look at the slopes of the two parallel lines. What do you notice? Look at the slopes of two perpendicular lines. What do you notice?
- Choose another equation and try again.
- As you try more equations, look for the relationship between the slopes of parallel lines, and the slopes for perpendicular lines. For the last equation you try, can you predict what the slopes of the parallel and perpendicular lines should be?



Note the following:

Parallel Lines

Two non-vertical lines in a plane are parallel if they have both:

- the same slope
- different y-intercepts

Any two vertical lines in a plane are parallel.

Example

Find the slope of a line parallel to the line $y=-3x+4$.

Solution

The given line is written in $y=mx+b$ form, with $m=-3$ and $b=4$. The slope is -3	Identify the slope of the given line.
The slope of the parallel line is -3.	A line parallel to the given line has the same slope.

Example

Determine whether the lines $y=6x+5$ and $y=6x-1$ are parallel.

Solution

The given lines are written in $y=mx+b$ form, with $m=6$ for the first line and $m=6$ for the second line. The slope of both lines is 6.	Identify the slopes of the given lines.
The first line has a y-intercept at $(0,5)$, and the second line has a y-intercept at $(0,-1)$. They are not the same line.	Look at b, the y-value of the y-intercept, to see if the lines are perhaps exactly the same line, in which case we don't say they are parallel.
The lines are parallel.	The slopes of the lines are the same and they have different y-intercepts, so they are not the same line and they are parallel.

Perpendicular Lines

Two non-vertical lines are perpendicular if the slope of one is the negative reciprocal of the slope of the other. If the slope of the first equation is 4, then the slope of the second equation will need to be $-\frac{1}{4}$ for the lines to be perpendicular.

You can also check the two slopes to see if the lines are perpendicular by multiplying the two slopes together. If they are perpendicular, the product of the slopes will be -1. For example, $4 \cdot -\frac{1}{4} = \frac{4}{1} \cdot -\frac{1}{4} = -1$.

✓ Example

Find the slope of a line perpendicular to the line $y = 2x - 6$.

Solution

The given line is written in $y = mx + b$ form, with $m = 2$ and $b = -6$. The slope is 2.	Identify the slope of the given line.
The slope of the perpendicular line is $-\frac{1}{2}$.	To find the slope of a perpendicular line, find the reciprocal, $\frac{1}{2}$, and then find the opposite of this reciprocal $-\frac{1}{2}$.

Note that the product $2\left(-\frac{1}{2}\right) = \frac{2}{1}\left(-\frac{1}{2}\right) = -1$, so this means the slopes are perpendicular.

In the case where one of the lines is vertical, the slope of that line is undefined and it is not possible to calculate the product with an undefined number. When one line is vertical, the line perpendicular to it will be horizontal, having a slope of zero $(m = 0)$.

✓ Example

Determine whether the lines $y = -8x + 5$ and $y = \frac{1}{8}x - 1$ are parallel, perpendicular, or neither.

Solution

The given lines are written in $y = mx + b$ form, with $m = -8$ for the first line and $m = \frac{1}{8}$ for the second line.	Identify the slopes of the given lines.
$-8 \neq \frac{1}{8}$, so the lines are not parallel. The opposite reciprocal of -8 is $\frac{1}{8}$, so the lines are perpendicular.	Determine if the slopes are the same or if they are opposite reciprocals.
The lines are perpendicular.	The slopes of the lines are opposite reciprocals, so the lines are perpendicular.

? Exercise

Which of the following lines are perpendicular to the line $y = 7x + \frac{2}{3}$?

$y = -\frac{1}{7}x$

$y = -\frac{1}{7}x - 12$

$y = \frac{1}{7}x - \frac{3}{2}$

$y = -7x + \frac{3}{2}$

A. $y = -\frac{1}{7}x$ and $y = -\frac{1}{7}x - 12$
B. $y = -\frac{1}{7}x - 12$ and $y = \frac{1}{7}x - \frac{3}{2}$
C. $y = -7x + \frac{3}{2}$
D. All of the lines are perpendicular.

Answer

A. Correct. These lines both have a slope of $-\frac{1}{7}$, which is the opposite reciprocal of the slope of 7 in the original equation. Both of these lines are perpendicular to the original line.
B. Incorrect. Perpendicular lines have slopes that are opposite of the reciprocal of the other. Only $-\frac{1}{7}$ is the opposite of the reciprocal of 7. The correct answer is $y=-\frac{1}{7}x$ and $y=-\frac{1}{7}x-12$.
C. Incorrect. Perpendicular lines have slopes that are the opposite of the reciprocal of the other. -7 is the opposite of 7, but it is not the opposite of the reciprocal of 7. The correct answer is $y=-\frac{1}{7}x$ and $y=-\frac{1}{7}x-12$.
D. Incorrect. Only the lines that have a slope of $-\frac{1}{7}$, which is the opposite of the reciprocal of 7, can be perpendicular to the original equation. The correct answer is $y=-\frac{1}{7}x$ and $y=-\frac{1}{7}x-12$.

Writing Parallel and Perpendicular Lines

The relationships between slopes of parallel and perpendicular lines can be used to write equations of parallel and perpendicular lines.

Let's start with an example involving parallel lines.

✓ Example

Write the equation of a line that is parallel to the line $x-y=5$ and goes through the point $(-2,1)$.

Solution

$\begin{aligned} x-y&=5\\ -y&=-x+5\\ y&=x-5 \end{aligned}$	Rewrite the line you want to be parallel to into the $y=mx+b$ form, if needed.
In the equation above, $m=1$ and $b=-5$. Since $m=1$, the slope is 1.	Identify the slope of the given line.
The slope of the parallel line is 1.	To find the slope of a parallel line, use the same slope.
$\begin{aligned} y&=mx+b\\ 1&=1(-2)+b \end{aligned}$	Use the method for writing an equation from the slope and a point on the line. Substitute 1 for m, and the point $(-2,1)$ for x and y.
$\begin{aligned} 1&=-2+b\\ 3&=b \end{aligned}$	Solve for b.
$y=x+3$	Write the equation using the new slope for m and the b you just found.

When you are working with perpendicular lines, you will usually be given one of the lines and an additional point.

✓ Example

Write the equation of a line that contains the point $(1,5)$ and is perpendicular to the line $y=2x-6$.

Solution

The given line is written in $y=mx+b$ form, with $m=2$ and $b=-6$. The slope is 2.	Identify the slope of the line you want to be perpendicular to.
The slope of the perpendicular line is $-\frac{1}{2}$.	To find the slope of a perpendicular line, find the reciprocal, $\frac{1}{2}$, then the opposite, $-\frac{1}{2}$.
$\begin{aligned} y&=mx+b\\ 5&=-\frac{1}{2}(1)+b \end{aligned}$	Use the method for writing an equation from the slope and a point on the line. Substitute $-\frac{1}{2}$ for m, and the point $(1,5)$ for x and y.
$\begin{aligned} 5&=-\frac{1}{2}+b\\ \frac{11}{2}&=b \end{aligned}$	Solve for b.

$y=-\frac{1}{2}x+\frac{11}{2}$	Write the equation using the new slope for m and the b you just found.

Exercise

Which of the following is the equation of the line that is parallel to $y=-2x-14$ and goes through the point $(-3,1)$?

A. $y=-2x+1$
B. $y=\frac{1}{2}x+2\frac{1}{2}$
C. $y=\frac{1}{2}x+1$
D. $y=-2x-5$

Answer

A. Incorrect. Checking $(-3,1)$ into the equation gives $1=-2(-3)+1$, which is $1=6+1$. Since $6+1=7$, not 1, this line cannot go through the point $(-3,1)$. The correct answer is $y=-2x-5$.
B. Incorrect. Parallel lines have the *same* slope, so this equation should also have a slope of -2. The correct answer is $y=-2x-5$.
C. Incorrect. Parallel lines have the *same* slope, so this equation should also have a slope of -2. The correct answer is $y=-2x-5$.
D. Correct. The line has the same slope as the original, so they are parallel. Checking $(-3,1)$ into the equation gives $1=-2(-3)-5$, or $1=6-5$, which is true. The line is parallel and goes through the point $(-3,1)$.

Example

Write the equation of a line that is parallel to the line $y=4$.

Solution

$y=4$ $y=0x+4$	Rewrite the line into $y=mx+b$ form, if needed. You may notice without doing this that $y=4$ is a horizontal line 4 units above the x-axis. Because it is horizontal, you know its slope is zero.
In the equation above, $m=0$ and $b=4$. Since $m=0$, the slope is 0. This is a horizontal line.	Identify the slope of the given line.
The slope of the parallel line is also 0.	To find the slope of a parallel line, use the same slope.
$y=10$	Since the parallel line will be a horizontal line, its form is $y=a$ constant. Pick a constant to find a parallel line.
$y=10$	This line is parallel to $y=4$ and intersects the y-axis at $(0,10)$.

Summary

When lines in a plane are parallel (that is, they never cross), they have the same slope. When lines are perpendicular (that is, they cross at a 90° angle), their slopes are opposite reciprocals of each other. The product of their slopes will be -1, except in the case where one of the lines is vertical, causing its slope to be undefined. You can use these relationships to find an equation of a line that goes through a particular point and is parallel or perpendicular to another line.

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13.2.4: Graphing Linear Inequalities

Learning Objectives

- Determine whether an ordered pair is a solution to an inequality.
- Graph an inequality in two variables.

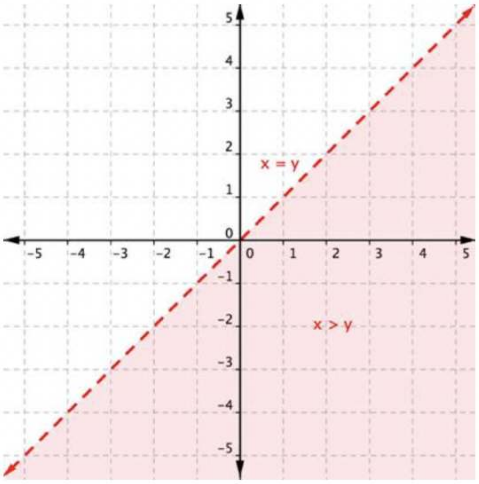
Introduction

Linear inequalities can be graphed on a **coordinate plane**. The solutions for a linear inequality are in a region of the coordinate plane. A **boundary line**, which is the related linear equation, serves as the boundary for the region. You can use a visual representation to figure out what values make the inequality true—and also which ones make it false. Let’s study inequalities by returning to the coordinate plane.

Linear Inequalities as Regions

Linear inequalities are different than linear equations, although you can apply what you know about equations to help you understand inequalities. Inequalities and equations are both math statements that compare two values. Equations use the symbol =; inequalities will be represented by the symbols $<, \leq, >, \text{ and } \geq$.

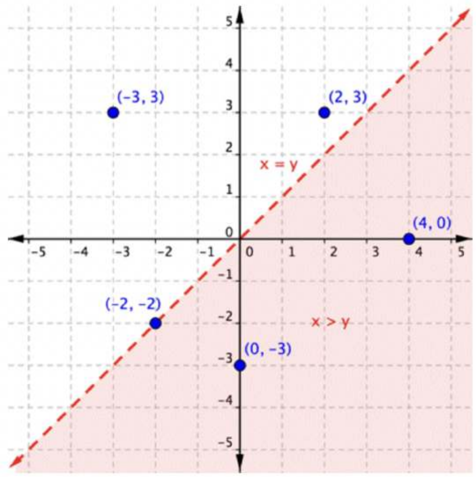
One way to visualize two-variable inequalities is to plot them on a coordinate plane. Here is what the inequality $x > y$ looks like. The solution is a region, which is shaded.



There are a few things to understand here. First, the dashed red boundary line is the graph of the related linear equation $x = y$. Next, the light red region that is to the right of the line (excluding the line $x = y$) represents the entire set of solutions for the inequality $x > y$. Remember how all points on a line are solutions to the linear equation of the line? Well, all points in a region are solutions to the **linear inequality** representing that region.

Let’s think about it for a moment—if $x > y$, then a graph of $x > y$ will show all ordered pairs (x, y) for which the x-coordinate is greater than the y-coordinate.

The graph below shows the region $x > y$ as well as some ordered pairs on the coordinate plane. Look at each ordered pair. Is the x-coordinate greater than the y-coordinate? Does the ordered pair sit inside or outside of the shaded region?

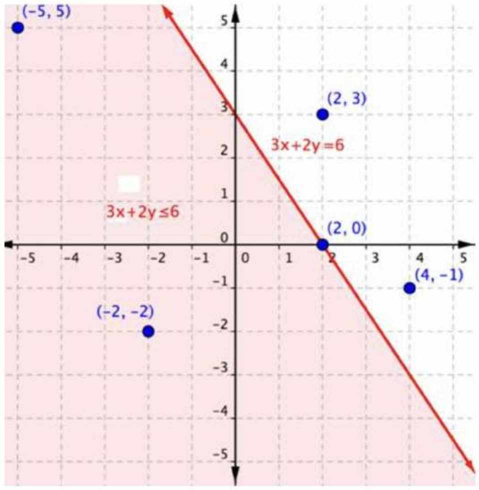


The ordered pairs $(4,0)$ and $(0,-3)$ lie inside the shaded region. In these ordered pairs, the x-coordinate is larger than the y-coordinate. These ordered pairs are in the solution set of the equation $x>y$.

The ordered pairs $(-3,3)$ and $(2,3)$ are outside of the shaded area. In these ordered pairs, the x-coordinate is *smaller* than the y-coordinate, so they are not included in the set of solutions for the inequality.

The ordered pair $(-2,-2)$ is on the boundary line. It is not a solution as -2 is not greater than -2. However, had the inequality been $x \geq y$ (read as " x is greater than or equal to y"), then $(-2,-2)$ would have been included (and the line would have been represented by a solid line, not a dashed line).

Let's take a look at one more example: the inequality $3x+2y \leq 6$. The graph below shows the region of values that makes this inequality true (shaded red), the boundary line $3x+2y=6$, as well as a handful of ordered pairs. The boundary line is solid this time, because points on the boundary line $3x+2y=6$ will make the inequality $3x+2y \leq 6$ true.



As you did with the previous example, you can substitute the x- and y-values in each of the (x,y) ordered pairs into the inequality to find solutions. While you may have been able to do this in your head for the inequality $x>y$, sometimes making a table of values makes sense for more complicated inequalities.

Ordered Pair	**Makes the inequality $3x+2y \leq 6$ a true statement**	**Makes the inequality $3x+2y \leq 6$ a false statement**
$(-5,5)$	$3(-5)+2(5) \leq 6$ $-15+10 \leq 6$ $-5 \leq 6$	

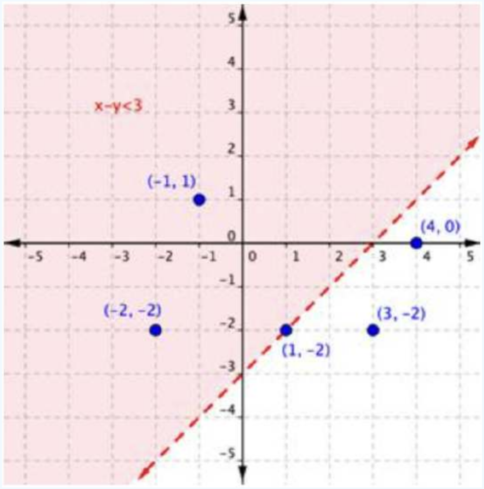
$(-2,-2)$	$3(-2)+2(-2) \leq 6$ $-6+(-4) \leq 6$ $-10 \leq 6$	
$(2,3)$		$3(2)+2(3) \leq 6$ $6+6 \leq 6$ $12 \leq 6$
$(2,0)$	$3(2)+2(0) \leq 6$ $6+0 \leq 6$ $6 \leq 6$	
$(4,-1)$		$3(4)+2(-1) \leq 6$ $12+(-2) \leq 6$ $10 \leq 6$

If substituting (x, y) into the inequality yields a true statement, then the ordered pair is a solution to the inequality, and the point will be plotted within the shaded region or the point will be part of a solid boundary line. A false statement means that the ordered pair is not a solution, and the point will graph outside the shaded region, or the point will be part of a dotted boundary line.

✓ Example

Use the graph to determine which ordered pairs plotted below are solutions of the inequality $x-y<3$.

Solution



	Solutions will be located in the shaded region. Since this is a "less than" problem, ordered pairs on the boundary line are not included in the solution set.
$(-1,1)$ $(-2,-2)$	These values are located in the shaded region, so are solutions. (When substituted into the inequality $x-y<3$, they produce true statements.)
$(1,-2)$ $(3,-2)$ $(4,0)$	These values are not located in the shaded region, so are not solutions. (When substituted into the inequality $x-y<3$, they produce false statements.)

$(-1,1)$,
$(-2,-2)$

Example

Is $(2,-3)$ a solution of the inequality $y<-3x+1$?

Solution

$y<-3x+1$	If $(2,-3)$ is a solution, then it will yield a true statement when substituted into the inequality $y<-3x+1$.
$-3<-3(2)+1$	Substitute $x=2$ and $y=-3$ into inequality.
$-3<-6+1$	Evaluate.
$-3<-5$	This statement is **not** true, so the ordered pair $(2,-3)$ is **not** a solution.

$(2,-3)$ is not a solution.

Exercise

Which ordered pair is a solution of the inequality $2y-5x<2$?

A. $(-5,1)$
B. $(-3,3)$
C. $(1,5)$
D. $(3,3)$

Answer

A. Incorrect. Substituting $(-5,1)$ into $2y-5x<2$, you find $2(1)-5(-5)<2$, or $2+25<2$.27 is not smaller than 2, so this cannot be correct. The correct answer is $(3,3)$.
B. Incorrect. Substituting $(-3,3)$ into $2y-5x<2$, you find $2(3)-5(-3)<2$, or $6+15<2$. 21 is not smaller than 2, so this cannot be correct. The correct answer is $(3,3)$.
C. Incorrect. Substituting $(1,5)$ into $2y-5x<2$, you find $2(5)-5(1)<2$, or $10-5<2$. 5 is not smaller than 2, so this cannot be correct. The correct answer is $(3,3)$.
D. Correct. Substituting $(3,3)$ into $2y-5x<2$, you find $2(3)-5(3)<2$, or $6-15<2$. This is a true statement, so it is a solution to the inequality.

Graphing Inequalities

So how do you get from the algebraic form of an inequality, like $y>3x+1$, to a graph of that inequality? Plotting inequalities is fairly straightforward if you follow a couple steps.

Graphing Inequalities

To graph an inequality:

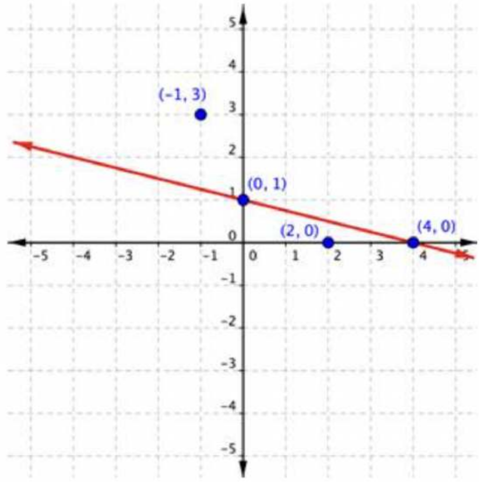
- Graph the related boundary line. Replace the <, >, ≤, or ≥ sign in the inequality with = to find the equation of the boundary line.
- Identify at least one ordered pair on either side of the boundary line and substitute those (x,y) values into the inequality. Shade the region that contains the ordered pairs that make the inequality a true statement.
- If points on the boundary line are solutions, then use a solid line for drawing the boundary line. This will happen for ≤ or ≥ inequalities.

Let’s graph the inequality $x+4y\leq 4$

To graph the boundary line, find at least two values that lie on the line $x+4y=4$. You can use the x- and y-intercepts for this equation by substituting 0 in for x first and finding the value of y; then substitute 0 in for y and find x.

x	y
0	1
4	0

Plot the points $(0,1)$ and $(4,0)$, and draw a line through these two points for the boundary line. The line is solid because ≤ means "less than or equal to," so all ordered pairs along the line are included in the solution set.



The next step is to find the region that contains the solutions. Is it above or below the boundary line? To identify the region where the inequality holds true, you can test a couple of ordered pairs, one on each side of the boundary line.

If you substitute $(-1,3)$ into $x+4y\leq 4$:

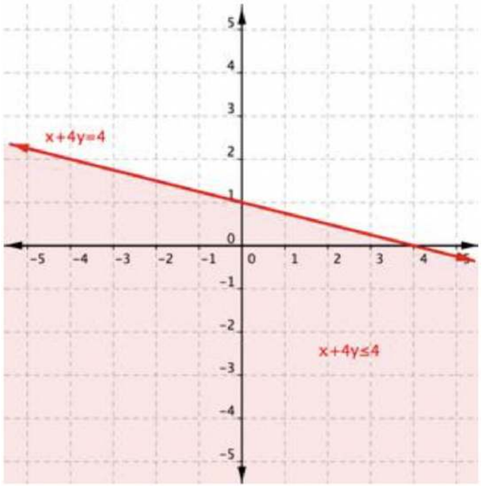
$$\begin{aligned}-1+4(3)&\leq 4\\ -1+12&\leq 4\\ 11&\leq 4\end{aligned}$$

This is a false statement, since 11 is not less than or equal to 4.

On the other hand, if you substitute $(2,0)$ into $x+4y\leq 4$:

$$\begin{aligned}2+4(0)&\leq 4\\ 2+0&\leq 4\\ 2&\leq 4\end{aligned}$$

This is true! The region that includes $(2,0)$ should be shaded, as this is the region of solutions.

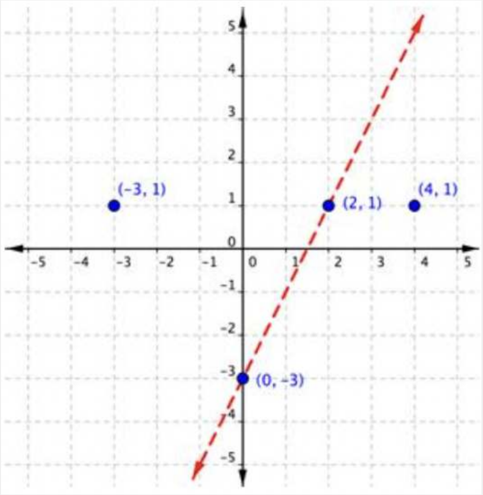


And there you have it—the graph of the set of solutions for $x+4y\leq 4$.

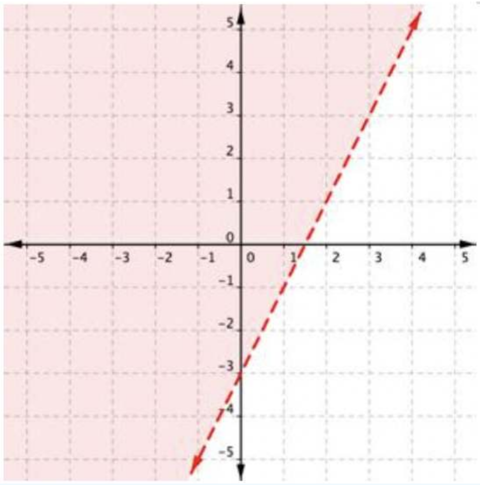
Example

Graph the inequality $2y > 4x - 6$.

Solution

<table>
<tr><td>$2y > 4x - 6$
$\frac{2y}{x} > \frac{4x}{2} - \frac{6}{2}$
$y > 2x - 3$
$y = 2x - 3$</td><td>Solve for y</td></tr>
<tr><td><table><tr><td>x</td><td>y</td></tr><tr><td>0</td><td>-3</td></tr><tr><td>2</td><td>1</td></tr></table></td><td>Create a table of values to find two points on the line $y = 2x - 3$, or graph it based on the slope-intercept method, the b value of the y-intercept is -3 and the slope is 2.</td></tr>
<tr><td></td><td>Plot the points, and graph the line. The line is dotted because the sign in the inequality is >, not ≥ and therefore points on the line are not solutions to the inequality.</td></tr>
<tr><td> </td><td></td></tr>
<tr><td>$2y > 4x - 6$
$\text{Test } 1 : (-3, 1)$
$2(1) > 4(-3) - 6$
$2 > -12 - 6$
$2 > -18 \text{ True!}$</td><td>Find an ordered pair on either side of the boundary line. Insert the x- and y-values into the inequality $2y > 4x - 6$ and see which ordered pair results in a true statement.</td></tr>
<tr><td>$\text{Test } 2 : (4, 1)$
$2(1) > 4(4) - 6$
$2 > 16 - 6$
$2 > 10 \text{ False!}$</td><td>Since $(-3, 1)$ results in a true statement, the region that includes $(-3, 1)$ should be shaded.</td></tr>
</table>

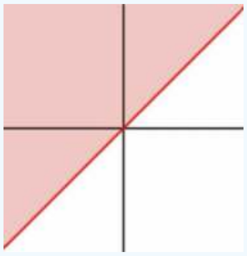
The graph of the inequality $2y > 4x - 6$ is:

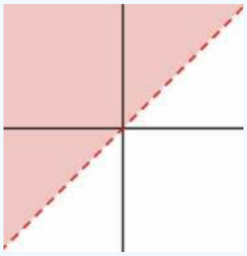


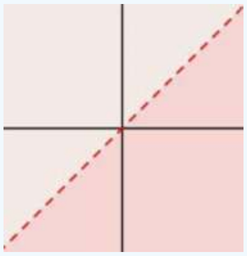
A quick note about the problem above. You can use the points $(0, -3)$ and $(2, 1)$ to graph the boundary line, but that these points are not included in the region of solutions, since the region does not include the boundary line!

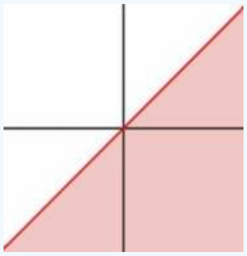
? Exercise

When plotted on a coordinate plane, what does the graph of $y \geq x$ look like?

A. 

B. 

C. 

D. 

Answer

A.
B.
C.
D.

Summary

When inequalities are graphed on a coordinate plane, the solutions are located in a region of the coordinate plane, which is represented as a shaded area on the plane. The boundary line for the inequality is drawn as a solid line if the points on the line itself do satisfy the inequality, as in the cases of ≤ and ≥. It is drawn as a dashed line if the points on the line do not satisfy the inequality, as in the cases of < and >. You can tell which region to shade by testing some points in the inequality. Using a coordinate plane is especially helpful for visualizing the region of solutions for inequalities with two variables.

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CHAPTER OVERVIEW

14: Systems of Equations and Inequalities

- 14.1: Graphing Systems of Equations and Inequalities
  - 14.1.1: Graphing Systems of Linear Equations
  - 14.1.2: Graphing Systems of Inequalities
- 14.2: Algebraic Methods to Solve Systems of Equations
  - 14.2.1: The Substitution Method
  - 14.2.2: The Elimination Method
- 14.3: Systems of Equations in Three or More Variables
  - 14.3.1: Solving Systems of Three Variables

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SECTION OVERVIEW

14.1: Graphing Systems of Equations and Inequalities

Topic hierarchy

14.1.1: Graphing Systems of Linear Equations

14.1.2: Graphing Systems of Inequalities

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14.1.1: Graphing Systems of Linear Equations

Learning Objectives

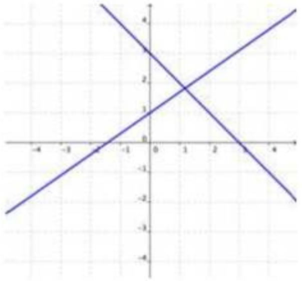
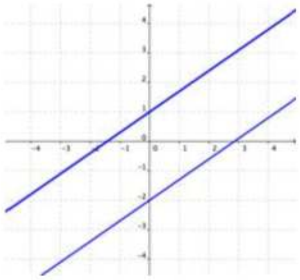
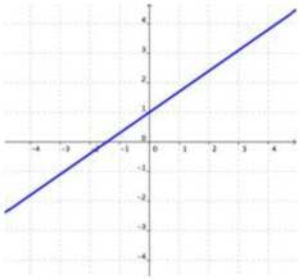
- Solve a system of linear equations by graphing.
- Determine whether a system of linear equations is consistent or inconsistent.
- Determine whether a system of linear equations is dependent or independent.
- Determine whether an ordered pair is a solution of a system of equations.
- Solve application problems by graphing a system of equations.

Introduction

Recall that a linear equation graphs as a line, which indicates that all of the points on the line are solutions to that linear equation. There are an infinite number of solutions. If you have a **system of linear equations**, the solution for the system is the value that makes all of the equations true. For two variables and two equations, this is the point where the two graphs intersect. The coordinates of this point will be the solution for the two variables in the two equations.

Systems of Equations

The solution for a system of equations is the value or values that are true for *all* equations in the system. The graphs of equations within a system can tell you how many solutions exist for that system. Look at the images below. Each shows two lines that make up a system of equations.

One Solution	**No Solutions**	**Infinite Solutions**
		
If the graphs of the equations intersect, then there is one solution that is true for both equations.	If the graphs of the equations do not intersect (for example, if they are parallel), then there are no solutions that are true for both equations.	If the graphs of the equations are the same, then there are an infinite number of solutions that are true for both equations.

When the lines intersect, the point of intersection is the only point that the two graphs have in common, so the coordinates of that point are the solution for the two variables used in the equations. When the lines are parallel, there are no solutions. Sometimes the two equations will graph as the same line, in which case we have an infinite number of solutions.

Special terms are sometimes used to describe these kinds of systems.

The following terms refer to how many solutions the system has.

- When a system has one solution (the graphs of the equations intersect once), the system is a **consistent system of linear equations** and the equations are independent.
- When a system has no solution (the graphs of the equations don’t intersect at all), the system is an **inconsistent system of linear equations** and the equations are independent.
- If the lines are the same (the graphs intersect at all points), the system is a consistent system of linear equations and the equations are dependent. That is, any solution of one equation must also be a solution of the other, so the equations *depend* on each other.

The following terms refer to whether the system has any solutions at all.

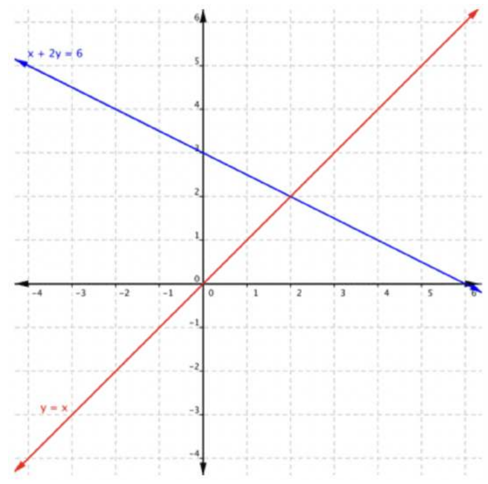
- The system is a consistent system of linear equations when it has solutions.
- The system is an inconsistent system of linear equations when it has no solutions.

> We can summarize this as follows:
>
> - A system with one or more solutions is consistent.
> - A system with no solutions is inconsistent.
> - If the lines are different, the equations are **independent linear equations**.
> - If the lines are the same, the equations are **dependent linear equations**.

✓ Example

Using the graph of $y = x$ and $x + 2y = 6$, shown below, determine how many solutions the system has. Then classify the system as consistent or inconsistent and the equations as dependent or independent.

Solution



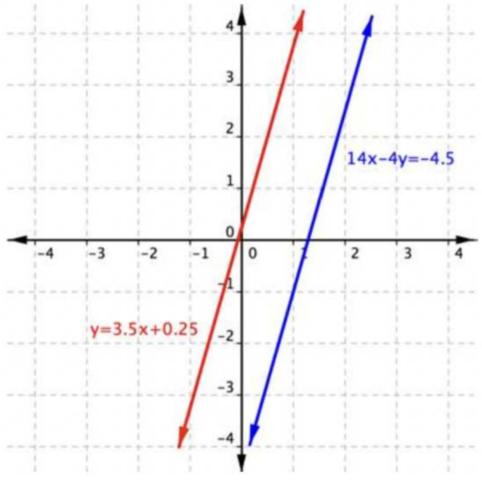
The lines intersect at one point, so the two lines have only one point in common. There is only one solution to the system.
Because the lines are not the same, the equations are independent.
Because there is only one solution, this system is consistent.

The system is consistent and the equations are independent.

✓ Advanced Example

Using the graph of $y = 3.5x + 0.25$ and $14x - 4y = -4.5$, shown below, determine how many solutions the system has. Then classify the system as consistent or inconsistent and the equations as dependent or independent.

Solution



The lines are parallel, meaning they do not intersect. There are no solutions to the system.
Because the lines are not the same, the equations are independent.
There are no solutions. Therefore, this system is inconsistent.

The system is inconsistent and the equations are independent.

? Advanced Question

Which of the following represents *dependent* equations and *consistent* systems?

A. $y=\frac{1}{2}x+5$
$-\frac{1}{2}x+y=-2$
B. $y=\frac{1}{5}x+5$
$\frac{1}{5}x+y=-\frac{3}{2}$
C. $y=-3x-1$
$3x+y=-1$
D. $y=3x-1$
$8x+y=-1$

Answer

A. $y=\frac{1}{2}x+5$
$-\frac{1}{2}x+y=-2$

Incorrect. The two lines in this system have the same slope, but different values for b. This means the lines are parallel. The lines don't intersect, so there are no solutions and the system is inconsistent. Because the lines are not the same, the equations are independent. The correct answer is C.

B. $y=\frac{1}{5}x+5$
$\frac{1}{5}x+y=-\frac{3}{2}$

Incorrect. The two lines in this system have different slopes and different values for b. This means the lines intersect at one point. Since there is a solution, this system is consistent. And because the lines are not the same, the equations are independent. The correct answer is C.

C. $y=-3x-1$
$3x+y=-1$

Correct. The two lines in this system are the same; $3x+y=-1$ can be rewritten as $y=-3x-1$. Since there are many solutions, this system is consistent. The lines are identical so the equations are dependent.

D. $y=3x-1$
$8x+y=-1$

Incorrect. The two lines in this system have different slopes and the same value for b. This means the lines intersect at one point—the *y*-intercept. Recall that intersecting lines have one solution and therefore the system is consistent. Because the lines are not the same, the equations are independent. The correct answer is C.

Verifying a Solution

From the graph above, you can see that there is one solution to the system $y=x$ and $x+2y=6$. The solution appears to be $(2,2)$. However, you must verify an answer that you read from a graph to be sure that it's not really $(2.001,2.001)$ or $(1.9943,1.9943)$.

One way of verifying that the point does exist on both lines is to substitute the x- and y--values of the ordered pair into the equation of each line. If the substitution results in a true statement, then you have the correct solution!

✓ Example

Is $(2,2)$ a solution of the system $y=x$ and $x+2y=6$?

Solution

$y=x$ $2=2$ TRUE $(2,2)$ is a solution of $y=x$.	$x+2y=6$ $2+2(2)=6$ $2+4=6$ $6=6$ TRUE $(2,2)$ is a solution of $x+2y=6$	Since the solution of the system must be a solution to *all* the equations in the system, check the point in each equation. Substitute 2 for x and 2 for y in each equation.
$(2,2)$ is a solution to the system.		Since $(2,2)$ is a solution to each of the equations in the system, $(2,2)$ is a solution to the system.

Example

Is $(3,9)$ a solution of the system $y=3x$ and $2x-y=6$?

Solution

$y=3x$ $9=3(3)$ TRUE $(3,9)$ is a solution of $y=3x$.	$2x-y=6$ $2(3)-9=6$ $6-9=6$ $-3=6$ FALSE $(3,9)$ is *not* a solution of $2x-y=6$.	Since the solution of the system must be a solution to *all* the equations in the system, check the point in each equation. Substitute 3 for x and 9 for y in each equation.
$(3,9)$ is not a solution to the system.		Since $(3,9)$ is not a solution of one of the equations in the system, it cannot be a solution to the system.

Example

Is $(-2,4)$ a solution of the system $y=2x$ and $3x+2y=1$?

Solution

$y=2x$ $4=2(-2)$ $4=-4$ FALSE $(-2,4)$ is not a solution of $y=2x$.	$3x+2y=1$ $3(-2)+2(4)=1$ $-6+8=1$ $2=1$ FALSE $(-2,4)$ is not a solution of $3x+2y=1$.	Since the solution of the system must be a solution to *all* the equations in the system, check the point in each equation. Substitute -2 for x and 4 for y in each equation.
$(-2,4)$ is not a solution to the system.		Since $(-2,4)$ is not a solution to either of the equations in the system, $(-2,4)$ is not a solution of the system.

Remember that, in order to be a solution to the system of equations, the value of the point must be a solution for both equations. Once you find one equation for which the point is false, you have determined that it is not a solution to the system.

Exercise

Which of the following statements is true for the system $2x-y=-3$ and $y=4x-1$?

A. $(2,7)$ is a solution of one equation but not the other, so it is a solution of the system
B. $(2,7)$ is a solution of one equation but not the other, so it is *not* a solution of the system
C. (\ (2,7)\) is a solution of both equations, so it is a solution of the system
D. (\ (2,7)\) is *not* a solution of either equation, so it is *not* a solution to the system

Answer

A. Incorrect. If the point is a solution of one equation but not the other, then it is *not* a solution of the system. In fact, the point $(2,7)$ is a solution of both equations, so it is a solution of the system. The two lines are not identical, so it is the only solution.
B. Incorrect. The point $(2,7)$ is a solution of both equations, so it is a solution of the system. The two lines are not identical, so it is the only solution.
C. Correct. Substituting 2 for x and 7 for y gives true statements in both equations, so the point is a solution to both equations. That means it is a solution to the system. The two lines are not identical, so it is the only solution.
D. Incorrect. Substituting 2 for x and 7 for y gives true statements in both equations, so the point lies on both lines. This means it is a solution to both equations. It is also the only solution to the system.

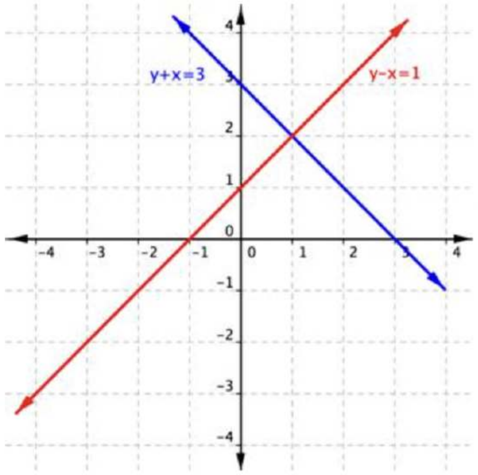
Graphing as a Solution Method

You can solve a system graphically. However, it is important to remember that you must check the solution, as the graph might not be accurate.

✓ Example

Find all solutions to the system $y-x=1$ and $y+x=3$.

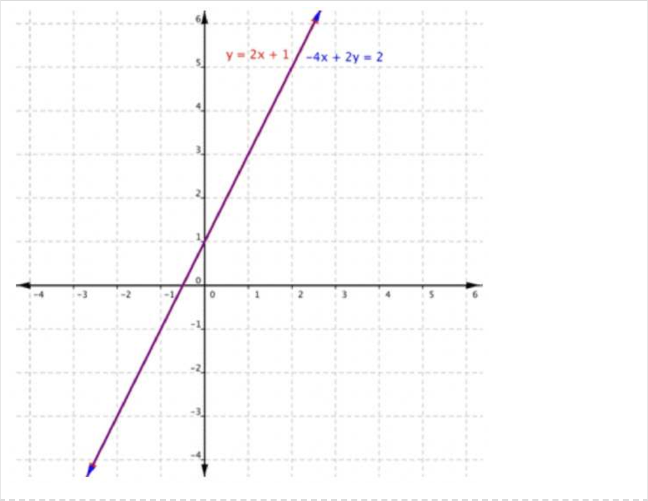
Solution

<table>
<tr><td> </td><td>First, graph both equations on the same axes.
The two lines intersect once. That means there is only one solution to the system.</td></tr>
<tr><td>The point of intersection appears to be $(1,2)$.</td><td>Read the point from the graph as accurately as possible.</td></tr>
<tr><td><table><tr><td>$y-x=1$
$2-1=1$
$1=1$</td><td>$y+x=3$
$2+1=3$
$3=3$</td></tr><tr><td>TRUE</td><td>TRUE</td></tr><tr><td>$(1,2)$ is a solution of $y-x=1$.</td><td>$(1,2)$ is a solution of $y+x=3$.</td></tr></table></td><td>Check the values in both equations. Substitute 1 for x and 2 for y. $(1,2)$ is a solution.</td></tr>
<tr><td>$(1,2)$ is the solution to the system $y-x=1$ and $y+x=3y+x=3$.</td><td>Since $(1,2)$ is a solution for each of the equations in the system, it is the solution for the system.</td></tr>
</table>

✓ Example

How many solutions does the system $y=2x+1$ and $-4x+2y=2$ have?

Solution

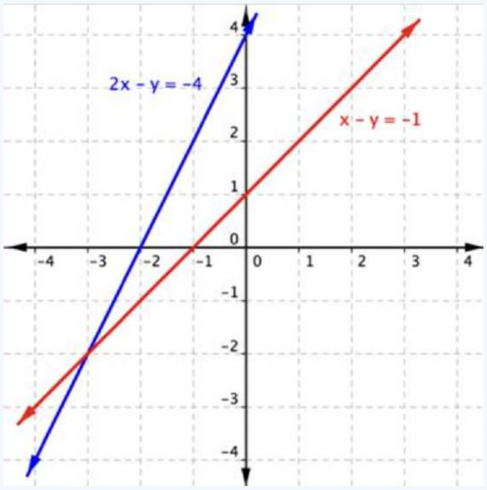


First, graph both equations on the same axes.
The two equations graph as the same line. So every point on that line is a solution for the system of equations.

The system $y = 2x + 1$ and $-4x + 2y = 2$ has an infinite number of solutions.

? Exercise

Which point is the solution to the system $x - y = -1$ and $2x - y = -4$? The system is graphed correctly below.



A. $(-1, 2)$
B. $(-4, -3)$
C. $(-3, -2)$
D. $(-1, 1)$

Answer

A. Incorrect. Substituting $(-1, 2)$ into each equation, you find that it is a solution for $2x - y = -4$, but not for $x - y = -1$. This means it cannot be a solution for the system. The correct answer is $(-3, -2)$.
B. Incorrect. Substituting $(-4, -3)$ into each equation, you find that it is a solution for $x - y = -1$, but not for $2x - y = -4$. This means it cannot be a solution for the system. The correct answer is $(-3, -2)$.
C. Correct. Substituting $(-3, -2)$ into each equation shows this point is a solution for both equations, so it is the solution for the system.
D. Incorrect. Substituting $(-1, -1)$ into each equation, you find that it is neither a solution for $2x - y = -4$, nor for $x - y = -1$. This means it cannot be a solution for the system. The correct answer is $(-3, -2)$.

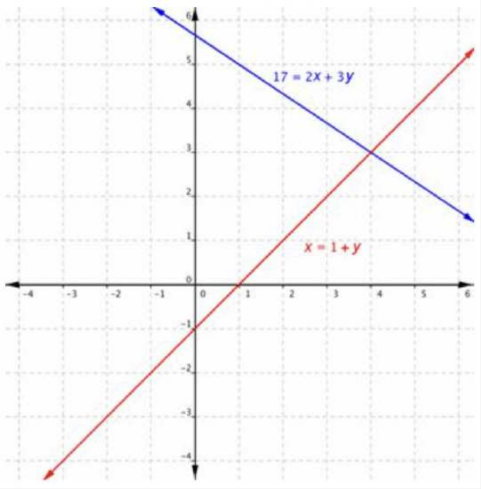
Graphing for a Real-World Context

Graphing a system of equations for a real-world context can be valuable in visualizing the problem. Let’s look at a couple of examples.

✓ Example

In yesterday’s basketball game, Cheryl scored 17 points with a combination of 2-point and 3-point baskets. The number of 2-point shots she made was one greater than the number of 3-point shots she made. How many of each type of basket did she score?

Solution

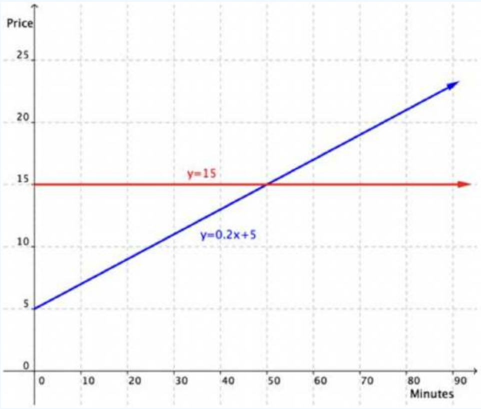
<table>
<tr><td>x = the number of 2-point shots made
y = the number of 3-point shots made</td><td>Assign variables to the two unknowns -- the number of each type of shots.</td></tr>
<tr><td>$2x$ = the points from 2-point baskets
$3y$ = the points from 3-point baskets</td><td>Calculate how many points are made from each of the two types of shots.</td></tr>
<tr><td>The number of points Cheryl scored (17) = the points from 2-point baskets + the points from 3-point baskets. $17=2x+3y$</td><td>Write an equation using information given in the problem.</td></tr>
<tr><td>The number of 2-point baskets $(x)=1+$ the number of 3-point baskets (y)</td><td>Write a second equation using additional information given in the problem.</td></tr>
<tr><td>$x=1+y$
$17=2x+3y$
$x=1+y$</td><td>Now you have a system of two equations with two variables.</td></tr>
<tr><td>
</td><td>Graph both equations on the same axes.
The two lines intersect, so they have only one point in common.
That means there is only one solution to the system.</td></tr>
<tr><td>The point of intersection appears to be $(4,3)$.</td><td>Read the point of intersection from the graph.</td></tr>
<tr><td><table>
<tr><td>$17=2x+3y$
$17=2(4)+3(3)$
$17=8+9$
$17=17$</td><td>$x=1+y$
$4=1+3$
$4=4$</td></tr>
<tr><td>TRUE</td><td>TRUE</td></tr>
<tr><td>$(4,3)$ is a solution of $17=2x+3y$.</td><td>$(4,3)$ is a solution of $x=1+y$</td></tr>
</table></td><td>Check $(4,3)$ in each equation to see if it is a solution to the system of equations.
$x=4$ and $y=3$</td></tr>
</table>

Cheryl made 4 two-point baskets and 3 three-point baskets.

✓ Example

Andres was trying to decide which of two mobile phone plans he should buy. One plan, TalkALot, charged a flat fee of \$15 per month for unlimited minutes. Another plan, FriendFone, charged a monthly fee of \$5 in addition to charging 20¢ per minute for calls.

To examine the difference in plans, he made a graph:



If he plans to talk on the phone for about 70 minutes per month, which plan should he purchase?

Solution

	Look at the graph. TalkALot is represented as $y=15$, while FriendFone is represented as $y=0.2x+5$. The number of minutes is listed on the x-axis. When $x=70$, TalkALot costs \$15, while FriendFone costs about \$19.
Andres should buy the TalkALot plan.	Since TalkALot costs less at 70 minutes, Andres should buy that plan.

Note that if the estimate for FriendFone had been incorrect, a new estimate could have been made. Regraphing to zoom in on the area where the lines cross would help make a better estimate.

? Exercise

Paco and Lisel spent \$30 going to the movies last night. Paco spent \$8 more than Lisel.

If P = the amount that Paco spent, and L = the amount that Lisel spent, which system of equations can you use to figure out how much each of them spent?

A. $\begin{array}{l}P+L=30\\P+8=L\end{array}$

B. $\begin{array}{l}P+L=30\\P=L+8\end{array}$

C. $\begin{array}{l}P+30=L\\P-8=L\end{array}$

D. $\begin{array}{l}L+30=P\\L-8=P\end{array}$

Answer

A. $\begin{array}{l}P+L=30\\P+8=L\end{array}$

Incorrect. $P+8=L$ reads: “Lisel spent \$8 more than Paco.” The correct system is: $\begin{array}{l}P+L=30\\P=L+8\end{array}$

B. $$\begin{aligned}P+L&=30\\P&=L+8\end{aligned}$$

Correct. The total amount spent $(P+L)$ is 30, so one equation should be $P+L=30$. Paco spent 8 dollars more than Lisel, so $L+8$ will give you the amount that Paco spent. This can be rewritten $P=L+8$.

C. $$\begin{aligned}P+30&=L\\P-8&=L\end{aligned}$$

Incorrect. $P+30=L$ reads: “Lisel spent $30 more than Paco.” The correct system is: $$\begin{aligned}P+L&=30\\P&=L+8\end{aligned}$$

D. $$\begin{aligned}L+30&=P\\L-8&=P\end{aligned}$$

Incorrect. $L+30=P$ reads: “Paco spent $30 more than Lisel.” The correct system is: $$\begin{aligned}P+L&=30\\P&=L+8\end{aligned}$$

Summary

A system of linear equations is two or more linear equations that have the same variables. You can graph the equations as a system to find out whether the system has no solutions (represented by parallel lines), one solution (represented by intersecting lines), or an infinite number of solutions (represented by two superimposed lines). While graphing systems of equations is a useful technique, relying on graphs to identify a specific point of intersection is not always an accurate way to find a precise solution for a system of equations.

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14.1.2: Graphing Systems of Inequalities

Learning Objectives

- Solve a system of linear inequalities by graphing.
- Determine whether an ordered pair is a solution of a system of inequalities.
- Solve application problems by graphing a system of inequalities.

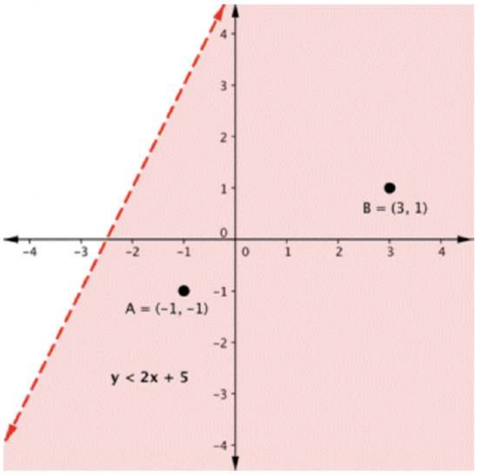
Introduction

A solution of a system of linear equalities is any ordered pair that is true for all of the equations in the system. Likewise, a solution of a **system of linear inequalities** is any ordered pair that is a solution for all of the inequalities in the system.

Graphs are used to show all of the values that are solutions for a system of linear inequalities.

Graphing a System of Two Inequalities

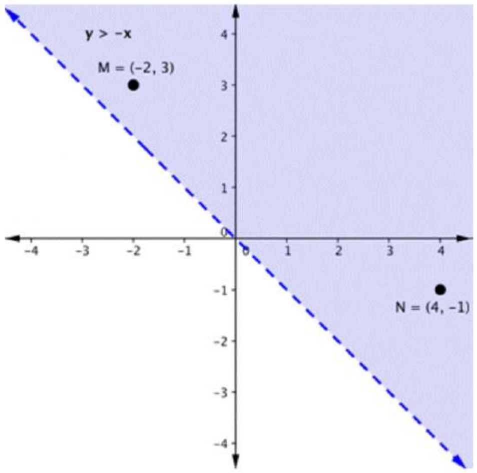
The graph of a single linear inequality splits the **coordinate plane** into two regions. On one side lie all the solutions to the inequality. On the other side, there are no solutions. Consider the graph of the inequality $y<2x+5$.



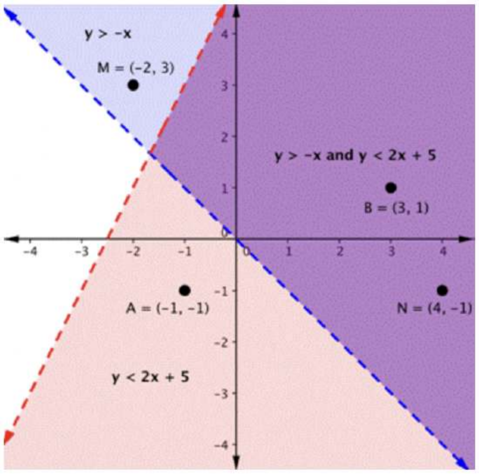
The dashed line is $y=2x+5$. Every ordered pair in the colored area below the line is a solution to $y<2x+5$, as all of the points below the line will make the inequality true. If you doubt that, try substituting the x and y coordinates of points A and B into the inequality—you'll see that they work. So, the shaded area shows all of the solutions for this inequality.

The **boundary line** divides the plane into two half planes. In this case, it is shown as a dashed line, as the points on the line don't satisfy the inequality. If the inequality had been $y\leq 2x+5$, then the boundary line would have been solid.

Let's graph another inequality: $y>-x$ You can check a couple of points to determine which side of the boundary line to shade. Checking points M and N yields true statements. So, we shade the area above the line. The line is dashed, as points on the line don't satisfy the inequality.



To create a system of inequalities, you need to graph two or more inequalities together. Let's use $y < 2x + 5$ and $y > -x$ since we have already graphed each of them.



The purple area shows where the solutions of the two inequalities overlap. This area is the solution to the *system of inequalities*. Any point within this purple region will be true for both $y > -x$ and $y < 2x + 5$.

On the graph, you can see that the points B and N are solutions for the system because their coordinates will make both inequalities true statements.

In contrast, points M and A both lie outside the solution region (purple). While point M is a solution for the inequality $y > -x$ and point A is a solution for the inequality $y < 2x + 5$, neither point is a solution for the *system*.

Example

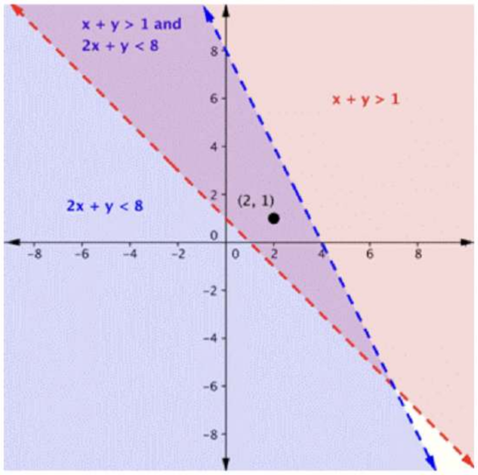
Is the point $(2, 1)$ a solution of the system $x + y > 1$ and $2x + y < 8$?

Solution

$x + y > 1$ $2 + 1 > 1$ $3 > 1$	$2x + y < 8$ $2(2) + 1 < 8$ $4 + 1 < 8$ $5 < 8$	Check the point with each of the inequalities. Substitute 2 for x and 1 for y. Is the point a solution of both inequalities?
TRUE	TRUE	
$(2, 1)$ is a solution for $x + y > 1$.	$(2, 1)$ is a solution for $2x + y < 8$.	

The point $(2,1)$ is a solution of the system $x+y>1$ and $2x+y<8$	Since $(2,1)$ is a solution of each inequality, it is also a solution of the system.

Here is a graph of the system in the example above. Notice that $(2,1)$ lies in the purple area, which is the overlapping area for the two inequalities.



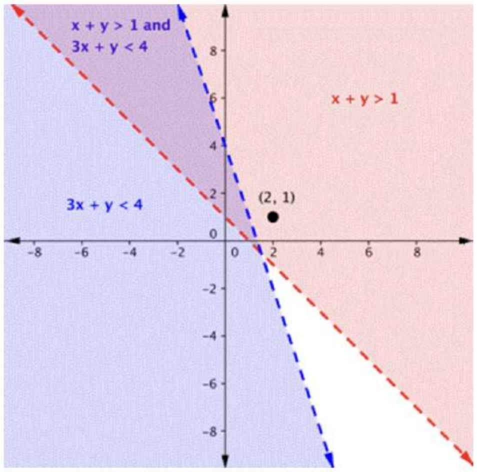
Example

Is the point $(2,1)$ a solution of the system $x+y>1$ and $3x+y<4$?

Solution

<table>
<tr>
<td>
<table>
<tr><td>$x+y>1$
$2+1>1$
$3>1$</td><td>$3x+y<4$
$3(2)+1<4$
$6+1<4$
$7<4$</td></tr>
<tr><td>TRUE</td><td>FALSE</td></tr>
<tr><td>$(2,1)$ is a solution for $x+y>1$.</td><td>$(2,1)$ is not a solution for $3x+y<4$.</td></tr>
</table>
</td>
<td>Check the point with each of the inequalities. Substitute 2 for x and 1 for y. Is the point a solution of both inequalities?</td>
</tr>
<tr>
<td>The point $(2,1)$ is not a solution of the system. $x+y>1$ and $3x+y<4$.</td>
<td>Since $(2,1)$ is not a solution of one of the inequalities, it is not a solution of the system.</td>
</tr>
</table>

Here is a graph of this system. Notice that $(2,1)$ is not in the purple area, which is the overlapping area; it is a solution for one inequality (the red region), but it is not a solution for the second inequality (the blue region).



Exercise

Which of the points listed below are solutions for the system?

$$y > x$$
$$x - 2 < 0$$

I. $(1,1)$
II. $(-5,9)$
III. $(0,7)$

A. I and II
B. II and III
C. I and III
D. II only

Answer

A. Incorrect. $(-5,9)$ is a solution to this system, but $(1,1)$ is not, because it is not a solution to the first inequality. The correct answer is B, $(-5,9)$ and $(0,7)$ are solutions to the system.
B. Correct. The points $(-5,9)$ and $(0,7)$ are both solutions for both inequalities in the system.
C. Incorrect. $(1,1)$ is a valid solution for the inequality $x-2<0$, but it is not a solution for $y>x$; consequently, it is not a solution for the system. The correct answer is B, $(-5,9)$ and $(0,7)$ are solutions to the system.
D. Incorrect. $(-5,9)$ is a valid solution for the system, but so is $(0,7)$. The correct answer is B, $(-5,9)$ and $(0,7)$ are solutions to the system.

Solving Systems of Inequalities by Graphing

As shown above, finding the solutions of a system of inequalities can be done by graphing each inequality and identifying the region they share.

Example

Find the solution to the system $x+y \geq 1$ and $y-x \geq 5$.

Solution

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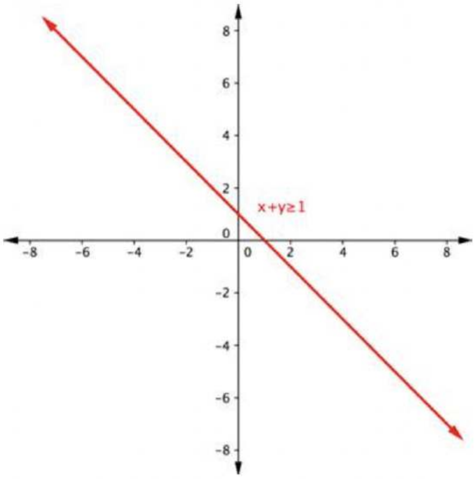
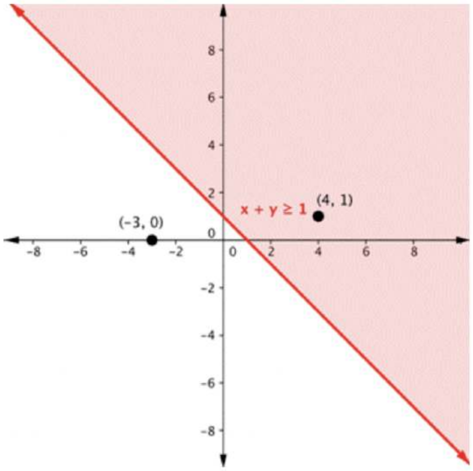
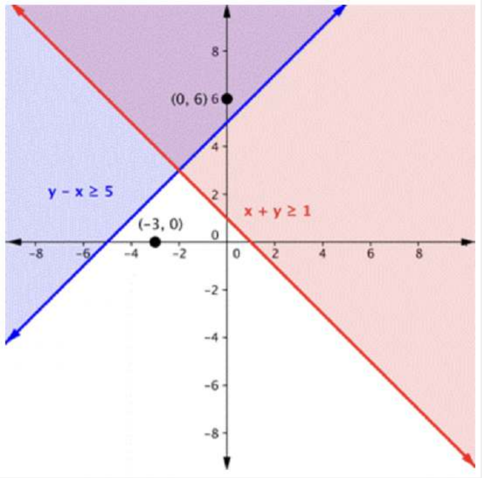
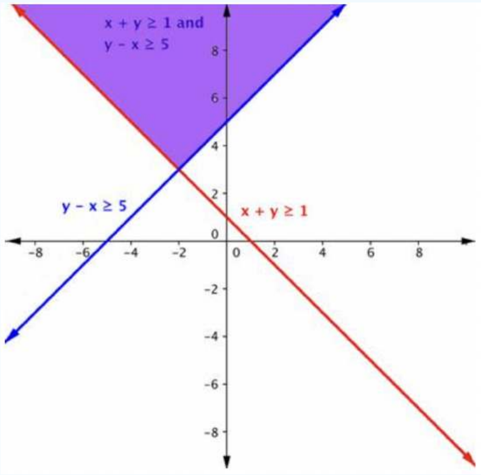
<tr>
<td>
</td>
<td>Graph one inequality. First, graph the boundary line using a table of values, intercepts, or any other method you prefer. The boundary line for $x+y\geq 1$ is $x+y=1$, or $y=-x+1$.
Since the equal sign is included with the greater than sign, the boundary line is solid.</td>
</tr>
<tr>
<td>
<table>
<tr><td>$$\begin{aligned} &\text{Test } 1:(-3,0)\\ &x+y\geq 1\\ &-3+0\geq 1\\ &-3\geq 1\end{aligned}$$</td><td>$$\begin{aligned} &\text{Test } 2:(4,1)\\ &x+y\geq 1\\ &4+1\geq 1\\ &5\geq 1\end{aligned}$$</td></tr>
<tr><td>FALSE</td><td>TRUE</td></tr>
</table>
</td>
<td>Find an ordered pair on either side of the boundary line. Insert the x- and y-values into the inequality $x+y\geq 1$ and see which ordered pair results in a true statement.</td>
</tr>
<tr>
<td>
</td>
<td>Since $(4,1)$ results in a true statement, the region that includes $(4,1)$ should be shaded.</td>
</tr>
<tr>
<td>
</td>
<td>Do the same with the second inequality. Graph the boundary line, then test points to find which region is the solution to the inequality. In this case, the boundary line is $y-x=5$ (or $y=x+5$) and is solid. Test point $(-3,0)$ is not a solution of $y-x\geq 5$, and test point $(0,6)$ is a solution.</td>
</tr>
</table>

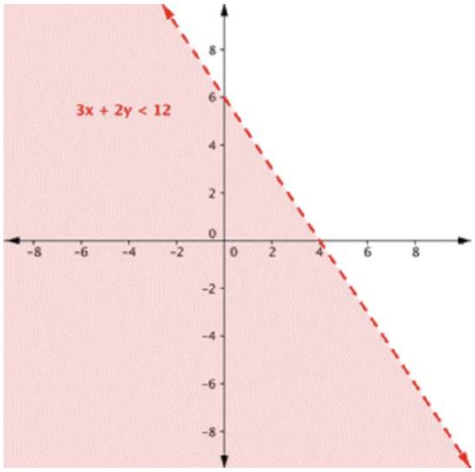
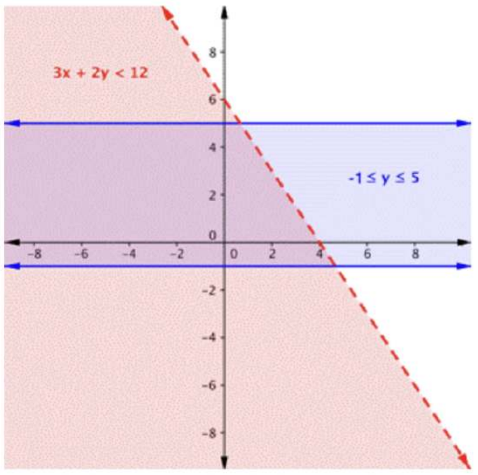
The purple region in this graph shows the set of all solutions of the system.



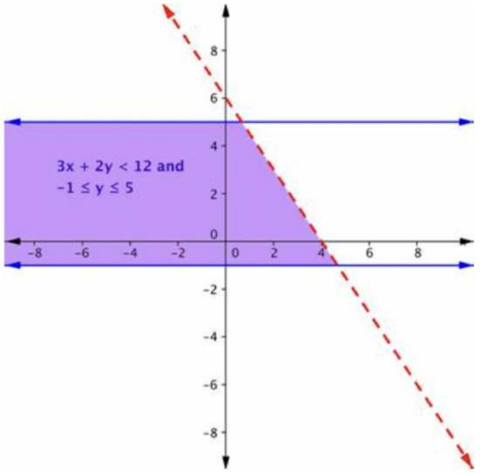
✓ Example

Find the solution to the system $3x+2y<12$ and $-1\le y\le 5$.

Solution

	Graph one inequality. First, graph the boundary line, then test points. Remember, because the inequality $3x+2y<12$ does not include the equal sign, draw a dashed border line. Testing a point such as $(0,0)$ will show that the area below the line is the solution to this inequality.
	The inequality $-1\le y\le 5$ is actually two inequalities: $-1\le y$, and $y\le 5$. Another way to think of this is that y must be between -1 and 5. The border lines for both are horizontal. The region between those two lines contains the solutions of $-1\le y\le 5$. We make the lines solid because we also want to include $y=-1$ and $y=5$. Graph this region on the same axes as the other inequality.

The purple region in this graph shows the set of all solutions of the system.

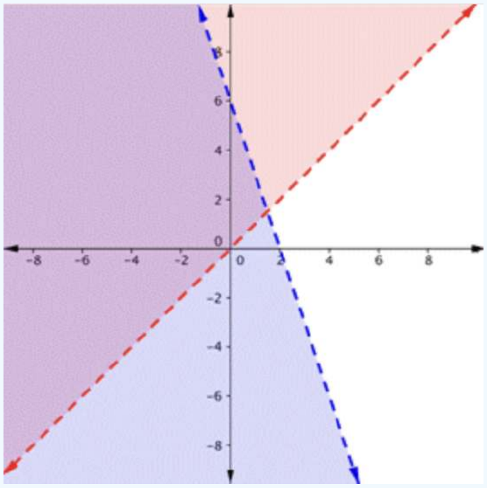


Exercise

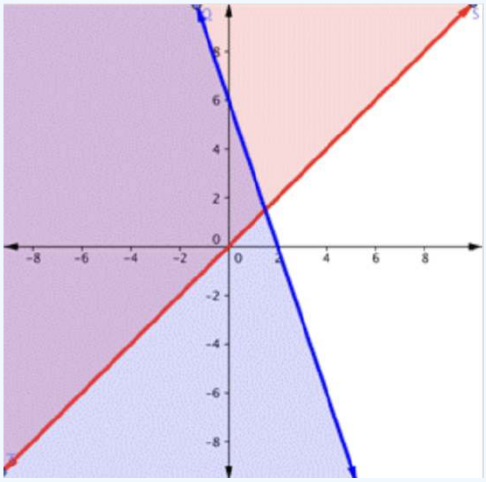
In which of the following is the purple region the solution for the system?

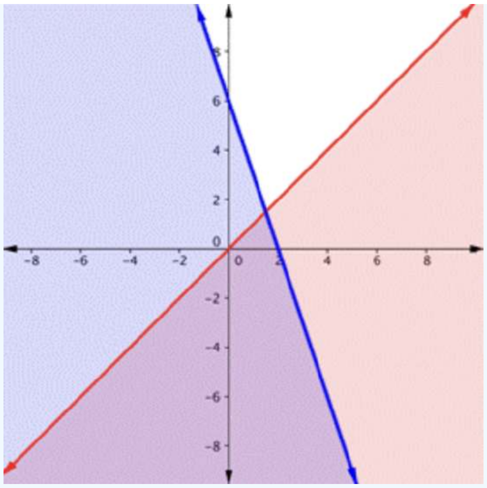
$$y > x$$
$$y < -3x + 6$$

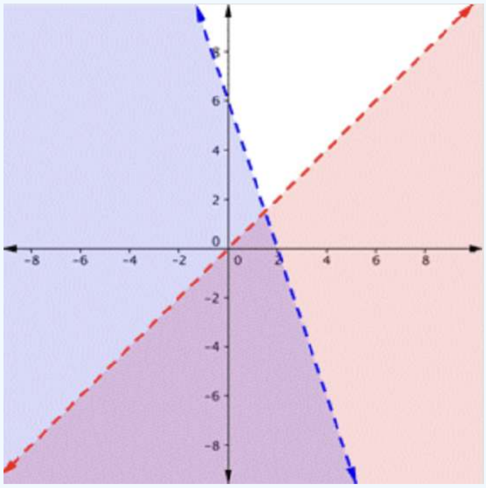
A.



B.



C. 

D. 

Answer

A.
B.
C.
D.

Graphing a system of inequalities can help you solve real-life situations.

✓ Example

Cathy is selling ice cream cones at a school fundraiser. She is selling two sizes: small (which has 1 scoop) and large (which has 2 scoops). She knows that she can get a maximum of 70 scoops of ice cream out of her supply. She charges $3 for a small cone and $5 for a large cone.

Cathy wants to earn at least $120 to give back to the school. Write and graph a system of inequalities that models this situation.

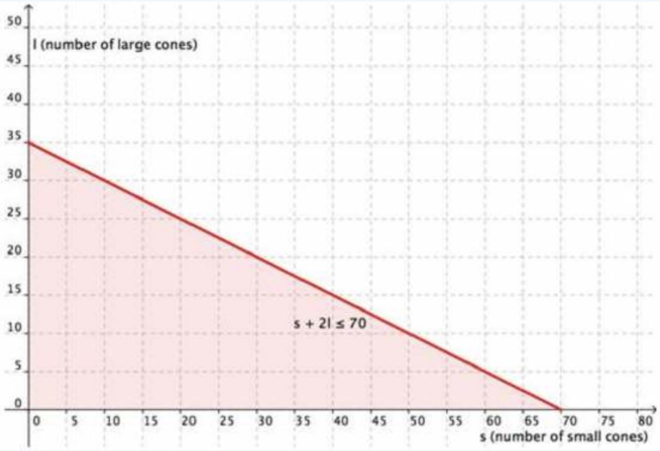
Solution

$s =$ small cone $l =$ large cone	First, identify the variables. There are two variables: the number of small cones and the number of large cones.

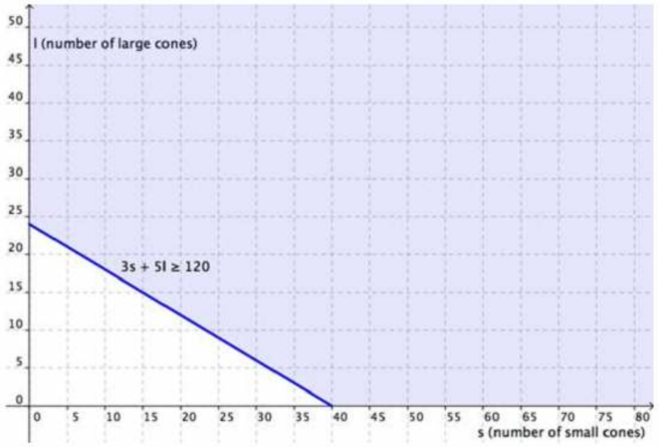
$s+2l \leq 70$	Write the first equation: the maximum number of scoops she can give out. The scoops she has available (70) must be greater than or equal to the number of scoops for the small cones (s) and the large cones $(2l)$ she sells.
$3s+5l \geq 120$	Write the second equation: the amount of money she raises. She wants the total amount of money earned from small cones $(3s)$ and large cones $(5l)$ to be at least $120.
$s+2l \leq 70$ $3s+5l \geq 120$	Write the system.

Now graph the system. The variables x and y have been replaced by s and l; graph s along the x-axis, and l along the y-axis.

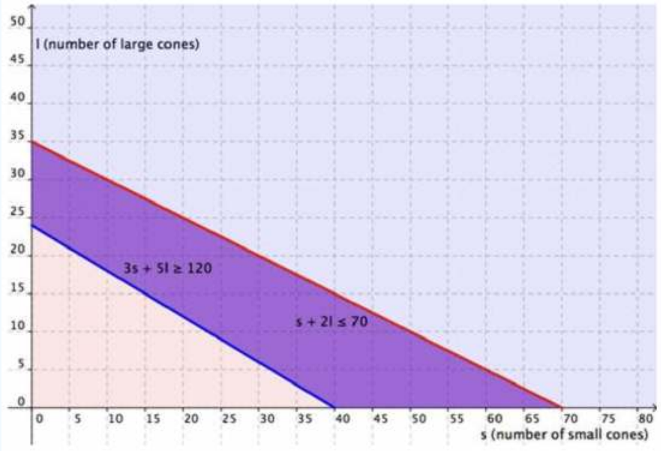
First, graph the region $s+2l \leq 70$. Graph the boundary line and then test individual points to see which region to shade. The graph is shown below.



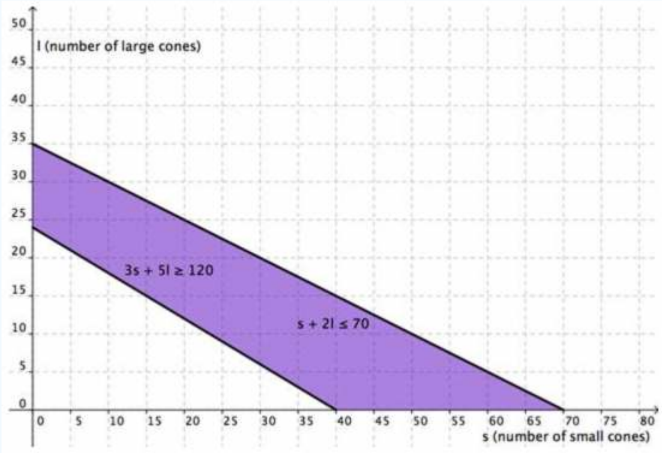
Now, graph the region $3s+5l \geq 120$. Graph the boundary line and then test individual points to see which region to shade. The graph is shown below.



Graphing the regions together, you find the following:



And represented just as the overlapping region, you have:



The region in purple is the solution. As long as the combination of small cones and large cones that Cathy sells can be mapped in the purple region, she will have earned at least $120 and not used more than 70 scoops of ice cream.

Summary

Systems of inequalities can be graphed on a coordinate plane. The solution set for a system of inequalities is not a single point, but rather an entire region defined by the overlapping areas of each individual inequality in the system. Every point within this region is a solution to both inequalities and thus a solution for the whole system.

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SECTION OVERVIEW

14.2: Algebraic Methods to Solve Systems of Equations

Topic hierarchy

14.2.1: The Substitution Method

14.2.2: The Elimination Method

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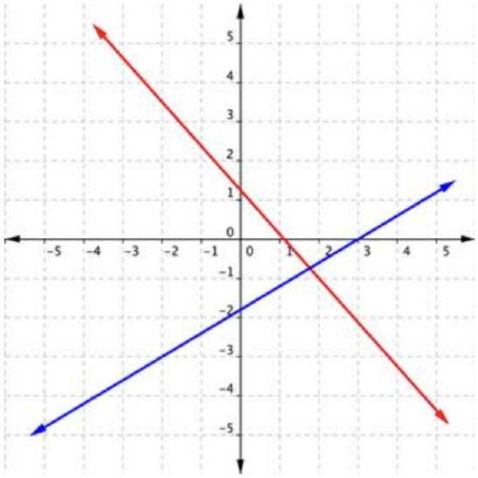
14.2.1: The Substitution Method

Learning Objectives

- Solve a system of equations using the substitution method.
- Recognize systems of equations that have no solution or an infinite number of solutions.
- Solve application problems using the substitution method.

Introduction

A graph can be used to show the solution for a system of two linear equations. However, accurately determining the solution from a graph is not always easy or accurate. For example, where do you think the two lines shown below intersect?



It looks like they might intersect at $(1.8, -0.7)$, though this is only an estimate. In cases like this, you can use algebraic methods to find exact answers. One method to look at is called the **substitution method**. You solve one equation for one variable and then substitute this expression into the other equation.

Using Substitution to Solve a System of Equations

In the substitution method, you solve for one variable and then substitute that expression into the other equation. The important thing here is that you are always substituting values that are equivalent.

For example:

Sean is 5 years older than four times his daughter's age. His daughter is 7. How old is Sean?

You might do this problem in your head. Sean's daughter is 7, so "four times his daughter's age" is 28, and 5 years added to that is 33. Sean is 33.

If you solved the problem like that, you used a simple substitution—you substituted in the value "7" for "his daughter's age." You learned in the second part of the problem that "his daughter is 7." So substituting in a value of "7" for "his daughter's age" in the first part of the problem was okay, because you knew these two quantities were equal.

Let's look at a simple system of equations that can be solved using substitution.

Example

Find the value of x for this system.

Equation A: $4x+3y=-14$

Equation B: $y=2$

Solution

$4x+3y=-14$ $y=2$	The problem asks to solve for x. Equation B gives you the value of y, $y=2$, so you can substitute 2 into Equation A for y.
$4x+3(2)=-14$	Substituting $y=2$ into Equation A.
$4x+6=-14$ $4x=-20$ $x=-5$	Simplify and solve the equation for x.

$x=-5$

You can substitute a value for a variable even if it is an expression. Here’s an example.

Example

Solve for x and y.

Equation A: $y+x=3$

Equation B: $x=y+5$

Solution

$y+x=3$ $x=y+5$	The goal of the substitution method is to rewrite one of the equations in terms of a single variable. Equation B tells us that $x=y+5$, so it makes sense to substitute that $y+5$ into Equation A for x.
$y+x=3$ $y+(y+5)=3$	Substitute $y+5$ into Equation A for x and you get $y+(y+5)=3$.
$\begin{array}{rcr} 2y+5 & = & 3 \\ -5 & & -5 \\ \hline 2y & & -2 \\ y & = & -1 \end{array}$	Simplify and solve the equation to get $y=-1$.
$\begin{array}{rcr} y+x & = & 3 \\ -1+x & = & 3 \\ +1 & & +1 \\ \hline x & = & 4 \end{array}$	To now find x, substitute this value for y into either equation and solve for x. We will use Equation A here to get $x=4$.
$y+x=3$ $-1+4=3$ $3=3$ TRUE $x=y+5$ $4=-1+5$ $4=4$ TRUE	Finally, check the solution $x=4$, $y=-1$ by substituting these values into each of the original equations.
$x=4$ and $y=-1$	The solution is $(4,-1)$.

Remember, a solution to a system of equations must be a solution to each of the equations within the system. The ordered pair $(4,-1)$ works for both equations, so you know that it is a solution to the system as well.

Let’s look at another example whose substitution involves the distributive property.

Example

Solve for x and y.

$$\begin{aligned} y&=3x+6 \\ -2x+4y&=4 \end{aligned}$$

Solution

$y=3x+6$ $-2x+4y=4$	Choose an equation to use for the substitution. The first equation tells you how to express y in terms of x, so it makes sense to substitute $3x+6$ into the second equation for y.
$-2x+4y=4$ $-2x+4(3x+6)=4$	Substitute $3x+6$ for y into the second equation.
$-2x+12x+24=4$ $10x+24=4$ $-24 \quad -24$ $10x=-20$ $x=-2$	Simplify and solve the equation for x.
$y=3x+6$ $y=3(-2)+6$ $y=-6+6$ $y=0$	To find y, substitute this value for x back into one of the original equations.
$y=3x+6$ $0=3(-2)+6$ $0=-6+6$ $0=0$ TRUE $-2x+4y=4$ $-2(-2)+4(0)=4$ $4+0=4$ $4=4$ TRUE	Check the solution $x=-2$, $y=0$ by substituting them into each of the original equations.

$x=-2$ and $y=0$.

The solution is $(-2,0)$.

In the examples above, one of the equations was already given to us in terms of the variable x or y. This allowed us to quickly substitute that value into the other equation and solve for one of the unknowns.

Sometimes you may have to rewrite one of the equations in terms of one of the variables first before you can substitute. Look at the example below.

✓ Example

Solve for x and y.

$$2x+3y=22$$
$$3x+y=19$$

Solution

$2x+3y=22$ $3x+y=19$	Choose an equation to use for the substitution. The second equation, $3x+y=19$, can easily be rewritten in terms of y, so it makes sense to start there.
$3x+y=19$ $y=19-3x$	Rewrite $3x+y=19$ in terms of y.
$2x+3y=22$ $2x+3(19-3x)=22$	Substitute $19-3x$ for y in the other equation as $2x+3(19-3x)$ $=22$.
$2x+57-9x=22$ $-7x+57=22$ $-7x=-35$ $x=5$	Simplify and solve the equation for x.

$3x+y=19$ $3(5)+y=19$ $15+y=19$ $y=19-15$ $y=4$	Substitute $x=5$ back into one of the original equations to solve for y.
$2x+3y=22$ $2(5)+3(4)=22$ $10+12=22$ $22=22$ TRUE $3x+y=19$ $3(5)+4=19$ $19=19$ TRUE	Check both solutions by substituting them into each of the original equations.

$x=5$ and $y=4$

The solution is $(5,4)$.

? Exercise

Solve the system for x and y.

$$2y=x+8$$
$$2y-10=2x$$

A. $x=-3, y=2$
B. $x=-2, y=3$
C. $x=-5, y=2$
D. $x=0, y=-5$

Answer

A. Incorrect. If you substitute the values $x=-3$ and $y=2$ into the first equation, you get a false statement: $2(2)=-3+9$. To solve this system, try rewriting the first equation as $x=2y-8$. Then substitute $2y-8$ in for x in the second equation, and solve for y. The correct answer is $x=-2$, $y=3$.
B. Correct. Substituting these values into either equation results in a true statement: $2(3)=-2+8$, and $2(3)-10=2(-2)$.
C. Incorrect. If you substitute the values $x=-5$ and $y=2$ into the second equation, you get a false statement: $2(2)-10=2(-5)$. To solve this system, try rewriting the first equation as $x=2y-8$. Then substitute $2y-8$ in for x in the second equation, and solve for y. The correct answer is $x=-2, y=3$.
D. Incorrect. If you substitute the values $x=0$ and $y=-5$ into the second equation, you get a false statement: $2(-5)-10=2(0)$. To solve this system, try rewriting the first equation as $x=2y-8$. Then substitute $2y-8$ in for x in the second equation, and solve for y. The correct answer is $x=-2, y=3$.

Special Situations

There are some cases where using the substitution method will yield results that, at first, do not make sense. Let's take a look at some of these and figure out what is going on.

✓ Example

Solve for x and y.

$$y=5x+4$$
$$10x-2y=4$$

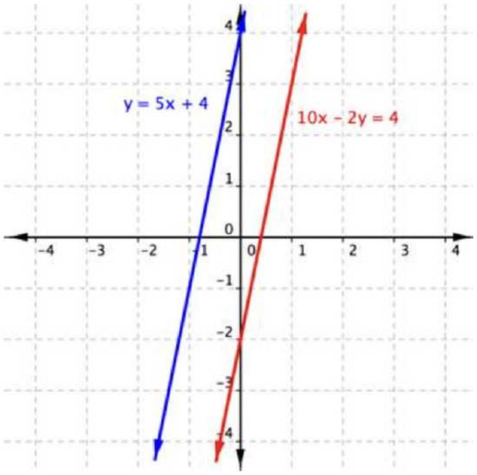
Solution

$y=5x+4$ $10x-2y=4$ $10x-2(5x+4)=4$	Since the first equation is $y=5x+4$, you can substitute $5x+4$ in for y in the second equation.

$10x - 10x - 8 = 4$	Expand the expression on the left.
$0 - 8 = 4$ $-8 = 4$	Combine like terms on the left side of equation. $10x - 10x = 0$, so you are left with $-8 = 4$.

The statement $-8 = 4$ is false, so there is no solution.

You get the false statement $-8 = 4$. What does this mean? The graph of this system sheds some light on what is happening.



The lines are parallel. They never intersect and there is no solution to this system of linear equations. Note that the result $-8 = 4$ is **not** a solution. It is simply a false statement and it indicates that there is **no** solution.

Now take this problem, which is interesting as well.

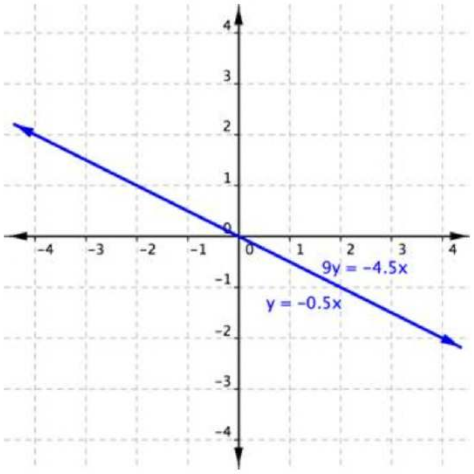
Solve for x and y.

$$y = -0.5x$$
$$9y = -4.5x$$

Substituting $-0.5x$ for y in the second equation, you find the following:

$$\begin{aligned} 9y &= -4.5x \\ 9(-0.5x) &= -4.5x \\ -4.5x &= -4.5x \end{aligned}$$

This time, you get a true statement: $-4.5x = -4.5x$. But what does this type of answer mean? Again, graphing can help you make sense of this system.



This system consists of two equations that both represent the same line; the two lines are collinear. Every point along the line will be a solution to the system, and that's why the substitution method yields a true statement. In this case, there are an infinite number of solutions.

? Exercise

Aubrey is using the substitution method to solve the following system of equations:

$$y-x=21$$
$$2y=2x+16$$

She arrives at an answer of $8=21$. She thinks that this answer means that the lines are parallel and that the system has no solution.

Aubrey wants to check her answer. Which of the following actions will best help her find out whether the two equations in the system are, in fact, parallel?

A. Check to see whether the slopes of both lines are the same, and the y-intercepts are different.
B. Check to see whether either line goes through the origin.
C. Check to see whether the lines have the same y-intercept.
D. Check to see whether both lines go through the point $(8,21)$.

Answer

A. Correct. Parallel lines have the same slope, but she also has to check whether they have different y-intercepts because the lines could be collinear (remember that 2 collinear lines are the same line). If Aubrey finds that the slopes of the lines are the same and the y-intercepts are different, then she can be confident that her answer is correct.
B. Incorrect. The origin has no bearing on whether two lines are parallel. In the case of this system, neither line goes through the origin, but the lines are still parallel. If Aubrey finds that the slopes of the lines are the same and the y-intercepts are different, then she can be confident that her answer is correct.
C. Incorrect. It is true that lines with the same y-intercept are never parallel, because parallel lines can never intersect. But just checking that the y-intercepts aren't the same is not enough. To be parallel, lines also must have the same slope. If Aubrey finds that the slopes of the lines are the same and the y-intercepts are different, then she can be confident that her answer is correct.
D. Incorrect. Although she arrived at an answer of $8=21$, this does not mean that the lines themselves intersect at the point $(8,21)$. If Aubrey finds that the slopes of the lines are the same and the y-intercepts are different, then she can be confident that her answer is correct.

Solving Application Problems Using Substitution

Systems of equations are a very useful tool for modeling real-life situations and answering questions about them. If you can translate the application into two linear equations with two variables, then you have a system of equations that you can solve to find the solution. You can use any method to solve the system of equations. Use the substitution method in this topic.

In order to sell more of its produce, a local farm sells bags of apples in two sizes: medium and large. A medium bag contains 4 Macintosh and 1 Granny Smith apples and costs \$2.80. A large bag contains 8 Macintosh and 4 Granny Smith apples and costs \$7.20. The price of one Granny Smith apple is the same in the medium bag as it is in the large bag. The price of one Macintosh apple is the same in the medium bag as it is in the large bag. What is the price of each kind of apple?

Let's start by creating a system of equations that represents what is happening in the problem. There are two types of apples and two sizes of bags. You can let m represent the cost of a Macintosh apple and g represent the cost of a Granny Smith apple. Let's make a table and see what is known.

	Cost of Macintosh apples	+	Cost of Granny Smith apples	=	Total cost of bag
Medium	$4m$	+	g	=	\$2.80
Large	$8m$	+	$4g$	=	\$7.20

Now that you have two equations in the same variables, you can solve the system. You will use substitution. The steps are shown in the example below:

Example

Solve for g and m using the substitution method.

$$\begin{aligned}4m+g&=2.80\\8m+4g&=7.20\end{aligned}$$

Solution

$4m+g=2.80$ $g=2.80-4m$	First, rewrite one of the equations in terms of one of the variables.
$8m+4g=7.20$ $8m+4(2.80-4m)=7.20$ $8m+11.20-16m=7.20$ $8m-16m=7.20-11.20$ $-8m=-4.00$ $m=0.50$	Substitute $(2.80-4m)$ for g in the second equation and solve for m.
$4m+g=2.80$ $4(0.5)+g=2.80$ $2+g=2.80$ $g=2.80-2$ $g=0.80$	Substitute the value of m, 0.50, into one of the original equations to solve for g.
$4m+g=2.80$ $4(.50)+.80=2.80$ $2.80=2.80$	Check both equations by substituting in the values of g and m.
$8m+4g=7.20$ $8(.50)+4(.80)=7.20$ $4.00+3.20=7.20$ $7.20=7.20$	

One Granny Smith apple costs \$0.80 and one Macintosh apple costs \$0.50.

Using the substitution method can be an effective approach to solving geometric problems.

Example

The perimeter of a rectangle is 60 inches. If the length is 10 inches longer than the width, find the dimensions using the substitution method.

Solution

$2l+2w=60$ $l=w+10$	Use the information provided to write a system of equations. Let $l=\text{ length }$ and $w=\text{ width }$.
$\begin{array}{rrrcr} 2l & & +2w & = & 60 \\ & 2(w+10) & +2w & = & 60 \\ 2w & +20 & +2w & = & 60 \\ & 4w & +20 & = & 60 \\ & & -20 & & -20 \\ \hline & & 4w & = & 40 \\ & & w & = & 10 \end{array}$	Substitute $w+10$ for l in the first equation and solve for w.
$l=w+10$ $l=10+10$ $l=20$	To find l, substitute 10 for w in one of the equations and solve for l.

$\begin{aligned} l &= w+10 \\ 20 &= 10+10 \\ 20 &= 20 \end{aligned}$	Check both solutions by substituting them into the two equations.
$\begin{aligned} 2l+2w &= 60 \\ 2(20)+2(10) &= 60 \\ 40+20 &= 60 \\ 60 &= 60 \end{aligned}$	Both of them are true, so this is a correct solution.

The length of the rectangle is 20 inches.

The width of the rectangle is 10 inches.

Summary

The substitution method is one way of solving systems of equations. To use the substitution method, use one equation to find an expression for one of the variables in terms of the other variable. Then substitute that expression in place of that variable in the second equation. You can then solve this equation as it will now have only one variable. Solving using the substitution method will yield one of three results: a single value for each variable within the system (indicating one solution), an untrue statement (indicating no solutions), or a true statement (indicating an infinite number of solutions).

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14.2.2: The Elimination Method

Learning Objectives

- Solve a system of equations when no multiplication is necessary to eliminate a variable.
- Solve a system of equations when multiplication is necessary to eliminate a variable.
- Recognize systems that have no solution or an infinite number of solutions.
- Solve application problems using the elimination method.

Introduction

The **elimination method** for solving systems of linear equations uses the addition property of equality. You can add the same value to each side of an equation.

So if you have a system: $x - y = -6$ and $x + y = 8$, you can add $x + y$ to the left side of the first equation and add 8 to the right side of the equation. And since $x + y = 8$, you are adding the same value to each side of the first equation.

Using Addition to Eliminate a Variable

If you add the two equations, $x - y = -6$ and $x + y = 8$ together, as noted above, watch what happens.

$$\begin{array}{rr} x - y = & -6 \\ x + y = & 8 \\ \hline 2x + 0 = & 2 \end{array}$$

You have eliminated the y term, and this equation can be solved using the methods for solving equations with one variable.

Let's see how this system is solved using the elimination method.

Example

Use elimination to solve the system.

$$\begin{array}{r} x - y = -6 \\ x + y = 8 \end{array}$$

Solution

$\begin{array}{rrr} & x - y = & -6 \\ + & x + y = & 8 \\ \hline & 2x \quad = & 2 \end{array}$	Add the equations.
$2x = 2$ $x = 1$	Solve for x.
$x + y = 8$ $1 + y = 8$ $y = 8 - 1$ $y = 7$	Substitute $x = 1$ into one of the original equations and solve for y.
$x - y = -6$ $1 - 7 = -6$ $-6 = -6$ TRUE $x + y = 8$ $1 + 7 = 8$ $8 = 8$ TRUE	Be sure to check your answer in both equations! The answers check.

The solution is $(1, 7)$.

Unfortunately, not all systems work out this easily. How about a system like $2x + y = 12$ and $-3x + y = 2$? If you add these two equations together, no variables are eliminated.

$$\begin{array}{r}2x+y=12\\ \underline{-3x+y=2}\\ -x+2y=14\end{array}$$

But you want to eliminate a variable. So let’s add the opposite of one of the equations to the other equation.

$$\begin{array}{rcrcr}2x+y=12 & \rightarrow & 2x+y=12 & \rightarrow & 2x+y=\ 12\\ -3x+y=2 & \rightarrow & -(-3x+y)=-(2) & \rightarrow & \underline{3x-y=-2}\\ & & & & 5x+0y=\ 10\end{array}$$

You have eliminated the y variable, and the problem can now be solved. See the example below.

✓ Example

Use elimination to solve the system.

$$\begin{array}{r}2x+y=12\\ -3x+y=2\end{array}$$

Solution

<table>
<tr><td>$2x+y=12$
$-3x+y=2$</td><td>You can eliminate the y-variable if you add the opposite of one of the equations to the other equation.</td></tr>
<tr><td>$2x+y=12$
$3x-y=-2$
$5x=10$</td><td>Rewrite the second equation as its opposite.
Add.</td></tr>
<tr><td>$x=2$</td><td>Solve for x.</td></tr>
<tr><td>$2(2)+y=12$
$4+y=12$
$y=8$</td><td>Substitute $x=2$ into one of the original equations and solve for y.</td></tr>
<tr><td><table><tr><td>$2x+y=12$
$2(2)+8=12$
$4+8=12$
$12=12$</td><td>$-3x+y=2$
$-3(2)+8=2$
$-6+8=2$
$2=2$</td></tr><tr><td>TRUE</td><td>TRUE</td></tr></table></td><td>Be sure to check your answer in both equations!
The answers check.</td></tr>
</table>

The solution is $(2,8)$.

The following are two more examples showing how to solve linear systems of equations using elimination.

✓ Example

Use elimination to solve the system.

$$\begin{array}{r}-2x+3y=-1\\ 2x+5y=25\end{array}$$

Solution

$-2x+3y=-1$ $2x+5y=25$	Notice the coefficients of each variable in each equation. If you add these two equations, the x term will be eliminated since $-2x+2x=0$.
$$\begin{array}{rr}-2x+3y= & -1\\ \underline{2x+5y=} & \underline{25}\\ 8y= & 24\\ y= & 3\end{array}$$	Add and solve for y.

$2x+5y=25$ $2x+5(3)=25$ $2x+15=25$ $2x=10$ $x=5$	Substitute $y=3$ into one of the original equations.
$-2x+3y=-1$ $-2(5)+3(3)=-1$ $-10+9=-1$ $-1=-1$ TRUE $2x+5y=25$ $2(5)+5(3)=25$ $10+15=25$ $25=25$ TRUE	Check solutions. The answers check.

The solution is $(5,3)(5,3)$.

✓ Example

Use elimination to solve for x and y.

Solution

$4x+2y=14$ $5x+2y=16$	Notice the coefficients of each variable in each equation. You will need to add the opposite of one of the equations to eliminate the variable y, as $2y+2y=4y$, but $2y+(-2y)=0$.
$\begin{array}{rl} 4x+2y= & 14 \\ -5x-2y= & -16 \\ \hline -x \quad\;\; = & -2 \\ x \quad\;\; = & 2 \end{array}$	Change one of the equations to its opposite, add and solve for x.
$4x+2y=14$ $4(2)+2y=14$ $8+2y=14$ $2y=6$ $y=3$	Substitute $x=2$ into one of the original equations and solve for y.

The solution is $(2,3)$.

Go ahead and check this last example—substitute $(2,3)$ into both equations. You get two true statements: $14=14$ and $16=16$!

Notice that you could have used the opposite of the first equation rather than the second equation and gotten the same result.

Using Multiplication and Addition to Eliminate a Variables

Many times, adding the equations or adding the opposite of one of the equations will not result in eliminating a variable. Look at the system below.

$$\begin{aligned} 3x+4y&=52 \\ 5x+y&=30 \end{aligned}$$

If you add the equations above, or add the opposite of one of the equations, you will get an equation that still has two variables. So let's now use the multiplication property of equality first. You can multiply both sides of one of the equations by a number that will result in the coefficient of one of the variables being the opposite of the same variable in the other equation.

This is where multiplication comes in handy. The first equation contains the term $4y$ and the second equation contains the term y. If you multiply the second equation by -4, when you add both equations, the y variables will add up to 0.

$$\begin{array}{lclcl} 3x+4y=52 & \rightarrow & 3x+4y=52 & \rightarrow & 3x+4y=52 \\ 5x+y=30 & \rightarrow & -4(5x+y)=-4(30) & \rightarrow & \underline{-20x-4y=-120} \\ & & & & -17x+0y=-68 \end{array}$$

See the example below.

✓ Example

Solve for x and y.

Equation A: $3x+4y=52$

Equation B: $5x+y=30$

Solution

$3x+4y=52$ $5x+\ \ y=30$	Look for terms that can be eliminated. The equations do not have any x or y terms with the same coefficients.
$3x+4y=52$ $\underline{-4(5x+y)=-4(30)}$	Multiply the second equation by -4 so they have the same coefficient.
$3x+4y=\quad 52$ $\underline{-20x-4y=\ -120}$	Rewrite the system, and add the equations.
$-17x=-68$ $x=4$	Solve for x.
$3x+4y=52$ $3(4)+4y=52$ $12+4y=52$ $4y=40$ $y=10$	Substitute $x=4$ into one of the original equations to find y.
$3x+4y=52$ $3(4)+4(10)=52$ $12+40=52$ $52=52$ TRUE $5x+y=30$ $5(4)+10=30$ $20+10=30$ $30=30$ TRUE	Check your answer. The answers check.

The solution is $(4,10)$.

There are other ways to solve this system. Instead of multiplying one equation in order to eliminate a variable when the equations were added, you could have multiplied *both* equations by different numbers.

Let's remove the variable x this time. Multiply Equation A by 5 and Equation B by -3.

✓ Example

Solve for x and y.

$$\begin{aligned}3x+4y&=52\\5x+\ \ y&=30\end{aligned}$$

Solution

$3x+4y=52$ $5x+\ \ y=30$	Look for terms that can be eliminated. The equations do not have any x and y terms with the same coefficient.
$5(3x+4y)=5(52)$ $5x+y\ \ =0$ $15x+20y=260$ $5x+y=30$	In order to use the elimination method, you have to create variables that have the same coefficient—then you can eliminate them. Multiply the top equation by 5.
$15x+20y=260$ $-3(5x+\ \ y)=-3(30)$ $15x+20y=\quad 260$ $-15x-\ \ 3y=\quad -90$	Now multiply the bottom equation by -3.

$\begin{array}{rcr} 15x+20y & = & +260 \\ -15x-3y & = & -90 \\ \hline 17y & = & 170 \\ y & = & 10 \end{array}$	Next add the equations, and solve for y.
$3x+4y=52$ $3x+4(10)=52$ $3x+40=52$ $3x=12$ $x=4$	Substitute $y=10$ into one of the original equations to find x.
The solution is $(4,10)$.	You arrive at the same solution as before.

These equations were multiplied by 5 and -3 respectively, because that gave you terms that would add up to 0. Be sure to multiply all of the terms of the equation.

? Exercise

Felix needs to find x and y in the following system.

Equation A: $7y-4x=5$

Equation B: $3y+4x=25$

If he wants to use the elimination method to eliminate one of the variables, which is the most efficient way for him to do so?

A. Add Equation A and Equation B
B. Add $4x$ to both sides of Equation A
C. Multiply Equation A by 5
D. Multiply Equation B by -1

Answer

A. Correct. If Felix adds the two equations, the terms $4x$ and $-4x$ will cancel out, leaving $10y=30$. Felix will then easily be able to solve for y.
B. Incorrect. Adding $4x$ to both sides of Equation A will not change the value of the equation, nor will it help eliminate either of the variables—Felix will end up with the rewritten equation $7y=5+4x$. The correct answer is to add Equation A and Equation B.
C. Incorrect. Multiplying Equation A by 5 yields $35y-20x=25$, which does not help Felix eliminate any of the variables in the system. Felix may notice that now both equations have a constant of 25, but subtracting one from another is not an efficient way of solving this problem. Instead, it would create another equation where both variables are present. The correct answer is to add Equation A and Equation B.
D. Incorrect. Multiplying Equation B by -1 yields $-3y-4x=-25$, which does not help Felix eliminate any of the variables in the system. Felix may notice that now both equations have a term of $-4x$, but adding them would not eliminate them, it would give him a $-8x$. The correct answer is to add Equation A and Equation B.

Special Situations

Just as with the substitution method, the elimination method will sometimes eliminate *both* variables, and you end up with either a true statement or a false statement. Recall that a false statement means that there is no solution.

Let’s look at an example.

✓ Example

Solve for x and y.

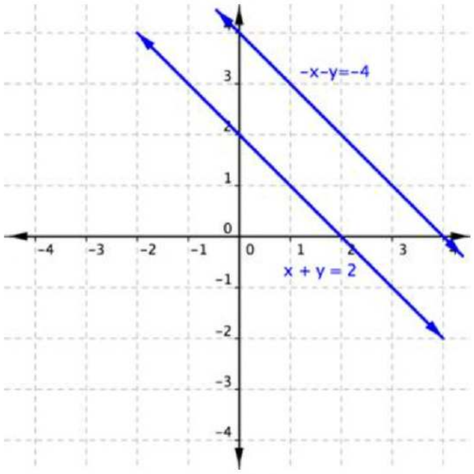
$$\begin{aligned} -x-y&=-4 \\ x+y&=2 \end{aligned}$$

Solution

$\begin{array}{rcr} -x-y & = & -4 \\ x+y & = & 2 \\ \hline 0 & = & -2 \end{array}$	Add the equations to eliminate the x-term.

There is no solution.

Graphing these lines shows that they are parallel lines and as such do not share any point in common, verifying that there is no solution.



If both variables are eliminated and you are left with a true statement, this indicates that there are an infinite number of ordered pairs that satisfy both of the equations. In fact, the equations are the same line.

✓ Example

Solve for x and y.

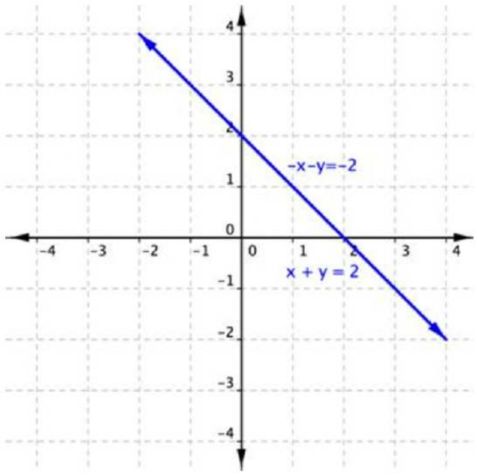
$$\begin{array}{rcr} x+y & = & 2 \\ -x-y & = & -2 \end{array}$$

Solution

$\begin{array}{rcr} x+y & = & 2 \\ -x-y & = & -2 \\ \hline 0 & = & 0 \end{array}$	Add the equations to eliminate the x-term.

There are an infinite number of solutions.

Graphing these two equations will help to illustrate what is happening.



Solving Application Problems Using the Elimination Method

The elimination method can be applied to solving systems of equations that model real situations. Two examples of using the elimination method in problem solving are shown below.

✓ Example

The sum of two numbers is 10. Their difference is 6. What are the two numbers?

Solution

$\begin{aligned} x+y &= 10 \\ x-y &= 6 \end{aligned}$	Write a system of equations to model the situation. x=one number y=the other number
$\begin{array}{rrcr} & x+y & = & 10 \\ + & x-y & = & 6 \\ \hline & 2x & = & 16 \\ & x & = & 8 \end{array}$	Add the equations to eliminate the y-term and then solve for x.
$\begin{aligned} x+y &= 10 \\ 8+y &= 10 \\ y &= 2 \end{aligned}$	Substitute the value for x into one of the original equations to find y.
$x+y=10$ $8+2=10$ $10=10$ TRUE $x-y=6$ $8-2=6$ $6=6$ TRUE	Check your answer by substituting $x=8$ and $y=2$ into the original system. The answers check.

The numbers are 8 and 2.

✓ Example

A theater sold 800 tickets for Friday night's performance. One child ticket costs $4.50 and one adult ticket costs $6.00. The total amount collected was $4,500. How many of each type of ticket were sold?

Solution

The total number of tickets sold is 800. $a+c=800$ *The amount of money collected* is $\$4,500$ $6a+4.5c=4,500$ System of equations: $a+c=800$ $6a+4.5c=4,500$	Write a system of equations to model the ticket sale situation. a=number of adult tickets sold c=number of child tickets sold
$6(a+c)=6(800)$ $6a+4.5c=4,500$ $6a+6c=4,800$ $6a+4.5c=4,500$	Use multiplication to re-write the first equation.
$\begin{array}{rcr} 6a+6c & = & 4,800 \\ -6a-4.5c & = & -4,500 \\ \hline 1.5c & = & 300 \end{array}$ $c=200$	Add the opposite of the second equation to eliminate a term and solve for c.
$\begin{array}{rcr} a+200 & = & 800 \\ -200 & & -200 \\ \hline a & = & 600 \end{array}$	Substitute 200 in for c in one of the original equations.
$a+c=800$ $600+200=800$ $800=800$ TRUE $6a+4.5c=4,500$ $6(600)+4.5(200)=4,500$ $3,600+900=4,500$ $4500=4,500$ TRUE	Check your answer by substituting $a=600$ and $c=200$ into the original system. The answers check.

600 adult tickets and 200 child tickets were sold.

Summary

Combining equations is a powerful tool for solving a system of equations. Adding or subtracting two equations in order to eliminate a common variable is called the elimination (or addition) method. Once one variable is eliminated, it becomes much easier to solve for the other one. Multiplication can be used to set up matching terms in equations before they are combined. When using multiplication in the elimination method, it is important to multiply all the terms on both sides of the equation—not just the one term you are trying to eliminate.

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SECTION OVERVIEW

14.3: Systems of Equations in Three or More Variables

Topic hierarchy

14.3.1: Solving Systems of Three Variables

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14.3.1: Solving Systems of Three Variables

Learning Objectives

- Solve a system of equations when no multiplication is necessary to eliminate a variable.
- Solve a system of equations when multiplication is necessary to eliminate a variable.
- Solve application problems that require the use of this method.
- Recognize systems that have no solution or an infinite number of solutions.

Introduction

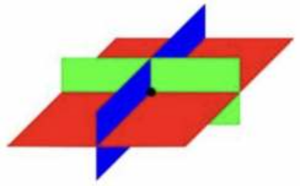
Equations can have more than one or two variables. You are going to look at equations with three variables. Equations with one variable graph on a line. Equations with two variables graph on a plane. Equations with three variables graph in a 3-dimensional space.

Equations with one variable require only one equation to have a unique (one) solution. Equations with two variables require two equations to have a unique solution (one ordered pair). So it should not be a surprise that equations with three variables require a system of three equations to have a unique solution (one ordered triplet).

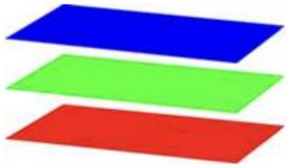
Solving A System of Three Variables

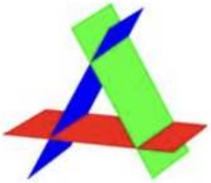
Just as when solving a system of two equations, there are three possible outcomes for the solution of a system of three variables. Let’s look at this visually, although you will not be graphing these equations.

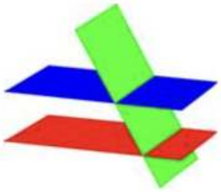
Case 1: There is one solution. In order for three equations with three variables to have one solution, the planes must intersect in a single point.



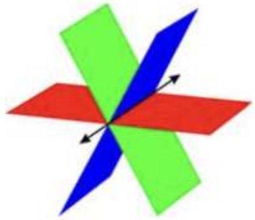
Case 2: There is no solution. The three planes do not have any points in common. (Note that two of the equations may have points in common with each other, but not all three.) Below are examples of some of the ways this can happen.

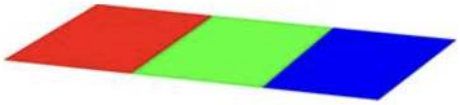






Case 3: There are an infinite number of solutions. This occurs when the three planes intersect in a line. And this can also occur when the three equations graph as the same plane.





Let’s start by looking at Case 1, where the system has a unique (one) solution. This is the case that you are usually most interested in.

Here is a system of linear equations. There are three variables and three equations.

$$\begin{aligned} 3x+4y-z&=8 \\ 5x-2y+z&=4 \\ 2x-2y+z&=1 \end{aligned}$$

You know how to solve a system with two equations and two variables. For the first step, use the elimination method to remove one of the variables. In this case, z can be eliminated by adding the first and second equations.

$$\begin{array}{rr} 3x+4y-z= & 8 \\ 5x-2y+z= & 4 \\ \hline 8x+2y\qquad = & 12 \end{array}$$

To solve the system, though, you need *two* equations using two variables. Adding the first and third equations in the original system will also give an equation with x and y but not z.

$$\begin{array}{r} 3x+4y-z=8 \\ 2x-2y+z=1 \\ \hline 5x+2y\qquad =9 \end{array}$$

Now you have a system of two equations and two variables.

$$\begin{aligned} 8x+2y&=12 \\ 5x+2y&=9 \end{aligned}$$

Solve the system using elimination again. In this case, you can eliminate y by adding the opposite of the second equation:

$$\begin{array}{rrcr} 8x+ & 2y & = & 12 \\ -5x+ & -2y & = & -9 \\ \hline 3x & & = & 3 \end{array}$$

Solve the resulting equation for the remaining variable.

$$\begin{aligned} 3x&=3 \\ x&=1 \end{aligned}$$

Now you use one of the equations in the two-variable system to find y.

$$\begin{aligned} 5x+2y&=9 \\ 5(1)+2y&=9 \\ 5+2y&=9 \\ 2y&=4 \\ y&=2 \end{aligned}$$

Finally, use *any* equation from the first system, along with the values already found, to solve for the last variable.

$$\begin{aligned} 2x-2y+z&=1 \\ 2(1)-2(2)+z&=1 \\ 2-4+z&=1 \\ -2+z&=1 \\ z&=3 \end{aligned}$$

Be sure to check your answer. With this many steps, there are a lot of places to make a simple error!

$$\begin{aligned} 3x+4y-z&=8 \\ 3(1)+4(2)-3&=8 \\ 3+8-3&=8 \\ 11-3&=8 \\ 8&=8 \\ \text{TRUE} \end{aligned}$$

$$\begin{aligned} 5x-2y+z&=4 \\ 5(1)-2(2)+3&=4 \\ 5-4+3&=4 \\ 1+3&=4 \\ 4&=4 \\ \text{TRUE} \end{aligned}$$

$$\begin{aligned} 2x-2y+z&=1\\ 2(1)-2(2)+3&=1\\ 2-4+3&=1\\ -2+3&=1\\ 1&=1\\ \text{TRUE}& \end{aligned}$$

Since $x=1$, $y=2$, and $z=3$ is a solution for all three equations, it's the solution for the system of equations. Just as two values can be written as an ordered pair, three values can be written as an ordered triplet: .

Solving a system of three variables

1. Choose two equations and use them to eliminate one variable.
2. Choose another pair of equations and use them to eliminate *the same* variable.
3. Use the resulting pair of equations from steps 1 and 2 to eliminate one of the two remaining variables.
4. Solve the final equation for the remaining variable.
5. Find the value of the second variable. Do this by using one of the resulting equations from steps 1 and 2 and the value of the found variable from step 4.
6. Find the value of the third variable. Do this by using one of the original equations and the values of the found variables from steps 4 and 5.
7. ***Check your answer in all three equations***!

Example

Solve for f, g, and h.

$$\begin{array}{rcr} f+g+h&=&13\\ f-h&=&-2\\ -2f+g&=&3 \end{array}$$

Solution

$\begin{array}{lcr} f+g+h&=&13\\ f\qquad -h&=&-2\\ \hline 2f\qquad +g&=&11 \end{array}$	Step 1: Choose two equations and eliminate a variable. The first two equations can be added to eliminate h.
$\begin{array}{rcr} 2f+g&=&11\\ -2f+g&=&3 \end{array}$	Step 2: The third equation has no h variable, so there's nothing to eliminate! You have a system of two equations and two variables.
$\begin{array}{rcr} 2f+\ g&=&11\\ -2f+\ g&=&3\\ \hline 2g&=&14 \end{array}$	Step 3: Eliminate a second variable. These equations can be added to eliminate f.
$\begin{aligned} 2g&=14\\ g&=7 \end{aligned}$	Step 4: Solve the resulting equation for the remaining variable.
$\begin{array}{lcr} 2f+g&=&11\\ 2f+7&=&11\\ 2f&=&4\\ f&=&2 \end{array}$	Step 5: Use that value and one of the equations from the system in step 3 that involves just two variables, one of which was g that you already know. Solve for the second variable.
$\begin{aligned} f+g+h&=13\\ 2+7+h&=13\\ 9+h&=13\\ h&=\ 4 \end{aligned}$	Step 6: Use the two found values and one of the original equations that had all three variables to solve for the third variable.

$\begin{array}{l} f+g+h=13 \\ 2+7+4=13 \\ 9+4=13 \\ 13=13 \\ \text{TRUE} \\ f-h=-2 \\ 2-4=-2 \\ -2=-2 \\ \text{TRUE} \\ -2f+g=3 \\ -2(2)+7=3 \\ -4+7=3 \\ 3=3 \\ \text{TRUE} \end{array}$	Step 7: ***Check your answer***.

The solution is $(f, g, h) = (2, 7, 4)$.

As with systems of two equations with two variables, you may need to add the opposite of one of the equations or even multiply one of the equations before adding in order to eliminate one of the variables.

Example

Solve for x, y, and z.

$$\begin{aligned} 3x-2y+\ \ z &= 12 \\ x+3y+\ \ z &= -4 \\ 2x+2y-4z &= 6 \end{aligned}$$

Solution

$\begin{array}{rcr} 3x-2y+z & = & 12 \\ -1(x+3y+z) & = & -1(-4) \\ \hline \end{array}$	Step 1: First, choose two equations and eliminate a variable. Multiply the second equation by -1, and then add it to the first equation. This will eliminate z.
$\begin{array}{rcr} 2x+2y-4z & = & 6 \\ 4(x+3y+z) & = & 4(-4) \\ \hline 2x+\ \ 2y-4z & = & 6 \\ 4x+12y+4z & = & -16 \\ \hline 6x+14y & = & -10 \end{array}$	Step 2: Next, combine the third equation and one of the first two to eliminate z again. However, the third equation has a coefficient of -4 on z while the coefficients in the first two equations are both 1. So, multiply the second equation by 4 and add.
$\begin{array}{rcr} 2x-\ \ 5y & = & 16 \\ 6x+14y & = & -10 \\ \hline \end{array}$	Step 3: Eliminate a second variable using the equations from steps 1 and 2. Again, they cannot be added as they are. Look at the coefficients on x. If you multiply the equation from step 1 by -3, the x terms will have the same coefficient.
$\begin{array}{rcr} -3(2x-5y) & = & -3(16) \\ 6x+14y & = & -10 \\ \hline -6x+\ \ 15y & = & -48 \\ 6x+\ \ 14y & = & -10 \\ \hline 29y & = & -58 \end{array}$	Multiply and then add. Be careful of the signs!
$\begin{array}{rcr} 29y & = & -58 \\ y & = & -2 \end{array}$	Step 4: Solve the resulting equation for the remaining variable.
$\begin{array}{l} 2x-5y=16 \\ 2x-5(-2)=16 \\ 2x+10=16 \\ 2x=6 \\ x=3 \end{array}$	Step 5: Use that value and one of the equations from the system in step 3, that involves just two variables, one of which was y. Solve for the second variable.

$x+3y+z=-4$ $3+3(-2)+z=-4$ $3+(-6)+z=-4$ $-3+z=-4$ $z=-1$	Step 6: Use the two found values and one of the original equations to solve for the third variable.
$3x-2y+z=12$ $3(3)-2(-2)+(-1)=12$ $9+4-1=12$ $13-1=12$ $12=12$ TRUE $x+3y+z=-4$ $3+3(-2)+(-1)=-4$ $3+(-6)+(-1)=-4$ $-3+(-1)=-4$ $-4=-4$ TRUE $2x+2y-4z=6$ $2(3)+2(-2)-4(-1)=6$ $6+(-4)+4=6$ $2+4=6$ $6=6$ TRUE	Step 7: ***Check your answer***.

The solution is $(x,y,z)=(3,-2,-1)$.

These systems can be helpful for solving real-world problems.

✓ Example

Andrea sells photographs at art fairs. She prices the photos according to size: small photos cost $10, medium photos cost $15, and large photos cost $40. She usually sells as many small photos as medium and large photos combined. She also sells twice as many medium photos as large. A booth at the art fair costs $300.

If her sales go as usual, how many of each size photo must she sell to pay for the booth?

Solution

S=number of small photos sold M=number of medium photos sold L=number of large photos sold $10S$=money received for small photos $15M$=money received for medium photos $40L$=money received for large photos	To set up the system, first choose the variables. In this case the unknown values are the number of small, medium, and large photos. The total of her sales must be $300 to pay for the booth.
$10S+15M+40L=300$ $S=M+L$ $M=2L$	The number of small photos is the same as the total of medium and large photos. She sells twice as many medium photos as large photos.
$10S+15M+40L=300$ $S-M-L=0$ $M-2L=0$	To make things easier, rewrite the equations to be in the same format, with all variables on the left side of the equal sign and only a constant number on the right. Now solve the system.
$\begin{array}{rrrcr} 10S+ & 15M+ & 40L & = & 300 \\ S- & M- & L & = & 0 \\ \hline \end{array}$ $\begin{array}{rrrcr} 10S+ & 15M+ & 40L & = & 300 \\ -10(S- & M- & L) & = & 10(0) \end{array}$	Step 1: First choose two equations and eliminate a variable. Since one equation has no S variable, it may be helpful to use the other two equations and eliminate the S variable from them.

$\begin{array}{rcl} 10S+15M+40L &=& 300 \\ -10S+10M+10L &=& 0 \\ \hline 25M+50L &=& 300 \end{array}$	Multiply the second equation by -10 and add.
$M-2L=0$	Step 2: The second equation for our two-variable system will be the remaining equation (that has no S variable).
$\begin{array}{rcl} 25M+50L &=& 300 \\ M-\ \ 2L &=& 0 \\ \hline \end{array}$	Step 3: Eliminate a second variable using the equations from steps 1 and 2.
$\begin{array}{lcl} \ \ M+2L &=& 12 \\ \ \ M-2L &=& 0 \\ \hline 2M &=& 12 \end{array}$	While you could multiply the second equation by 25 to eliminate L, the numbers will be easier to work with if you divide the first equation by 25. Don't forget to be careful of the signs!
$\begin{array}{l} 2M=12 \\ M=6 \end{array}$	Step 4: Solve the resulting equation for the remaining variable.
$\begin{array}{l} M=2L \\ 6=2L \\ 3=L \end{array}$	Step 5: Use that value and one of the equations containing just two variables, one of those variables being L, that you already know, to solve for the second variable.t's best to use one of the original equations—in case an error was made in multiplication.
$\begin{array}{l} S=M+L \\ S=6+3 \\ S=9 \end{array}$	Step 6: Use the two found values and one of the original equations to solve for the third variable. You can even use one of the equations before you rewrote it for the system.
She usually sells as many small photos as medium and large photos combined. Medium and large photos combined $=6+3=9$, which is the number of small photos.	
She also sells twice as many medium photos as large. Medium photos is 6, which is twice the number of large photos (3).	Step 7: ***Check your answer***. With application problems, it's sometimes easier (and better) to use the original wording of the problem rather than the equations you write.
A booth at the art fair costs \$300. Andrea receives \$10(9) or \$90 for the 9 small photos, \$15(6) or \$90 for the 6 medium photos, and \$40(3) or \$120 for the large photos. $\$90+\$90+\$120=\$300$.	

If Andrea sells small photos, medium photos, and large photos, she'll receive exactly the amount of money needed to pay for the booth.

? Exercise

In the solution to this system, what is the value of x?

$$\begin{array}{rcl} 7x-4y+3z &=& 28 \\ 3x+3y-\ \ z &=& 19 \\ 3x+2y+\ \ z &=& 16 \end{array}$$

A. 5
B. 16
C. -31
D. 1

Answer

A. Correct. Eliminate z by adding the last two equations together to get $6x+5y=35$. Now, multiply the second equation by 3 and add it to the first equation to get $16x+5y=85$. This creates a smaller system of two equations and two variables: $6x+5y=35$ and $16x+5y=85$. Multiply $6x+5y=35$ by -1 to create $-6x-5y=-35$ and now add this to $16x+5y=85$. This eliminates y, giving $10x=50$, so $x=5$.

B. Incorrect. Eliminate z by adding the last two equations together, to get $6x+5y=35$. Now, multiply the second equation by 3 and add it to the first equation to get $16x+5y=85$. This creates a smaller system of two equations and two variables: $6x+5y=35$ and $16x+5y=85$. Multiply $6x+5y=35$ by -1 to create $-6x-5y=-35$ and now add this to $16x+5y=85$. This eliminates y, giving $10x=50$, so $x=5$.

C. Incorrect. Eliminate z by adding the last two equations together, to get $6x+5y=35$. Now, multiply the second equation by 3 and add it to the first equation to get $16x+5y=85$. This creates a smaller system of two equations and two variables: $6x+5y=35$ and $16x+5y=85$. Multiply $6x+5y=35$ by -1 to create $-6x-5y=-35$ and now add this to $16x+5y=85$. This eliminates y, giving $10x=50$, so $x=5$.

D. Incorrect. Eliminate z by adding the last two equations together, to get $6x+5y=35$. Now, multiply the second equation by 3 and add it to the first equation to get $16x+5y=85$. This creates a smaller system of two equations and two variables: $6x+5y=35$ and $16x+5y=85$. Multiply $6x+5y=35$ by -1 to create $-6x-5y=-35$ and now add this to $16x+5y=85$. This eliminates y, giving $10x=50$, so $x=5$.

Systems with No Solutions or an Infinite Number of Solutions

Now let’s look at Case 2 (no solution) and Case 3 (an infinite number of solutions).

Since you will not graph these equations, as it is difficult to graph in three dimensions on a 2-dimensional sheet of paper, you will look at what happens when you try to solve systems with no solutions or an infinite number of solutions.

Let’s look at a system that has no solutions.

$$\begin{aligned}5x-2y+z&=3\\4x-4y-8z&=2\\-x+y+2z&=-3\end{aligned}$$

Suppose you wanted to solve this system, and you started with the last two equations. Multiply the third equation by 4 and add it to the second equation to eliminate x.

$$\begin{aligned}4x-4y-8z\quad&=\quad 2\\4(-x+y+2z)\quad&=4(-3)\end{aligned}$$

$$\begin{array}{r}4x-4y-8z=2\\-4x+4y+8z=-12\\\hline 0=-10\end{array}$$

In this case, the result is a *false* statement. This means there are no solutions to the two equations and therefore there can be no solutions for the system of three equations. If this occurs for any two of the three equations, then there is no solution for the system of equations.

Now let’s look at a system that has an infinite number of solutions.

$$\begin{aligned}x-2y+z&=3\\-3x+6y-3z&=-9\\4x-8y+4z&=12\end{aligned}$$

For the first step, you would choose two equations and combine them to eliminate a variable. You can eliminate x by multiplying the first equation by 3 and adding to the second equation.

$$\begin{array}{r}3(x-2y+z)=3(3)\\-3x+6y-3z=-9\\\hline\end{array}$$

$$\begin{array}{r}3x-6y+3z=9\\-3x+6y-3z=-9\\\hline 0=0\end{array}$$

Notice that when the two equations are added, all variables are eliminated! The final equation is a true statement: $0=0$.

When this happens, it's because the two equations are *equivalent*. These two equations would graph as the same plane. But in order for the solution to the system of three equations to be infinite, you need to continue to check with the third equation.

Since the first two equations are equivalent, the system of equations could be written with only two equations. Continue as before. Multiply the first equation by -4 and add the third equation.

$$4x-8y+4z=-12$$

$$\begin{array}{r}-4x+8y-4z=\quad -12\\ 4x-8y+4z=\quad 12\\ \hline 0=\quad 0\end{array}$$

Again, the final equation is the true statement $0=0$. So the third equation is the same plane as the first two. Now you can confirm that there are an infinite number of solutions—all of the points that are on the plane that these three equations each describe.

This is one type of situation where there are an infinite number of solutions. There are others, which you will not examine at this time.

✓ Example

How many solutions does the following system of equations have?

$$\begin{aligned}x+y+z&=2\\ 2x+2y+2z&=4\\ -3x-3y-3z&=-6\end{aligned}$$

Solution

$\begin{array}{r}-2(x+y+z)=-2(2)\\ 2x+2y+2z=\qquad 4\\ \hline\end{array}$	Multiply the first equation by -2 and then add that resulting equation to the second equation.
$\begin{array}{r}-2x-2y-2z=-4\\ 2x+2y+2z=\quad 4\\ \hline 0=\quad 0\end{array}$	$0=0$ is a true statement, which leads us to believe that you may have an infinite number of solutions. This outcome indicates that the first pair of equations is really the same equation. The values of x, y, and z that will make the first equation work will also work for the second.
$\begin{array}{r}3(x+y+z)=\ 3(2)\\ -3x-3y-3z=\ -6\\ \hline\end{array}$	Now add the third equation with the first.
$\begin{array}{r}3x+3y+3z=\quad 6\\ -3x-3y-3z=-6\\ \hline 0=\quad 0\end{array}$	Again, the result is another true statement. The first and third equations are the same. So you have three equations that will all graph as the same plane.

There are an infinite number of solutions to this system.

✓ Example

Solve the following system of equations.

$$\begin{aligned}x-y-2z&=4\\ 4x-4y-z&=2\\ -x+y+2z&=-3\end{aligned}$$

Solution

$\begin{array}{r}-4(x-y-2z)=-4(4)\\ 4x-4y-8z=\qquad 2\\ \hline\end{array}$	Compare the coefficients on the x terms. Multiply the first equation by -4 and then add that resulting equation to the second equation.
$\begin{array}{r}-4x+4y+8z=-16\\ 4x-4y-8z=\qquad 2\\ \hline 0=-14\end{array}$	Notice that a false statement is produced: $0=-14$. This means that there is no solution to this system of equations; you do not have to complete any further steps.

The system has no solutions.

Exercise

How many solutions does this system have?

$$\begin{aligned} 6x + 4y + 2z &= 32 \\ 3x - 3y - z &= 19 \\ 3x + 2y + z &= 32 \end{aligned}$$

A. No solutions
B. One
C. An infinite number of solutions

Answer

A. Correct. Multiply the last equation by -2 to get $-6x - 4y - 2z = -64$. If you add this equation to the first one, you will get $0 = -32$, a false statement. This means that this system has no solutions.
B. Incorrect. If you multiply the last equation by -2 and then add it to the first equation, you get $0 = -32$, a false statement. This system has no solutions.
C. Incorrect. If you multiply the last equation by -2 and then add it to the first equation, you get $0 = -32$, a false statement. This system has no solutions.

Summary

Combining equations is a powerful tool for solving a system of equations, including systems with three equations and three variables. Sometimes, you must multiply one of the equations before you add so that you can eliminate a variable. You continue the process of combining equation and eliminating variables until you have found the value of all of the variables. Occasionally this process leads to all of the variables being eliminated (eliminated not solved for). When all the variables are eliminated by combining equations, if it leads to a false statement, then the system will have no solutions. When all the variables are eliminated by combining equations, if one of the resulting equations is true, the system *may* have an infinite number of solutions. However, all the equations must be compared and found to true for there to be an infinite number of solutions, not just two of the three equations.

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CHAPTER OVERVIEW

15: Rational Expressions

15.1: Operations with Rational Expressions
  15.1.1: Introduction to Rational Expressions
  15.1.2: Multiplying and Dividing Rational Expressions
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SECTION OVERVIEW

15.1: Operations with Rational Expressions

Topic hierarchy

15.1.1: Introduction to Rational Expressions

15.1.2: Multiplying and Dividing Rational Expressions

15.1.3: Adding and Subtracting Rational Expressions

15.1.4: Complex Rational Expressions

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15.1.1: Introduction to Rational Expressions

Learning Objectives

- Find values of a variable that make a rational expression undefined.
- Simplify rational expressions.

Introduction

Rational expressions are fractions that have a polynomial in the numerator, denominator, or both. Although rational expressions can seem complicated because they contain variables, they can be simplified in the same way that numeric fractions, also called numerical fractions, are simplified.

Finding the Domain of an Expression

The first step in simplifying a rational expression is to determine the **domain**, the set of all possible values of the variables. The denominator in a fraction cannot be zero because division by zero is undefined. The reason $\frac{6}{3}=2$ is that when you multiply the answer 2, times the divisor 3, you get back 6. To be able to divide any number c by zero $\left(\frac{c}{0}=?\right)$, you would have to find a number that when you multiply it by 0 you would get back c (what times zero equals c?). There are no numbers that can do this, so we say "division by zero is undefined". In simplifying rational expressions, you need to pay attention to what values of the variable(s) in the expression would make the denominator equal zero. These values cannot be included in the domain, so they're called **excluded values**. Discard them right at the start, before you go any further.

(Note that although the *denominator* cannot be equivalent to 0, the *numerator* can. This is why you only look for excluded values in the denominator of a rational expression.)

For rational expressions, the domain will exclude values for which the value of the denominator is 0. Two examples to illustrate finding the domain of an expression are shown below.

Example

Identify the domain of the expression.

$\frac{3x+2}{x-4}$

Solution

$x-4=0$	Find any values for x that would make the denominator equal 0.
$x=4$	When $x=4$, the denominator is equal to 0.

The domain is all real numbers, except 4.

You found that x cannot be 4. (Sometimes you may see this idea presented as "$x\neq 4$.") What happens if you do substitute that value into the expression?

$\frac{3x+2}{x-4}$

$\frac{3(4)+2}{(4)-4}$

$\frac{12+2}{0}$

$\frac{14}{0}$

You find that when $x=4$, the numerator evaluates to 14, but the denominator evaluates to 0. And since division by 0 is undefined, this must be an excluded value.

Let's try one that's a little more challenging.

Example

Identify the domain of the expression.

$\frac{x+7}{x^2+8x-9}$

Solution

$x^2+8x-9=0$	Find any values for x that would make the denominator equal to 0 by setting the denominator equal to 0 and solving the equation.
$(x+9)(x-1)=0$ $x=-9$ or $x=1$	Solve the equation by factoring. The solutions are the values that are excluded from the domain.

The domain is all real numbers except -9 and 1.

Exercise

Find the domain of the rational expression $\frac{5x}{2x+8}$.

A. all real numbers except -4
B. all real numbers except 4
C. all real numbers except 0
D. all real numbers

Answer

A. Correct. When $x=-4$, the denominator is $2(-4)+8=-8+8=0$. Division by 0 is undefined, so the domain must exclude $x=-4$.
B. Incorrect. When $x=4$, the denominator does not equal 0, therefore it is not an excluded value. Set the denominator equal to 0 and solve for x. The correct answer is all real numbers except -4.
C. Incorrect. When $x=0$, the numerator equals 0 but the denominator does not, therefore it is not an excluded value. Set the denominator equal to 0 and solve for x. The correct answer is all real numbers except -4.
D. Incorrect. There is one value of x that will make the denominator 0. Set the denominator equal to 0 and solve for x. The correct answer is all real numbers except -4.

Simplifying Rational Expressions

Once you've figured out the excluded values, the next step is to simplify the rational expression. To simplify a rational expression, follow the same approach you use to simplify numeric fractions: find common factors in the numerator and denominator. Let’s start by simplifying a numeric fraction.

Example

Simplify. $\frac{15}{27}$

Solution

$\frac{15}{27}=\frac{3\cdot 5}{3\cdot 3\cdot 3}$	Factor the numerator and denominator.
$\frac{3\cdot 5}{3\cdot 3\cdot 3}$	Identify fractions that equal 1, and then pull them out of the fraction.
$\frac{5}{3\cdot 3}\cdot\frac{3}{3}$ $\frac{5}{3\cdot 3}\cdot 1$	In this fraction, the factor 3 is in both the numerator and denominator. Recall that $\frac{3}{3}$ is another name for 1.
$\frac{5}{3\cdot 3}=\frac{5}{9}$	Simplify.

$\frac{15}{27}=\frac{5}{9}$

Now, you could have done that problem in your head, but it was worth writing it all down because that's exactly how you simplify a rational expression.

So let's simplify a rational expression, using the same technique you applied to the fraction $\frac{15}{27}$. Only this time, the numerator and denominator are both monomials with variables.

✓ Example

Simplify. $\frac{5x^2}{25x}$

Solution

$\frac{5x^2}{25x}=\frac{5\cdot x\cdot x}{5\cdot 5\cdot x}$	Factor the numerator and denominator.
$\frac{5\cdot x\cdot x}{5\cdot 5\cdot x}$ $\frac{x}{5}\cdot\frac{5\cdot x}{5\cdot x}$ $\frac{x}{5}\cdot 1$	Identify fractions that equal 1, and then pull them out of the fraction.
$\frac{x}{5}$	Simplify.

$\frac{5x^2}{25x}=\frac{x}{5}$

The same steps worked again. Factor the numerator, factor the denominator, identify factors that are common to the numerator and denominator and write as a factor of 1, and simplify.

When simplifying rational expressions, it is a good habit to always consider the domain, and to find the values of the variable (or variables) that make the expression undefined. (This will come in handy when you begin *solving* for variables a bit later on.)

✓ Example

Identify the domain of the expression.

$\frac{5x^2}{25x}$

Solution

$25x=0$	Find any values for x that would make the denominator equal to 0 by setting the denominator equal to 0 and solving the equation.
$\frac{25x}{25}=\frac{0}{25}$ $x=0$	The values for x that make the denominator equal 0 are excluded from the domain.

The domain is all real numbers except 0.

Notice that you started with the *original* expression, and identified values of x that would make $25x$ equal to 0. Why does this matter? When $\frac{5x^2}{25x}$ is simplified, it is the fraction $\frac{x}{5}$. Since 5 is the denominator, it seems that no values need to be excluded from the domain. When finding the domain of an expression, you always start with the original expression because variable terms may be factored out as part of the simplification process.

In the examples that follow, the numerator and the denominator are polynomials with more than one term, but the same principles of simplifying will once again apply. Factor the numerator and denominator to simplify the rational expression.

✓ Example

Simplify and state the domain for the expression.

$\frac{x+3}{x^2+12x+27}$

Solution

$x^2+12x+27=0$	

$(x+3)(x+9)=0$ $x+3=0 \quad \text{or} \quad x+9=0$ $x=0-3 \quad \text{or} \quad x=0-9$ $x=-3 \quad \text{or} \quad x=-9$ $x=-3$ or $x=-9$ domain is all real numbers except -3 and -9	To find the domain (and the excluded values), find the values for which the denominator is equal to 0. Factor the quadratic to find the values.
$\frac{x+3}{x^2+12x+27}$	Factor the numerator and denominator.
$\frac{x+3}{(x+3)(x+9)}$	Identify the factors that are the same in the numerator and denominator.
$\frac{x+3}{x+3} \cdot \frac{1}{x+9}$ $1 \cdot \frac{1}{x+9}$	Write as separate fractions, pulling out fractions that equal 1. Simplify.

$\frac{x+3}{x^2+12x+27} = \frac{1}{x+9}$

The domain is all real numbers except -3 and -9.

Example

Simplify and state the domain for the expression.

$\frac{x^2+10x+24}{x^3-x^2-20x}$

Solution

$x^3-x^2-20x=0$	
$x\left(x^2-x-20\right)=0$ $x(x-5)(x+4)=0$	To find the domain, determine the values for which the denominator is equal to 0.
domain is all real numbers except 0, 5, and -4	
$\frac{x^2+10x+24}{x^3-x^2-20x}$ $\frac{(x+4)(x+6)}{x(x-5)(x+4)}$ $\frac{x+6}{x(x-5)} \cdot \frac{(x+4)}{(x+4)}$	To simplify, factor the numerator and denominator of the rational expression. Identify the factors that are the same in the numerator and denominator. Write as separate fractions, pulling out fractions that equal 1.
$\frac{x+6}{x(x-5)}$ or $\frac{x+6}{x^2-5x}$	Simplify. It is acceptable to either leave the denominator in factored form or to distribute with multiplication.

$\frac{x+6}{x(x-5)}$ or $\frac{x+6}{x^2-5x}$

The domain is all real numbers except 0, 5, and -4.

Steps for Simplifying a Rational Expression

To simplify a rational expression, follow these steps:

- Determine the domain. The excluded values are those values for the variable that result in the expression having a denominator of 0.
- Factor the numerator and denominator.
- Find common factors for the numerator and denominator and simplify.

Exercise

Simplify the rational expression below.

$\frac{2x^2+13x+15}{2x^2+23x+30}$

[Note: While the domain and excluded values of a rational expression are important, you will not always be asked to find them when simplifying a rational expression. In this expression, the domain is all real numbers except $\frac{3}{2}$ and -10.]

A. $\frac{14}{25}$
B. $\frac{x+5}{x+10}$
C. $\frac{2x}{3}$
D. $\frac{1}{2}$

Answer

A. Incorrect. You must first factor the polynomials in the numerator and the denominator and then express like factors in the numerator and denominator as 1 to simplify. The expression can be factored as $\frac{(2x+3)(x+5)}{(2x+3)(x+10)}$, so the correct answer is $\frac{x+5}{x+10}$.
B. Correct. The rational expression can be simplified by factoring the numerator and denominator as $\frac{(2x+3)(x+5)}{(2x+3)(x+10)}$. Since $\frac{2x+3}{2x+3}=1$, simplify the expression to $\frac{x+5}{x+10}$.
C. Incorrect. You must first factor the polynomials in the numerator and the denominator and then express like factors in the numerator and denominator as 1 to simplify. The expression can be factored as $\frac{(2x+3)(x+5)}{(2x+3)(x+10)}$, so the correct answer is $\frac{x+5}{x+10}$.
D. Incorrect. You cannot simplify $\frac{x+5}{x+10}$ to $\frac{1}{2}$ because the x in the numerator and the x in the denominator are not being multiplied, they are being added. The correct answer is $\frac{x+5}{x+10}$.

Summary

Rational expressions are fractions containing polynomials. They can be simplified much like numeric fractions. To simplify a rational expression, first determine common factors of the numerator and denominator, and then remove them by rewriting them as expressions equal to 1.

An additional consideration for rational expressions is to determine what values are excluded from the domain. Since division by 0 is undefined, any values of the variables that result in a denominator of 0 must be excluded. Excluded values must be identified in the original equation, not from its factored form.

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15.1.2: Multiplying and Dividing Rational Expressions

Learning Objectives

- Multiply rational expressions and simplify.
- Divide rational expressions and simplify.

Introduction

Just as you can multiply and divide fractions, you can multiply and divide **rational expressions**. In fact, you use the same processes for multiplying and dividing rational expressions as you use for multiplying and dividing numeric fractions. The process is the same even though the expressions look different!

Multiplication of Rational Expressions

Remember that there are two ways to multiply numeric fractions.

One way is to multiply the numerators and the denominators and then simplify the product, as shown here.

$\frac{4}{5} \cdot \frac{9}{8} = \frac{36}{40} = \frac{3 \cdot 3 \cdot 2 \cdot 2}{5 \cdot 2 \cdot 2 \cdot 2} = \frac{3 \cdot 3}{5 \cdot 2} \cdot \frac{2 \cdot 2}{2 \cdot 2} = \frac{3 \cdot 3}{5 \cdot 2} \cdot 1 = \frac{9}{10}$

A second way is to factor and simplify the fractions *before* performing the multiplication.

$\frac{4}{5} \cdot \frac{9}{8} = \frac{2 \cdot 2}{5} \cdot \frac{3 \cdot 3}{2 \cdot 2 \cdot 2} = \frac{2 \cdot 2}{2 \cdot 2} \cdot \frac{3 \cdot 3}{5 \cdot 2} = 1 \cdot \frac{3 \cdot 3}{5 \cdot 2} = \frac{9}{10}$

Notice that both methods result in the same product. In some cases, you may find it easier to multiply and then simplify, while in others it may make more sense to simplify fractions before multiplying.

The same two approaches can be applied to rational expressions. In the following examples, both techniques are shown. First, let's multiply and then simplify.

Example

Multiply. State the product in simplest form.

$\frac{5a^2}{14} \cdot \frac{7}{10a^3}$

Solution

$\frac{5a^2}{14} \cdot \frac{7}{10a^3} = \frac{35a^2}{140a^3}$	Multiply the numerators, and then multiply the denominators.
$\frac{35a^2}{140a^3} = \frac{5 \cdot 7 \cdot a^2}{5 \cdot 7 \cdot 2 \cdot 2 \cdot a^2 \cdot a}$	Simplify by finding common factors in the numerator and denominator.
$= \frac{5 \cdot 7 \cdot a^2}{5 \cdot 7 \cdot a^2} \cdot \frac{1}{2 \cdot 2 \cdot a}$ $= 1 \cdot \frac{1}{4a}$	Use the common factors to rewrite as multiplication by 1.
$\frac{1}{4a}$	Simplify.

$\frac{5a^2}{14} \cdot \frac{7}{10a^3} = \frac{1}{4a}$

Okay, that worked. But this time, let's simplify first, then multiply. When using this method, it helps to look for the **greatest common factor**. You can factor out *any* common factors, but finding the greatest one will take fewer steps.

Example

Multiply.

State the product in simplest form.

$\frac{5a^2}{14} \cdot \frac{7}{10a^3}$

Solution

$\frac{5 \cdot a^2}{7 \cdot 2} \cdot \frac{7}{5 \cdot 2 \cdot a^2 \cdot a}$	Factor the numerators and denominators. Look for the greatest common factors.
$\frac{5}{5} \cdot \frac{a^2}{a^2} \cdot \frac{7}{7} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{a}$ $1 \cdot 1 \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{a}$	Regroup the fractions to express common factors as multiplication by 1, and then multiply.

$$\frac{5a^2}{14} \cdot \frac{7}{10a^3} = \frac{1}{4a}$$

Both methods produced the same answer.

Also, remember that when working with rational expressions, you should get into the habit of identifying any values for the variables that would result in division by 0. These **excluded values** must be eliminated from the **domain**, the set of all possible values of the variable. In the example above, $\frac{5a^2}{14} \cdot \frac{7}{10a^3}$, the domain is all real numbers where a is not equal to 0. When $a = 0$, the denominator of the fraction $\frac{7}{10a^3}$ equals 0, which will make the fraction undefined.

Some rational expressions contain quadratic expressions and other multi-term polynomials. To multiply these rational expressions, the best approach is to first factor the polynomials and then look for common factors. (Multiplying the terms before factoring will often create complicated polynomials, and then you will have to factor these polynomials anyway! For this reason, it is easier to factor, simplify, and then multiply.) Just take it step by step, like in the examples below.

✓ Example

Multiply. State the product in simplest form.

$\frac{a^2-a-2}{5a} \cdot \frac{10a}{a+1}, a \neq -1, 0$

Solution

$\frac{(a-2)(a+1)}{5 \cdot a} \cdot \frac{5 \cdot 2 \cdot a}{(a+1)}$	Factor the numerators and denominators.
$\frac{(a+1)}{(a+1)} \cdot \frac{5}{5} \cdot \frac{a}{a} \cdot \frac{(a-2)}{1} \cdot \frac{2}{1}$	Regroup to express rational expressions equivalent to 1.
$1 \cdot 1 \cdot 1 \cdot \frac{(a-2)}{1} \cdot \frac{2}{1}$ $2(a-2)$	Multiply simplified rational expressions. This expression can be left with the numerator in factored form or multiplied out.

$$\frac{a^2-a-2}{5a} \cdot \frac{10a}{a+1} = 2a - 4$$

✓ Example

Multiply. State the product in simplest form.

$\frac{a^2+4a+4}{2a^2-a-10} \cdot \frac{a+5}{a^2+2a}, a \neq -2, 0, \frac{5}{2}$

Solution

$\frac{(a+2)(a+2)}{(2a-5)(a+2)} \cdot \frac{a+5}{a(a+2)}$	Factor the numerators and denominators.
$\frac{a+2}{a+2} \cdot \frac{a+2}{a+2} \cdot \frac{1}{(2a-5)} \cdot \frac{a+5}{a}$	Regroup to express rational expressions equivalent to 1.
$1 \cdot 1 \cdot \frac{1}{(2a-5)} \cdot \frac{a+5}{a} = \frac{a+5}{a(2a-5)}$	Multiply simplified rational expressions. This expression can be left with the denominator in factored form or multiplied out.

$$\frac{a^2+4a+4}{2a^2-a-10} \cdot \frac{a+5}{a^2+2a} = \frac{a+5}{a(2a-5)}$$

Note that in the answer above, you cannot simplify the rational expression any further. It may be tempting to express the $5s$ in the numerator and denominator as the fraction $\frac{5}{5}$, but these $5s$ are terms because they are being added or subtracted. Remember that only common factors, not terms, can be regrouped to form factors of 1!

Exercise

Multiply, and express the product as a simplified rational expression.

$\frac{y^2-4}{y^2-2y} \cdot \frac{y}{y^2+10y+16}, y \neq -8, -2, 0, 2$

A. $\frac{2}{y^2+26y}$
B. $\frac{y}{y(y+8)}$
C. $\frac{(y-2)y}{(y^2-2y)(y+8)}$
D. $\frac{1}{y+8}$

Answer

A. Incorrect. You simplify by factoring the numerators and denominators. The fraction $\frac{y^2-4}{y^2-2y}$ can be written as $\frac{(y+2)(y-2)}{y(y-2)}$, and the fraction $\frac{y}{y^2+10y+16}$ can be written as $\frac{y}{(y+8)(y+2)}$.
B. Incorrect. This expression is equivalent, but it can be further simplified since there is a common factor of y in the numerator and denominator. The correct answer is $\frac{1}{y+8}$.
C. Incorrect. This expression is equivalent, but it can be further simplified since there are common factors in the numerator and denominator: $y^2-2y=y(y-2)$. The correct answer is $\frac{1}{y+8}$.
D. Correct. Factoring the numerators and denominators, you get $\frac{(y-2)(y+2)}{y(y-2)} \cdot \frac{y}{(y+8)(y+2)}$. Regrouping, you get $\frac{y-2}{y-2} \cdot \frac{y+2}{y+2} \cdot \frac{y}{y} \cdot \frac{1}{y+8} = \frac{1}{y+8}$.

Dividing Rational Expressions

You've seen that you multiply rational expressions as you multiply numeric fractions. It should come as no surprise that you also divide rational expressions the same way you divide numeric fractions. Specifically, to divide rational expressions, keep the first rational expression, change the division sign to multiplication, and then take the reciprocal of the second rational expression.

Let’s begin by recalling division of numerical fractions.

$\frac{2}{3} \div \frac{5}{9} = \frac{2}{3} \cdot \frac{9}{5} = \frac{18}{15} = \frac{6}{5}$

Use the same process to divide rational expressions. You can think of division as multiplication by the reciprocal, and then use what you know about multiplication to simplify.

You do still need to think about the domain, specifically the variable values that would make either denominator equal zero. But there's a new consideration this time because you divide by multiplying by the reciprocal of one of the rational expressions, you also need to find the values that would make the *numerator* of that expression equal zero. Have a look.

Example

Identify the domain of the expression.

$\frac{5x^2}{9} \div \frac{15x^3}{27}$

Solution

$15x^3=0$ $x=0$ is an excluded value.	Find excluded values. 9 and 27 can never equal 0. Because $15x^3$ becomes the denominator in the reciprocal of $\frac{15x^3}{27}$, you must find the values of x that would make $15x^3$ equal 0.

The domain is all real numbers except 0.

Knowing how to find the domain may seem unimportant here, but it will help you when you learn how to solve rational equations. To divide, multiply by the reciprocal.

✓ Example

Divide. State the quotient in simplest form.

$\frac{5x^2}{9}\div\frac{15x^3}{27}$

Solution

$\frac{5x^2}{9}\cdot\frac{27}{15x^3}$	Rewrite division as multiplication by the reciprocal.
$\frac{5\cdot x\cdot x}{3\cdot 3}\cdot\frac{3\cdot 3\cdot 3}{5\cdot 3\cdot x\cdot x\cdot x}$	Factor the numerators and denominators.
$\frac{5\cdot x\cdot x\cdot 3\cdot 3}{5\cdot x\cdot x\cdot 3\cdot 3}\cdot\frac{1}{x}$	Reorder factors and express as multiplication by 1.
$1\cdot\frac{1}{x}$	Simplify.

$\frac{5x^2}{9}\div\frac{15x^3}{27}=\frac{1}{x}, x\neq 0$

✓ Example

Divide. State the quotient in simplest form, and express the domain of the expression.

$\frac{3x^2}{x+2}\div\frac{6x^4}{x^2+5x+6}$

Solution

$(x+2)=0$ $x=-2$	
$(x^2+5x+6)=0$ $(x+3)(x+2)=0$ $x=-3 \text{ or } -2$ $6x^4=0$ $x=0$	Determine the excluded values that make the denominators and the numerator of the divisor equal to 0.
Domain is all real numbers except 0, -2, and -3.	
$\frac{3x^2}{x+2}\cdot\frac{x^2+5x+6}{6x^4}$	Rewrite division as multiplication by the reciprocal.
$\frac{3\cdot x\cdot x}{x+2}\cdot\frac{(x+2)(x+3)}{2\cdot 3\cdot x\cdot x\cdot x\cdot x}$	Factor the numerators and denominators.
$\frac{3\cdot x\cdot x\cdot(x+2)}{3\cdot x\cdot x\cdot(x+2)}\cdot\frac{x+3}{2\cdot x\cdot x}$ $1\cdot\frac{(x+3)}{2x^2}$	Reorder factors and express as multiplication by 1. Simplify.

$\frac{3x^2}{x+2}\div\frac{6x^4}{x^2+5x+6}=\frac{x+3}{2x^2}$

The domain is all real numbers except 0, -2, and -3.

Notice that once you rewrite the division as multiplication by a reciprocal, you follow the same process you used to multiply rational expressions.

? Exercise

Find the quotient and express as a simplified rational expression.

$\frac{(y+2)(y+1)}{4}\div\frac{y+2}{y+1}$, domain is all real numbers except -2 and -1.

A. $\frac{(y+2)^2}{4}$
B. $\frac{(y+1)^2}{4}$
C. $\frac{y^2+y+1}{4}$
D. $\frac{y^2+3y+3}{4y+4}$

Answer

A. Incorrect. This operation is division, not multiplication. The first step in division is to rewrite it as multiplication by the reciprocal. The correct answer is $\frac{(y+1)^2}{4}$.
B. Correct. Division is the same as multiplication by the reciprocal, so this problem can be written: $\frac{(y+2)(y+1)}{4} \cdot \frac{y+1}{y+2} = \frac{(y+1)^2}{4}$.
C. Incorrect. You appear to have taken the reciprocal, factored correctly, and simplified your common factors to 1, getting $\frac{(y+2)}{(y+2)} \cdot \frac{(y+1)}{4} \cdot \frac{y+1}{1} = 1 \cdot \frac{(y+1)}{4} \cdot \frac{y+1}{1}$. But you need to multiply the two numerators together $\frac{(y+1)}{4} \cdot \frac{y+1}{1} = \frac{(y+1)}{4} \cdot \frac{(y+1)}{1}$. The correct answer is $\frac{(y+1)^2}{4}$.
D. Incorrect. This is a division problem. The first step in division is to rewrite it as multiplication by the reciprocal. You also appear to have simplified incorrectly. The correct answer is $\frac{(y+1)^2}{4}$.

Summary

Rational expressions are multiplied and divided the same way as numeric fractions. To multiply, first find the greatest common factors of the numerator and denominator. Next, regroup the factors to make fractions equivalent to one. Then, multiply any remaining factors. To divide, first rewrite the division as multiplication by the reciprocal of the denominator. The steps are then the same as for multiplication.

When expressing a product or quotient, it is important to state the excluded values. These are all values of a variable that would make a denominator equal zero at any step in the calculations.

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15.1.3: Adding and Subtracting Rational Expressions

Learning Objectives

- Add rational expressions and simplify.
- Subtract rational expressions and simplify.
- Find the Least Common Multiple of several algebraic expressions.
- Simplify problems that combine both adding and subtracting.

Introduction

In beginning math, students usually learn how to add and subtract whole numbers before they are taught multiplication and division. However, with fractions and **rational expressions**, multiplication and division are sometimes taught first because these operations are easier to perform than addition and subtraction. Addition and subtraction of rational expressions are not as easy to perform as multiplication because, as with numeric fractions, the process involves finding common denominators. By working carefully and writing down the steps along the way, you can keep track of all of the numbers and variables and perform the operations accurately.

Adding and Subtracting Rational Expressions with Like Denominators

Adding rational expressions with the same denominator is the simplest place to start, so let’s begin there.

To add fractions with like denominators, add the numerators and keep the same denominator. Then simplify the sum. You know how to do this with numeric fractions.

$$\frac{2}{9}+\frac{4}{9}=\frac{6}{9}$$
$$\frac{6}{9}=\frac{3\cdot 2}{3\cdot 3}=\frac{3}{3}\cdot\frac{2}{3}=1\cdot\frac{2}{3}=\frac{2}{3}$$

Follow the same process to add rational expressions with like denominators. Let's try one.

Example

Add. State the sum in simplest form.

$\frac{2x^2}{x+4}+\frac{8x}{x+4}$

Solution

$\frac{2x^2+8x}{x+4}$	Since the denominators are the same, add the numerators. Remember that x cannot be -4 because the denominators would be 0.
$\frac{2x(x+4)}{x+4}$	Factor the numerator.
$\frac{x+4}{x+4}\cdot 2x$ $1\cdot 2x$	Rewrite the common factor as multiplication by 1 and simplify.

$\frac{2x^2}{x+4}+\frac{8x}{x+4}=2x, x\neq -4$

Remember that you should also describe the **domain**, the set of all possible values for the variables. The **excluded values** of the domain are any values of the variable(s) that result in any denominator being equal to 0. In the problem above, the domain is all real numbers except -4, since a value of $x=-4$ will create a denominator of 0. Sometimes, when we simplify an expression, the reader looking only at the simplified answer would not realize that there are excluded values. In the example above, just looking at the simplified form of $2x$ as a replacement for the original $\frac{2x^2}{x+4}+\frac{8x}{x+4}$, the reader would have no way of knowing that a value of -4 can’t be used for x. So when we claim that $2x$ is the equivalent of $\frac{2x^2}{x+4}+\frac{8x}{x+4}$, we need to state that -4 is an excluded value.

To subtract rational expressions with like denominators, follow the same process you use to subtract fractions with like denominators. The process is just like the addition of rational expressions, except that you subtract instead of add.

Example

Subtract. State the difference in simplest form.

$\frac{4x+7}{x+6}-\frac{2x+8}{x+6}$

Solution

$\frac{4x+7-(2x+8)}{x+6}$	Subtract the second numerator from the first and keep the denominator the same. Remember that x cannot be -6 because the denominators would be 0.
$\frac{4x+7-2x-8}{x+6}$	Be careful to distribute the negative to both terms of the second numerator.
$\frac{2x-1}{x+6}$	Combine like terms. This rational expression cannot be simplified any further.

$\frac{4x+7}{x+6}-\frac{2x+8}{x+6}=\frac{2x-1}{x+6}$

Exercise

Subtract, and state the difference in simplest form.

$\frac{x^2}{x-5}-\frac{25}{x-5}, x\neq 5$

A. $\frac{x^2-25}{x-5}$
B. $x+5$
C. $x-5$
D. 5

Answer

A. Incorrect. You performed the subtraction properly, but this rational expression can be simplified because the numerator and denominator have a common factor of $(x-5)$. The correct answer is $x+5$.
B. Correct. Since there is a common denominator, subtract the numerators to get $\frac{x^2-25}{x-5}$. The numerator can be factored and a common factor of $(x-5)$ is present in numerator and denominator. $\frac{(x-5)(x+5)}{x-5}=\frac{x-5}{x-5}\cdot(x+5)=x+5$.
C. Incorrect. The common factor present in the numerator and denominator is $x-5$, not $x+5$. After factoring, you get: $\frac{(x-5)(x+5)}{x-5}=\frac{x-5}{x-5}\cdot(x+5)$. The correct answer is $x+5$.
D. Incorrect. To find the difference, subtract the numerator of the second fraction from the numerator of the first, like this: $\frac{x^2-25}{x-5}$. Then factor the numerator and simplify. The correct answer is $x+5$.

Adding and Subtracting Rational Expressions with Unlike Denominators

Before adding and subtracting rational expressions with *unlike* denominators, you need to find a common denominator. Once again, this process is similar to the one used for adding and subtracting numeric fractions with unlike denominators. Let's look at a numeric example to start.

$\frac{5}{6}+\frac{8}{10}+\frac{3}{4}$

Since the denominators are 6, 10, and 4, you want to find the **least common denominator** and express each fraction with this denominator before adding. (By the way, you can add fractions by finding *any* common denominator; it does not have to be the least. You focus on using the *least* because then there is less simplifying to do. But either way works.)

Finding the least common denominator is the same as finding the **least common multiple** of 4, 6, and 10. There are a couple of ways to do this. The first is to list the multiples of each number and determine which multiples they have in common. The least of these numbers will be the least common denominator.

Multiples of 4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64
Multiples of 6	12	18	24	30	36	42	48	54	60	66	68				
Multiples of 10	20	30	40	50	60										

The other method is to use **prime factorization**, the process of finding the prime factors of a number. This is how the method works with numbers.

Example

Use prime factorization to find the least common multiple of 6, 10, and 4.

Solution

$6=3\cdot 2$ $10=5\cdot 2$ $4=2\cdot 2$	First, find the prime factorization of each denominator.
$6=3\cdot 2$ $10=5\cdot 2$ $4=2\cdot 2$	The LCM will contain factors of 2, 3, and 5. Multiply each number the maximum number of times it appears in a single factorization.
$\text{Least Common Multiple }=3\cdot 5\cdot 2\cdot 2$	In this case, 3 appears once, 5 appears once, and 2 is used twice because it appears twice in the prime factorization of 4.
	Therefore, the LCM of 6, 10, and 4 is $3\cdot 5\cdot 2\cdot 2$, or 60.

The least common multiple of 6, 10, and 4 is 60.

You found the same least common multiple using both methods. Prime factorization was faster, though, because you did not have to make a chart full of multiples to find a common multiple.

Now that you have found the least common multiple, you can use that number as the least common denominator of the fractions. Multiply each fraction by the fractional form of 1 that will produce a denominator of 60:

$$\begin{aligned}\frac{5}{6}\cdot\frac{10}{10}&=\frac{50}{60}\\ \frac{8}{10}\cdot\frac{6}{6}&=\frac{48}{60}\\ \frac{3}{4}\cdot\frac{15}{15}&=\frac{45}{60}\end{aligned}$$

Now that you have like denominators, add the fractions:

$$\frac{50}{60}+\frac{48}{60}+\frac{45}{60}=\frac{143}{60}$$

You can find least common denominators for rational expressions, too, and use them to add rational expressions with unlike denominators:

Example

Add. State the sum in simplest form.

$$\frac{2n}{15m^2}+\frac{3n}{21m}$$

Solution

$15m^2=3\cdot 5\cdot m\cdot m$ $21m=3\cdot 7\cdot m$	Find the prime factorization of each denominator.

$15m^2 = 3 \cdot 5 \cdot \mathbf{m} \cdot \mathbf{m}$ $21\ \mathrm{m} = 3 \cdot 7 \cdot \mathrm{m}$ $\mathrm{LCM}: 3 \cdot 5 \cdot 7 \cdot \mathrm{m} \cdot \mathrm{m}$ $\text{LCM: } 105\ \mathrm{m}^2$	Find the least common multiple. 3 appears exactly once in both of the expressions, so it will appear once in the least common multiple. Both 5 and 7 appear at most once. For the variables, the most m appears is twice. Use the least common multiple for your new common denominator, it will be the LCD. Compare each original denominator and the new common denominator. Now rewrite the rational expressions to each have the common denominator of $105m^2$. Remember that m cannot be 0 because the denominators would be 0.
$\frac{2n}{15m^2} \cdot \frac{7}{7} = \frac{14n}{105m^2}$	The first denominator is $15m^2$ and the LCD is $105m^2$. You need to multiply $15m^2$ by 7 to get the LCD, so multiply the entire rational expression by $\frac{7}{7}$.
$\frac{3n}{21m} \cdot \frac{5m}{5m} = \frac{15mn}{105m^2}$	The second denominator is $21m$ and the LCD is $105m^2$. You need to multiply $21m$ by $5m$ to get the LCD, so multiply the entire rational expression by $\frac{5m}{5m}$.
$\frac{14n}{105m^2} + \frac{15mn}{105m^2} = \frac{14n+15mn}{105mn^2}$	Add the numerators and keep the denominator the same.
$\frac{n(14+15m)}{105m^2}$	If possible, simplify by finding common factors in the numerator and denominator. This rational expression is already in simplest form because the numerator and denominator have no factors in common.

$$\frac{2n}{15m^2} + \frac{3n}{21m} = \frac{n(14+15m)}{105m^2}, m \neq 0$$

That took a while, but you got through it. Adding rational expressions can be a lengthy process, but taken one step at a time, it can be done.

Now let's try subtracting rational expressions. You'll use the same basic technique of finding the least common denominator and rewriting each rational expression to have that denominator.

✓ Example

Subtract. State the difference in simplest form.

$$\frac{2}{t+1} - \frac{t-2}{t^2-t-2}$$

Solution

$t+1 = t+1$ $t^2-t-2 = (t-2)(t+1)$	Find the prime factorization of each denominator. $t+1$ cannot be factored any further, but t^2-t-2 can be. Remember that t cannot be -1 or 2 because the denominators would be 0.
$t+1 = t+1$	Find the least common multiple. $t+1$ appears exactly once in both of the expressions, so it will appear once in the least common denominator. $t-2$ also appears once.
$t^2-t-2 = (t-2)(t+1)$	This means that $(t-2)(t+1)$ is the least common multiple. In this case, it is easier to leave the common multiple in terms of the factors, so you will not multiply it out.
$\text{LCM: } (t+1)(t-2)$	Use the least common multiple for your new common denominator, it will be the LCD. Compare each original denominator and the new common denominator. Now rewrite the rational expressions to each have the common denominator of $(t+1)(t-2)$.

$\frac{2}{t+1} \cdot \frac{t-2}{t-2} = \frac{2(t-2)}{(t+1)(t-2)}$	You need to multiply $t+1$ by $t-2$ to get the LCD, so multiply the entire rational expression by $\frac{t-2}{t-2}$.
$\frac{t-2}{t^2-t-2} = \frac{t-2}{(t+1)(t-2)}$	The second expression already has a denominator of $(t+1)(t-2)$, so you do not need to multiply it by anything.
$\frac{2(t-2)}{(t+1)(t-2)} - \frac{t-2}{(t+1)(t-2)}$	Then rewrite the subtraction problem with the common denominator.
$\frac{2(t-2)-(t-2)}{(t+1)(t-2)}$ $\frac{2t-4-t+2}{(t+1)(t-2)}$ $\frac{t-2}{(t+1)(t-2)}$	Subtract the numerators and simplify. Remember that parentheses need to be included around the second $(t-2)$ in the numerator because the whole quantity is subtracted. Otherwise you would be subtracting just the t.
$\frac{t-2}{(t-2)(t+1)}$ $\frac{t-2}{t-2} \cdot \frac{1}{t+1}$ $1 \cdot \frac{1}{t+1}$	The numerator and denominator have a common factor of $t-2$, so the rational expression can be simplified.

$$\frac{2}{t+1} - \frac{t-2}{t^2-t-2} = \frac{1}{t+1}, t \neq -1, 2$$

So far, all the rational expressions you've added and subtracted have shared some factors. What happens when they don't have factors in common?

✓ Example

Subtract. State the difference in simplest form.

$$\frac{3y}{2y-1} - \frac{4}{y-5}$$

Solution

$\text{LCM} = (2y-1)(y-5)$	Neither $2y-1$ nor $y-5$ can be factored. Because they have no common factors, the least common multiple, which will become the least common denominator, is the product of these denominators. Remember that y cannot be $\frac{1}{2}$ or 5 because the denominators would be 0.
$\frac{3y}{2y-1} \cdot \frac{y-5}{y-5} = \frac{3y(y-5)}{(2y-1)(y-5)}$	Multiply each expression by the equivalent of 1 that will give it the common denominator.
$\frac{4}{y-5} \cdot \frac{2y-1}{2y-1} = \frac{4(2y-1)}{(2y-1)(y-5)}$ $\frac{3y(y-5)}{(2y-1)(y-5)} - \frac{4(2y-1)}{(2y-1)(y-5)}$ $\frac{3y^2-15y}{(2y-1)(y-5)} - \frac{8y-4}{(2y-1)(y-5)}$	Then rewrite the subtraction problem with the common denominator. It makes sense to keep the denominator in factored form in order to check for common factors.
$\frac{3y^2-15y-(8y-4)}{(2y-1)(y-5)}$ $\frac{3y^2-15y-8y+4}{(2y-1)(y-5)}$	Subtract and simplify.

$$\frac{3y}{2y-1} - \frac{4}{y-5} = \frac{3y^2-23y+4}{2y^2-11y+5}, y \neq \frac{1}{2}, 5$$

? Exercise

Add. State the sum in simplest form. $\frac{x}{x+4} + \frac{3}{x-3}$

A. $\frac{x(x-3)+3(x+4)}{(x+4)(x-3)}$
B. $\frac{x+3}{2x+1}$
C. $\frac{1}{x}$
D. $\frac{x^2+12}{(x+4)(x-3)}$

Answer

A. Incorrect. The approach is correct, but the answer has not been simplified. The numerator of the rational expression can be simplified by multiplying and combining like terms. The correct answer is $\frac{x^2+12}{(x+4)(x-3)}$.

B. Incorrect. To add rational expressions with unlike denominators, you must first find a common denominator. The common denominator for these rational expressions is $(x+4)(x-3)$ because the denominators do not have any common factors. Write both addends with a common denominator, $\frac{x(x-3)}{(x+4)(x-3)}+\frac{3(x+4)}{(x+4)(x-3)}$, and then simplify. The correct answer is $\frac{x^2+12}{(x+4)(x-3)}$.

C. Incorrect. You can only simplify numerator and denominator when there are like *factors*, not like *terms*. You cannot cancel the x^2 terms and $12s$. The correct answer is $\frac{x^2+12}{(x+4)(x-3)}$.

D. Correct. First find a common denominator, $(x+4)(x-3)$, and rewrite each addend using that denominator: $\frac{x(x-3)}{(x+4)(x-3)}$ $+\frac{3(x+4)}{(x+4)(x-3)}$. Multiply and add the numerators: $\frac{x^2-3x+3x+12}{(x+4)(x-3)}$.

Combining Multiple Rational Expressions

You may need to combine more than two rational expressions. While this may seem pretty straightforward if they all have the same denominator, what happens if they do not?

In the example below, notice how a common denominator is found for three rational expressions. Once that is done, the addition and subtraction of the terms looks the same as earlier, when you were only dealing with two terms.

✓ Example

Simplify. State the result in simplest form.

$\frac{2x^2}{x^2-4}+\frac{x}{x-2}-\frac{1}{x+2}$

Solution

$x^2-4=(x+2)(x-2)$ $x-2=x-2$ $x+2=x+2$	Find the least common multiple by factoring each denominator. Multiply each factor the maximum number of times it appears in a single factorization. Remember that x cannot be 2 or -2 because the denominators would be 0.
$\text{Least Common Multiple } =(x+2)(x-2)$	$(x+2)$ appears a maximum of one time, as does $(x-2)$. This means the LCM is $(x+2)(x-2)$.
$\frac{2x^2}{x^2-4}=\frac{2x^2}{(x+2)(x-2)}$ $\frac{x}{x-2}\cdot\frac{x+2}{x+2}=\frac{x(x+2)}{(x+2)(x-2)}$ $\frac{1}{x+2}\cdot\frac{x-2}{x-2}=\frac{1(x-2)}{(x+2)(x-2)}$	The LCM becomes the common denominator. Multiply each expression by the equivalent of 1 that will give it the common denominator.
$\frac{2x^2}{(x+2)(x-2)}+\frac{x(x+2)}{(x+2)(x-2)}-\frac{1(x-2)}{(x+2)(x-2)}$	Rewrite the original problem with the common denominator. It makes sense to keep the denominator in factored form in order to check for common factors.
$\frac{2x^2+x(x+2)-1(x-2)}{(x+2)(x-2)}$ $\frac{2x^2+x^2+2x-x+2}{(x+2)(x-2)}$	Combine the numerators.
$\frac{3x^2+x+2}{(x+2)(x-2)}$	Check for simplest form. Since neither $(x+2)$ nor $(x-2)$ is a factor of $3x^2+x+2$, this expression is in simplest form.

$\frac{2x^2}{x^2-4}+\frac{x}{x-2}-\frac{1}{x+2}=\frac{3x^2+x+2}{(x+2)(x-2)}, x \neq 2,-2$

✓ Example

Simplify. State the result in simplest form.

$\frac{y^2}{3y}-\frac{2}{x}-\frac{15}{9}$

Solution

$3y = 3 \cdot y$ $x = x$ $9 = 3 \cdot 3$ $\text{Least Common Multiple } = 3 \cdot 3 \cdot x \cdot y$ $\text{Least Common Multiple } = 9xy$	Find the least common multiple by factoring each denominator. Multiply each factor the maximum number of times it appears in a single factorization. Remember that x and y cannot be 0 because the denominators would be 0.
$\frac{y^2}{3y} \cdot \frac{3x}{3x} = \frac{3xy^2}{9xy}$ $\frac{2}{x} \cdot \frac{9y}{9y} = \frac{18y}{9xy}$ $\frac{15}{9} \cdot \frac{xy}{xy} = \frac{15xy}{9xy}$	The LCM becomes the common denominator. Multiply each expression by the equivalent of 1 that will give it the common denominator.
$\frac{3xy^2}{9xy} - \frac{18y}{9xy} - \frac{15xy}{9xy}$	Rewrite the original problem with the common denominator.
$\frac{3xy^2-18y-15xy}{9xy}$ $\frac{3y(xy-6-5x)}{9xy}$	Combine the numerators.
$\frac{3y(xy-6-5x)}{3y \cdot 3x}$ $\frac{3y}{3y} \cdot \frac{xy-6-5x}{3x}$ $1 \cdot \frac{xy-6-5x}{3x}$	Check for simplest form.

$$\frac{y^2}{3y} - \frac{2}{x} - \frac{15}{9} = \frac{xy-5x-6}{3x}, y \neq 0, x \neq 0$$

Summary

To add and subtract rational expressions, apply the same ideas you use to add and subtract numeric fractions: first find a common denominator. The least common denominator is the same as the least common multiple and can be found by listing multiples of each denominator or through prime factorization. Then use that common denominator to rewrite the fractions so that you can add or subtract them.

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15.1.4: Complex Rational Expressions

Learning Objectives

- Simplify complex rational expressions.

Introduction

Fractions and rational expressions can be interpreted as quotients. When both the dividend (numerator) and divisor (denominator) include fractions or rational expressions, you have something more *complex* than usual. Don't fear! You have all the tools you need to simplify these quotients!

Complex Fractions

A **complex fraction** is the quotient of two fractions. These complex fractions are never considered to be in simplest form, but they can always be simplified using division of fractions.

Example

Simplify. $\frac{\frac{3}{4}}{\frac{9}{10}}$

Solution

$\frac{\frac{3}{4}}{\frac{9}{10}}=\frac{3}{4}\div\frac{9}{10}$	Rewrite the complex fraction as a division problem.
$=\frac{3}{4}\cdot\frac{10}{9}$ $=\frac{30}{36}$	Rewrite the division as multiplication, using the reciprocal of the divisor.
$=\frac{5\cdot 6}{6\cdot 6}$ $=\frac{5}{6}\cdot\frac{6}{6}$ $=\frac{5}{6}\cdot 1$	Multiply, and simplify if possible.

$$\frac{\frac{3}{4}}{\frac{9}{10}}=\frac{5}{6}$$

Before you multiply the numbers, it's often helpful to factor the numbers. You can then use the factors to create a fraction equal to 1.

Example

Simplify. $\frac{\frac{12}{35}}{\frac{6}{7}}$

Solution

$\frac{\frac{12}{35}}{\frac{6}{7}}=\frac{12}{35}\div\frac{6}{7}$	Rewrite the complex fraction as a division problem.
$=\frac{12}{35}\cdot\frac{7}{6}$	Rewrite the division as multiplication, using the reciprocal of the divisor.
$=\frac{2.6\cdot 7}{5\cdot 7\cdot 6}$ $=\frac{2}{5}\cdot\frac{6\cdot 7}{6\cdot 7}$ $=\frac{2}{5}\cdot 1$	Factor the numerator and denominator, looking for common factors, before multiplying numbers together.

$$\frac{\frac{12}{35}}{\frac{6}{7}}=\frac{2}{5}$$

If two fractions appear in the numerator or denominator (or both), first combine them. Then simplify the quotient as shown above.

Example

Simplify. $\frac{\frac{3}{5}+\frac{1}{5}}{\frac{4}{7}-\frac{1}{7}}$

Solution

$\frac{\frac{3}{5}+\frac{1}{5}}{\frac{4}{7}-\frac{1}{7}}=\frac{\frac{4}{5}}{\frac{3}{7}}$	First combine the numerator and denominator by adding or subtracting.
$\frac{\frac{4}{5}}{\frac{3}{7}}=\frac{4}{5}\div\frac{3}{7}$	Rewrite the complex fraction as a division problem.
$=\frac{4}{5}\cdot\frac{7}{3}$	Rewrite the division as multiplication, using the reciprocal of the divisor.
$=\frac{28}{15}$	Multiply and simplify as needed.

$$\frac{\frac{3}{5}+\frac{1}{5}}{\frac{4}{7}-\frac{1}{7}}=\frac{28}{15}$$

Exercise

Simplify. $\frac{\frac{21}{52}}{\frac{10}{13}}$

A. $\frac{105}{338}$
B. $\frac{273}{520}$
C. $\frac{21}{40}$
D. $\frac{21}{10}$

Answer

A. Incorrect. You may have multiplied $\frac{21}{52}$ by $\frac{10}{13}$ instead of dividing. The complex fraction is equivalent to $\frac{21}{52}\div\frac{10}{13}$, which is $\frac{21}{52}\cdot\frac{13}{10}$. The correct answer is $\frac{21}{40}$.
B. Incorrect. You correctly found that the complex fraction is equivalent to $\frac{21}{52}\cdot\frac{13}{10}$, but this can be further simplified by removing the common factor of 13 in the numerator and denominator: $\frac{21}{4(13)}\cdot\frac{13}{10}$. The correct answer is $\frac{21}{40}$.
C. Correct. The complex fraction is equivalent to $\frac{21}{52}\div\frac{10}{13}$, which is $\frac{21}{52}\cdot\frac{13}{10}$. Now remove the common factor of 13 from the numerator and denominator: $\frac{21}{4(13)}\cdot\frac{13}{10}$. This leaves $\frac{21}{40}$ as the simplified fraction.
D. Incorrect. Before you try to remove any common factors, rewrite the complex fraction as $\frac{21}{52}\div\frac{10}{13}$, which is $\frac{21}{52}\cdot\frac{13}{10}$. Then remove the common factor of 13 from the numerator and denominator: $\frac{21}{4(13)}\cdot\frac{13}{10}$. The correct answer is $\frac{21}{40}$.

Complex Rational Expressions

A **complex rational expression** is a quotient with rational expressions in the dividend, divisor, or in both. Simplify these in the exact same way as you would a complex fraction.

Example

Simplify. $\frac{\frac{x+5}{x^2-16}}{\frac{x^2-25}{x-4}}$

Solution

$=\frac{x+5}{x^2-16}\div\frac{x^2-25}{x-4}$	Rewrite the complex expression as a division problem.
$=\frac{x+5}{x^2-16}\cdot\frac{x-4}{x^2-25}$	Rewrite the division as multiplication, using the reciprocal of the divisor. Note that the excluded values for this are -4, 4, -5 and 5, because those values make the denominators of one of the fractions zero.

$=\frac{(x+5)(x-4)}{(x+4)(x-4)(x+5)(x-5)}$ $=\frac{(x+5)(x-4)}{(x+5)(x-4)}\cdot\frac{1}{(x+4)(x-5)}$	Factor the numerator and denominator, looking for common factors. In this case, $x+5$ and $x-4$ are common factors of the numerator and denominator. Notice that $\frac{(x+5)(x-4)}{(x+5)(x-4)}$ is equal to 1.
$\frac{\frac{x+5}{x^2-16}}{\frac{x^2-25}{x-4}}=\frac{1}{(x+4)(x-5)}$	$x\neq-4,4,-5,5$

The same ideas can be used when simplifying complex rational expressions that include more than one rational expression in the numerator or denominator. However, there is a shortcut that can be used. Compare these two examples of simplifying a complex fraction.

✓ Example

Simplify. $\frac{1-\frac{9}{x^2}}{1+\frac{5}{x}+\frac{6}{x^2}}$

Solution

$=\frac{\frac{x^2}{x^2}-\frac{9}{x^2}}{\frac{x^2}{x^2}+\frac{5x}{x^2}+\frac{6}{x^2}}$ $=\frac{\frac{x^2-9}{x^2}}{\frac{x^2+5x+6}{x^2}}$	Combine the expressions in the numerator and denominator. To do this, rewrite the expressions using a common denominator. There is an excluded value of 0 because this makes the denominators of the fractions zero.
$=\frac{x^2-9}{x^2}\div\frac{x^2+5x+6}{x^2}$	Rewrite the complex rational expression as a division problem. (When you are comfortable with the step of rewriting the complex rational fraction as a division problem, you might skip this step and go straight to rewriting it as multiplication.)
$=\frac{x^2-9}{x^2}\cdot\frac{x^2}{x^2+5x+6}$	Rewrite the division as multiplication, using the reciprocal of the divisor.
$=\frac{(x+3)(x-3)x^2}{x^2(x+3)(x+2)}$ $=\frac{x-3}{x+2}\cdot\frac{x^2(x+3)}{x^2(x+3)}$	Factor the numerator and denominator, looking for common factors. In this case, $x+3$ and x^2 are common factors. We can now see there are two additional excluded values, -2 and -3.

$\frac{1-\frac{9}{x^2}}{1+\frac{5}{x}+\frac{6}{x^2}}=\frac{x-3}{x+2}\quad x\neq-3,-2,0$

✓ Example

Simplify. $\frac{1-\frac{9}{x^2}}{1+\frac{5}{x}+\frac{6}{x^2}}$

Solution

$=\frac{1-\frac{9}{x^2}}{1+\frac{5}{x}+\frac{6}{x^2}}\cdot\frac{x^2}{x^2}$	Before combining the expressions, find a common denominator for all of the rational expressions. (In this case, x^2 is a common denominator.)
$=\frac{\left(1-\frac{9}{x^2}\right)x^2}{\left(1+\frac{5}{x}+\frac{6}{x^2}\right)x^2}$ $=\frac{x^2-9}{x^2+5x+6}$	Multiply by 1 in the form of a fraction with the common denominator in both numerator and denominator. (In this case, multiply by $\frac{x^2}{x^2}$.) There is an excluded value of 0 because this makes the denominators of the fractions zero.
$=\frac{(x+3)(x-3)}{(x+3)(x+2)}$ $=\frac{x+3}{x+3}\cdot\frac{x-3}{x+2}$ $=1\cdot\frac{x-3}{x+2}$	Notice that the expression is no longer complex! You can simplify by factoring and identifying common factors. We can now see there are two additional excluded values, -2 and -3.
$\frac{1-\frac{9}{x^2}}{1+\frac{5}{x}+\frac{6}{x^2}}=\frac{x-3}{x+2}$	$x\neq-3,-2,0$

You may find the second method easier to use. Try both methods on the following problem.

? Exercise

Simplify. $\frac{1+\frac{2}{3x}}{\frac{2}{x^2}+\frac{3}{x}}$

A. 1
B. $\frac{x}{3}$
C. $\frac{3x^2+2x}{6+9x}$
D. $\frac{x^2+2x}{2+3x}$

Answer

A. Incorrect. You probably tried to multiply the expression using a common denominator, but used one multiplier $(3x)$ for the expressions in the numerator and a different one (x^2) for the expressions in the denominator. The common denominator of all terms in both the numerator and denominator is $3x^2$. Multiplying the expression by $\frac{3x^2}{3x^2}$ gives $\frac{3x^2+2x}{6+9x}$. The numerator and denominator in this expression have a common factor of $(3x+2)$, so the correct answer is $\frac{x}{3}$.
B. Correct. The common denominator of all terms in both numerator and denominator is $3x^2$. Multiplying the expression by $\frac{3x^2}{3x^2}$ gives $\frac{3x^2+2x}{6+9x}$. The numerator and denominator in this expression have a common factor of $(3x+2)$, so the correct answer is $\frac{x}{3}$.
C. Incorrect. You have correctly simplified the complex rational expression to the simple rational expression $\frac{3x^2+2x}{6+9x}$, but this can be simplified further. The numerator and denominator in this expression have a common factor of $(3x+2)$, so the correct answer is $\frac{x}{3}$.
D. Incorrect. You may have used x^2 rather than $3x^2$ as the common denominator of all the rational expressions, and forgotten about the 3 in the denominator of $\frac{2}{3x}$. Multiplying the expression by $\frac{3x^2}{3x^2}$ gives $\frac{3x^2+2x}{6+9x}$. The numerator and denominator in this expression have a common factor of $(3x+2)$, so the correct answer is $\frac{x}{3}$.

Summary

Complex rational expressions are quotients with rational expressions in the divisor, dividend, or both. When written in fractional form, they appear to be fractions within a fraction. These can be simplified by first treating the quotient as a division problem. Then you can rewrite the division as multiplication using the reciprocal of the divisor. Or you can simplify the complex rational expression by multiplying both the numerator and denominator by a denominator common to all rational expressions within the complex expression. This can help simplify the complex expression even faster.

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SECTION OVERVIEW

15.2: Rational Equations

Topic hierarchy

15.2.1: Solving Rational Equations and Applications

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15.2.1: Solving Rational Equations and Applications

Learning Objectives

- Solve rational equations.
- Check for extraneous solutions.
- Solve application problems involving rational equations.

Introduction

Equations that contain rational expressions are called **rational equations**. For example, $\frac{2x+1}{4}=\frac{x}{3}$ is a rational equation.

You can solve these equations using the techniques for performing operations with rational expressions and the procedures for solving algebraic equations. Rational equations can be useful for representing real-life situations and for finding answers to real problems. In particular, they are quite good for describing distance-speed-time relationships and for modeling work problems that involve more than one person.

Solving Rational Equations

One method for solving rational equations is to rewrite the rational expressions in terms of a common denominator. Then, since you know the numerators are equal, you can solve for the variable. To illustrate this, let's look at a very simple equation.

$$\frac{2}{5}=\frac{x}{5}$$

Since the denominator of each expression is the same, the numerators must be equivalent. This means that $x=2$.

This is true for rational equations with polynomials too.

$$\frac{x-5}{x+4}=\frac{11}{x+4}$$

Since the denominators of each rational expression are the same, $x+4$, the numerators must be equivalent for the equation to be true. So, $x-5=11$ and $x=16$.

Just as with other algebraic equations, you can check your solution in the original rational equation by substituting the value for the variable back into the equation and simplifying.

$$\begin{aligned}\frac{16-5}{16+4}&=\frac{11}{16+4}\\ \frac{11}{20}&=\frac{11}{20}\end{aligned}$$

When the terms in a rational equation have unlike denominators, solving the equation will involve some extra steps. One way of solving rational equations with unlike denominators is to multiply both sides of the equation by the least common multiple of the denominators of all the fractions contained in the equation. This eliminates the denominators and turns the rational equation into a polynomial equation. Here's an example.

Example

Solve the equation $\frac{x+5}{8}=\frac{7}{4}$

Solution

$4=2\cdot2$ $8=2\cdot2\cdot2$ Least Common Multiple $=2\cdot2\cdot2$ Least Common Multiple $=8$	Find the least common multiple (LCM) of 4 and 8. Remember, to find the LCM, identify the greatest number of times each factor appears in each factorization. Here, 2 appears 3 times, so $2\cdot2\cdot2$, or 8, will be the LCM.
$8\cdot\frac{x+5}{8}=\frac{7}{4}\cdot8$ $\frac{8(x+5)}{8}=\frac{7(8)}{4}$	The LCM of 4 and 8 is also the lowest common denominator for the two fractions.
$\frac{8}{8}\cdot(x+5)=\frac{7(4\cdot2)}{4}$ $\frac{8}{8}\cdot(x+5)=7\cdot2\cdot\frac{4}{4}$ $1\cdot(x+5)=14\cdot1$	Multiply both sides of the equation by the common denominator, 8, to keep the equation balanced and to eliminate the denominators.

$x+5=14$ $x=9$	Simplify and solve for x.
$\frac{x+5}{8}=\frac{7}{4}$ $\frac{9+5}{8}=\frac{7}{4}$ $\frac{14}{8}=\frac{7}{4}$ $\frac{7}{4}=\frac{7}{4}$	Check the solution by substituting 9 for x in the original equation.

$x=9$

Another way to solve a rational equation with unlike denominators is to rewrite each term with a common denominator and then just create an equation from the numerators. This works because if the denominators are the same, the numerators must be equal. The next example shows this approach with the same equation you just solved:

✓ Example

Solve the equation $\frac{x+5}{8}=\frac{7}{4}$.

Solution

$\frac{x+5}{8}=\frac{7}{4}\cdot\frac{2}{2}$ $\frac{x+5}{8}=\frac{14}{8}$	Multiply the right side of the equation by $\frac{2}{2}$ to get a common denominator of 8. (Multiplying by $\frac{2}{2}$ is the same as multiplying by 1, so the equation stays balanced.)
$x+5=14$ $x=9$	Since the denominators are the same, the numerators must be equal for the equation to be true. Solve for x.

$x=9$

In some instances, you’ll need to take some additional steps in finding a common denominator. Consider the example below, which illustrates using what you know about denominators to rewrite one of the expressions in the equation.

✓ Example

Solve the equation $\frac{x}{3}+1=\frac{4}{3}$.

Solution

$\frac{x}{3}+1=\frac{4}{3}$ $\frac{x}{3}+\frac{3}{3}=\frac{4}{3}$ $\frac{x+3}{3}=\frac{4}{3}$	Rewrite the expression using a common denominator.
$x+3=4$ $x=1$	Since the denominator for each expression is 3, the numerators must be equal.
$\frac{1}{3}+1=\frac{4}{3}$ $\frac{1}{3}+\frac{3}{3}=\frac{4}{3}$	Check the solution in the original equation.
$\frac{4}{3}=\frac{4}{3}$	

$x=1$

You could also solve this problem by multiplying each term in the equation by 3 to eliminate the fractions altogether. Here is how it would look.

Example

Solve the equation $\frac{x}{3}+1=\frac{4}{3}$**.**

Solution

$3\left(\frac{x}{3}+1\right)=3\left(\frac{4}{3}\right)$	Both fractions in the equation have a denominator of 3. Multiply both *sides* of the equation (not just the fractions!) by 3 to eliminate the denominators.
$\begin{aligned} 3\left(\frac{x}{3}\right)+3(1)&=3\left(\frac{4}{3}\right)\\ \frac{3x}{3}+3&=\frac{3(4)}{3}\\ \frac{3}{3}\cdot x+3&=4\cdot\frac{3}{3}\\ 1\cdot x+3&=4\cdot 1\\ x+3&=4 \end{aligned}$	Apply the distributive property and multiply 3 by each term within the parentheses. Then simplify and solve for x.

$x=1$

Excluded Values and Extraneous Solutions

Some rational expressions have a variable in the denominator. When this is the case, there is an extra step in solving them. Since division by 0 is undefined, you must exclude values of the variable that would result in a denominator of 0. These values are called **excluded values**. Let's look at an example.

Example

Solve the equation $\frac{2x-5}{x-5}=\frac{15}{x-5}$**.**

Solution

$\frac{2x-5}{x-5}=\frac{15}{x-5}$ 5 is an excluded value because it makes the denominator $x-5$ equal to 0.	Determine any values for x that would make the denominator 0.
$\begin{aligned} 2x-5&=15\\ 2x&=20\\ x&=10 \end{aligned}$	Since the denominator of each expression in the equation is the same, the numerators must be equal. Set the numerators equal to one another and solve for x.
$\begin{aligned} \frac{2x-5}{x-5}&=\frac{15}{x-5}\\ \frac{2(10)-5}{10-5}&=\frac{15}{10-5}\\ \frac{20-5}{10-5}&=\frac{15}{10-5}\\ \frac{15}{5}&=\frac{15}{5} \end{aligned}$	Check the solution in the original equation.

$x=10$

Exercise

Give the excluded values for $\frac{7}{p+2}+\frac{5}{p-2}=\frac{10p-2}{p^2-4}$. Do not solve.

A. $\frac{2}{10}$
B. 2
C. $-2,2$
D. $-2,2,4$

Answer

A. Incorrect. Excluded values are those values of the variable which result in a 0 in the denominator, not in the numerator. The correct answer is $-2,2$.
B. Incorrect. 2 is an excluded value, but -2 also results in a 0 in the denominator. The correct answer is $-2,2$.

C. Correct. -2 and 2, when substituted into the equation, result in a 0 in the denominator. Since division by 0 is undefined, both of these values are excluded from the solution.
D. Incorrect. While -2 and 2 are excluded, 4 is not excluded because it does not cause the denominator to be 0. The correct answer is -2, 2.

Let’s look at an example with a more complicated denominator.

✓ Example

Solve the equation $\frac{x}{x^2-9}+\frac{3}{x-3}=\frac{3}{x+3}$.

Solution

$\frac{x}{x^2-9}+\frac{3}{x-3}=\frac{3}{x+3}$ 3 is an excluded value because it makes $x-3$ and x^2-9 equal to 0. -3 is an excluded value because it makes $x+3$ and x^2-9 equal to 0.	Determine any values for x that would make the denominator 0.
$[(x+3)(x-3)]\left[\frac{x}{x^2-9}+\frac{3}{x-3}\right]=\frac{3}{x+3}(x+3)(x-3)$ $\frac{x(x+3)(x-3)}{x^2-9}+\frac{3(x+3)(x-3)}{x-3}=\frac{3(x+3)(x-3)}{x+3}$ $\left[\frac{(x+3)(x-3)}{x^2-9}\cdot x\right]+\left[\frac{x-3}{x-3}\cdot 3(x+3)\right]=\left[\frac{x+3}{x+3}\cdot 3(x-3)\right]$ $[1\cdot x]+[1\cdot 3(x+3)]=[1\cdot 3(x-3)]$ $x+3(x+3)=3(x-3)$ $x+3x+9=3x-9$ $4x+9=3x-9$ $x+9=-9$ $x=-18$	Since x^2-9 or $(x-3)(x+3)$ is a common multiple of $x-3$ as well as $x+3$, you can multiply both sides of the equation by $(x-3)(x+3)$ to clear the denominator from the equation. Solve for x.
$\frac{x}{x^2-9}+\frac{3}{x-3}=\frac{3}{x+3}$ $\frac{-18}{(-18)^2-9}+\frac{3}{-18-3}=\frac{3}{-18+3}$ $\frac{-18}{324-9}+\frac{3}{-21}=\frac{3}{-15}$ $\frac{-18}{315}+\frac{3}{-21}=\frac{3}{-15}$ $\frac{-18}{315}+\frac{-3(15)}{21(15)}=\frac{-3(21)}{15(21)}$ $\frac{-18}{315}+\frac{-45}{315}=\frac{-63}{315}$ $\frac{-63}{315}=\frac{-63}{315}$	Check the solution in the original equation.

$x=-18$

? Exercise

Solve the equation: $\frac{4}{m}=\frac{3}{m-2}, m\neq 0 \text{ or } 2$

A. $m=2$
B. no solution
C. $m=8$

Answer

A. Incorrect. You probably found the common denominator correctly, but forgot to distribute when you were simplifying. You also forgot to check your solution or note the excluded values; $m\neq 2$ because it makes the expression on the right side undefined. Multiplying both sides by the common denominator gives $\frac{4}{m}\cdot m(m-2)=\frac{3}{m-2}\cdot m(m-2)$, so $4m-8=3m$. The correct answer is $m=8$.
B. Incorrect. $\frac{4}{m}\cdot m(m-2)=\frac{3}{m-2}\cdot m(m-2)$, so $4m-8=3m$. The solution, 8, is not an excluded value. The correct answer is $m=8$.

C. Correct. Multiplying both sides of the equation by the common denominator gives $\frac{4}{m}\cdot m(m-2)=\frac{3}{m-2}\cdot m(m-2)$, so $4m-8=3m$. The correct answer is $m=8$.

You've seen that there is more than one way to solve rational equations. Because both of these techniques manipulate and rewrite terms, sometimes they can produce solutions that don't work in the original form of the equation. These types of answers are called **extraneous solutions**. That's why it is always important to check all solutions in the original equations: you may find that they yield untrue statements or produce undefined expressions.

✓ Example

Solve the equation $\frac{16}{m+4}=\frac{m^2}{m+4}$.

Solution

-4 is an excluded value because it makes $m+4$ equal to 0.	Determine any values for m that would make the denominator 0.
$\begin{aligned} 16&=m^2\\ 0&=m^2-16\\ 0&=(m+4)(m-4)\\ 0=m+4 \quad \text{or} \quad 0&=m-4\\ m=-4 \quad \text{or} \quad m&=4\\ m&=4,-4 \end{aligned}$	Since the denominator of each expression in the equation is the same, the numerators must be equal. Set the numerators equal to one another and solve for m.
$\frac{16}{m+4}=\frac{m^2}{m+4}$	Check the solutions in the original equation.
$\frac{16}{-4+4}=\frac{(-4)^2}{-4+4}$ $\frac{16}{0}=\frac{16}{0}$	Since $m=-4$ leads to division by 0, it is an extraneous solution.
-4 is excluded because it leads to division by 0.	
$\frac{16}{4+4}=\frac{(4)^2}{4+4}$ $\frac{16}{8}=\frac{16}{8}$	

$m=4$

Solving Application Problems

A "work problem" is an example of a real-life situation that can be modeled and solved using a rational equation. Work problems often ask you to calculate how long it will take different people working at different speeds to finish a task. The algebraic models of such situations often involve rational equations derived from the work formula, $W=rt$. (Notice that the work formula is very similar to the relationship between distance, rate, and time, or $d=rt$.) The amount of work done (W) is the product of the rate of work (r) and the time spent working (t). The work formula has 3 versions.

$$W=rt$$
$$t=\frac{W}{r}$$
$$r=\frac{W}{t}$$

Some work problems include multiple machines or people working on a project together for the same amount of time but at different rates. In that case, you can add their individual work rates together to get a total work rate. Let's review an example.

✓ Example

Myra takes 2 hours to plant 50 flower bulbs. Francis takes 3 hours to plant 45 flower bulbs. Working together, how long should it take them to plant 150 bulbs?

Solution

Myra: $\frac{50 \text{ bulbs}}{2 \text{ hours}}$, or $\frac{25 \text{ bulbs}}{1 \text{ hour}}$ Francis: $\frac{45 \text{ bulbs}}{3 \text{ hours}}$, or $\frac{15 \text{ bulbs}}{1 \text{ hour}}$	Think about how many bulbs each person can plant in one hour. This is their planting rate.

Myra and Francis together: $\frac{25\text{ bulbs}}{1\text{ hour}}+\frac{15\text{ bulbs}}{1\text{ hour}}=\frac{40\text{ bulbs}}{1\text{ hour}}$	Combine their hourly rates to determine the rate they work together.
$\frac{40}{1}=\frac{150}{t}$	Use one of the work formulas to write a rational equation, for example $r=\frac{W}{t}$. You know r, the combined work rate, and you know W, the amount of work that must be done. What you don't know is how much time it will take to do the required work at the designated rate.
$\frac{40}{1}\cdot 1t=\frac{150}{t}\cdot 1t$ $40t=150$ $t=\frac{150}{40}=\frac{15}{4}$ $t=3\frac{3}{4}$ hours	Solve the equation by multiplying both sides by the common denominator, then isolating t.

It should take 3 hours 45 minutes for Myra and Francis to plant 150 bulbs together.

Other work problems go the other way. You can calculate how long it will take one person to do a job alone when you know how long it takes people working together to complete the job.

✓ Example

Joe and John are planning to paint a house together. John thinks that if he worked alone, it would take him 3 times as long as it would take Joe to paint the entire house. Working together, they can complete the job in 24 hours. How long would it take each of them, working alone, to complete the job?

Solution

Let $x=$ time it takes Joe to complete the job $3x=$ time it takes John to complete the job	Choose variables to represent the unknowns. Since John takes 3 times as long as Joe to paint the house, his time is represented as $3x$.
Joe's rate: $\frac{1}{x}$ John's rate: $\frac{1}{3x}$	The work is painting 1 house or 1. Write an expression to represent each person's rate using the formula $r=\frac{W}{t}$.
combined rate: $\frac{1}{x}+\frac{1}{3x}$	Their combined rate is the sum of their individual rates. Use this rate to write a new equation using the formula $W=rt$.
$1=\left(\frac{1}{x}+\frac{1}{3x}\right)24$ $1=\frac{24}{x}+\frac{24}{3x}$	The problem states that it takes them 24 hours together to paint a house, so if you multiply their combined hourly rate $\left(\frac{1}{x}+\frac{1}{3x}\right)$ by 24, you will get 1, which is the number of houses they can paint in 24 hours.
$1=\frac{3}{3}\cdot\frac{24}{x}+\frac{24}{3x}$ $1=\frac{3\cdot 24}{3x}+\frac{24}{3x}$ $1=\frac{72}{3x}+\frac{24}{3x}$ $1=\frac{72+24}{3x}$ $1=\frac{96}{3x}$ $3x=96$ $x=32$	Now solve the equation for x. (Remember that x represents the number of hours it will take Joe to finish the job.)
$1=\left(\frac{1}{x}+\frac{1}{3x}\right)24$ $1=\left[\frac{1}{32}+\frac{1}{3(32)}\right]24$ $1=\frac{24}{32}+\frac{24}{3(32)}$ $1=\frac{24}{32}+\frac{24}{96}$ $1=\frac{3}{3}\cdot\frac{24}{32}+\frac{24}{96}$ $1=\frac{72}{96}+\frac{24}{96}$ $1=\frac{96}{96}$	Check the solutions in the original equation. The solution checks. Since $x=32$ it takes Joe 32 hours to paint the house by himself. John's time is $3x$, so it takes him 96 hours to do the same amount of work.

It takes 32 hours for Joe to paint the house by himself and 96 hours for John the paint the house himself.

As shown above, many work problems can be represented by the equation $\frac{t}{a}+\frac{t}{b}=1$, where t is the time to do the job together, a is the time it takes person A to do the job, and b is the time it takes person B to do the job. The 1 refers to the total work done; in this case, the work was to paint 1 house.

The key idea here is to figure out each worker's individual rate of work. Then, once those rates are identified, add them together, multiply by the time t, set it equal to the amount of work done, and solve the rational equation.

? Exercise

Mari and Liam can each wash a car and vacuum its interior in 2 hours. Zach needs 3 hours to do this same job alone. If Zach, Liam, and Mari work together, how long will it take them to clean a car?

A. 20 minutes
B. 45 minutes
C. 1.2 hours
D. 1 hour

Answer

A. Incorrect. It looks like you divided one hour by 3 to arrive at 20 minutes. Remember that Zach is working at a different rate than Mari and Liam, so you cannot do straight division. The correct answer is 45 minutes.
B. Correct. According to the formula, $r=\frac{W}{t}$. Mari and Liam each have a rate of $\frac{1}{2}$ car in one hour, and Zach's rate is $\frac{1}{3}$ car in one hour. Working together, they have a rate of $\frac{1}{2}+\frac{1}{2}+\frac{1}{3}$, or $\frac{4}{3}$. W is one car, so the formula becomes $\frac{4}{3}=\frac{1}{t}$. This means $3t\cdot\frac{4}{3}=\frac{1}{t}\cdot 3t$, so $4t=3$, and $t=\frac{3}{4}$. It takes three-quarters of an hour, or 45 minutes, to clean one car.
C. Incorrect. Mari and Liam clean a car in two hours each, not as a team. They each have a rate of $\frac{1}{2}$ car in one hour, and Zach's rate is $\frac{1}{3}$ car in one hour. Working together, they have a rate of $\frac{1}{2}+\frac{1}{2}+\frac{1}{3}$. The correct answer is 45 minutes.
D. Incorrect. This is the time it would take for Mari and Liam to clean one car together. Since Zach is helping, it will take less time than that. The correct answer is 45 minutes.

Summary

You can solve rational equations by finding a common denominator. By rewriting the equation so that all terms have the common denominator, you can solve for the variable using just the numerators. Or, you can multiply both sides of the equation by the least common multiple of the denominators so that all terms become polynomials instead of rational expressions. Rational equations can be used to solve a variety of problems that involve rates, times and work. Using rational expressions and equations can help you answer questions about how to combine workers or machines to complete a job on schedule.

An important step in solving rational equations is to reject any extraneous solutions from the final answer. Extraneous solutions are solutions that don't satisfy the original form of the equation because they produce untrue statements or are excluded values that make a denominator equal to 0.

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SECTION OVERVIEW

15.3: Formulas and Variation

Topic hierarchy

15.3.1: Rational Formulas and Variation

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15.3.1: Rational Formulas and Variation

Learning Objectives

- Solve a formula for a specified letter.
- Identify direct, inverse, and joint variation.
- Find the unknown in a variation problem.
- Solve application problems involving direct variation.
- Solve application problems involving inverse variation.
- Solve application problems involving joint variation.

Introduction

Rational formulas can be useful tools for representing real-life situations and for finding answers to real problems. Equations representing direct, inverse, and joint variation are examples of rational formulas that can model many real-life situations. As you will see, if you can find a formula, you can usually make sense of a situation.

Working with Formulas

When solving problems using rational formulas, it is often helpful to first solve the formula for the specified variable. For example, work problems ask you to calculate how long it will take different people working at different speeds to finish a task. The algebraic models of such situations often involve rational equations derived from the work formula, $W=rt$. The amount of work done (W) is the product of the rate of work (r) and the time spent working (t). Using algebra, you can write the work formula 3 ways:

$$\begin{aligned} & W=rt \\ \text{Find the time } (t).\ & t=\frac{W}{r} (\text{ divide both sides by } r) \\ \text{Find the rate } (r).\ & r=\frac{W}{t} (\text{divide both sides by } t) \end{aligned}$$

Example

The formula for finding the density of an object is $D=\frac{m}{v}$, where D is the density, m is the mass of the object and v is the volume of the object. Rearrange the formula to solve for the mass (m) and then for the volume (v).

Solution

$D=\frac{m}{v}$	Start with the formula for density.
$v\cdot D=\frac{m}{v}\cdot v$	Multiply both sides of the equation by v to isolate m.
$v\cdot D=m\cdot\frac{v}{v}$ $v\cdot D=m\cdot 1$ $v\cdot D=m$	Simplify and rewrite the equation, solving for m.
$\frac{v\cdot D}{D}=\frac{m}{D}$ $\frac{D}{D}\cdot v=\frac{m}{D}$ $1\cdot v=\frac{m}{D}$ $v=\frac{m}{D}$	To solve the equation $D=\frac{m}{v}$ in terms of v, you will need do the same steps to this point, and then divide both sides by D.

Now let's look at an example using the formula for the volume of a cylinder.

Example

The formula for finding the volume of a cylinder is $V=\pi r^2h$, where V is the volume, r is the radius and h is the height of the cylinder. Rearrange the formula to solve for the height (h).

Solution

$V=\pi r^2h$	Start with the formula for the volume of a cylinder.

$\frac{V}{\pi r^2} = \frac{\pi r^2 h}{\pi r^2}$ $\frac{V}{\pi r^2} = h$	Divide both sides by πr^2 to isolate h. Simplify. You find the height, h, Simplify. You find the height, $\frac{V}{\pi r^2}$.

$h = \frac{V}{\pi r^2}$

Direct Variation

Variation equations are examples of rational formulas and are used to describe the relationship between variables. For example, imagine a parking lot filled with cars. The total number of tires in the parking lot is dependent on the total number of cars. Algebraically, you can represent this relationship with an equation.

$$\text{Number of tires } = 4 \cdot \text{ number of cars}$$

The number 4 tells you the rate at which cars and tires are related. You call the rate the **constant of variation**. It’s a constant because this number does not change. Because the number of cars and the number of tires are linked by a constant, changes in the number of cars cause the number of tires to change in a proportional, steady way. This is an example of **direct variation**, where the number of tires varies directly with the number of cars.

You can use the car and tire equation as the basis for writing a general algebraic equation that will work for all examples of direct variation. In the example, the number of tires is the output, 4 is the constant, and the number of cars is the input. Let’s enter those generic terms into the equation. You get:

$$\text{output } = \text{ constant } \cdot \text{ input}$$

That’s the formula for all direct variation equations.

$$\begin{aligned} \text{Number of tires } &= 4 \cdot \text{ number of cars} \\ \text{output} &= \text{constant} \cdot \text{input} \end{aligned}$$

Let’s find a general way to represent direct variation. When we start talking about *input* and *output* in an equation, we often call the equation a *function*. The output of a function (equation) is also known as the dependent variable and is generally represented symbolically as y. The input is called the independent variable, represented by the symbol x. Let’s represent the constant with the letter k. Now put those symbols into the equation.

$$\begin{aligned} \text{Number of tires } &= 4 \cdot \text{ number of cars} \\ \text{output} &= \text{constant} \cdot \text{input} \\ y &= kx \end{aligned}$$

And you’ve done it! All direct variation equations can be described by the equation $y = kx$.

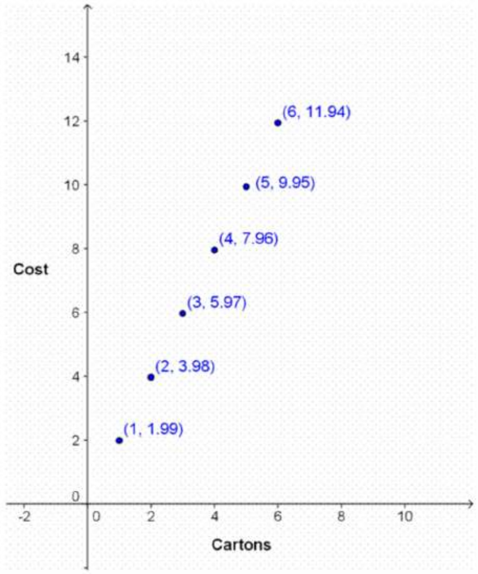
Let’s look at another example of direct variation. Mary works at a roadside stand on the family chicken farm, selling eggs for $1.99 per carton on busy weekends. When customers buy a lot of cartons at once, she has to add up the totals with a pencil and paper, and she worries about making mistakes. Lucky for Mary, this is a direct variation relationship: the output (total cost) equals the input (number of cartons) times a constant (the price per carton). She can use a direct variation equation to make a pricing table to use as a shortcut.

In this case:

$$\text{total cost } = \text{ carton price } \cdot \text{ number of cartons. Total cost } = 1.99 \cdot \text{ number of cartons.}$$

Number of cartons	**Total price**
1	$1.99
2	$3.98
3	$5.97
4	$7.96
5	$9.95
6	$11.94

Let’s graph the egg cost/carton function.



This function is made up of individual points because the farm stand only sells whole cartons of eggs. But you can see that all of the points are evenly spaced, and appear to lie on a straight line. You can also see that although it isn’t plotted, the point $(0,0)$ satisfies the function. The cost of 0 cartons would be 0 dollars.

Now let’s look at the graph of a continuous direct variation equation and see how it differs from the graph above. (In the graph of a continuous function, individual data points exist at every point along the line, unlike in the egg example above where it makes no sense to talk about the price of 1.235 cartons of eggs!)

Imagine a faucet pours water into a tub at a rate of 2.5 gallons per minute. The amount of water in the tub varies directly with the amount of time the faucet has been running. You can represent the relationship between the time and the water in the tub with the following formula.

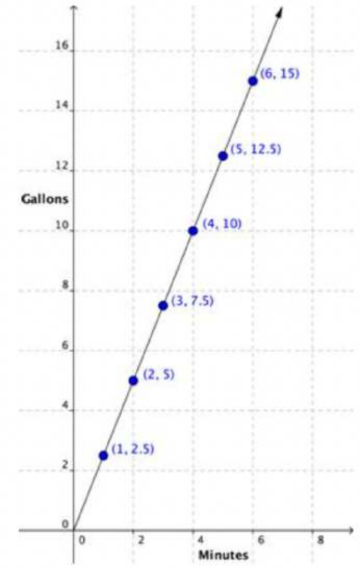
$$\text{Total gallons of water } = 2.5 \text{ gallons of water}/\text{minute} \cdot \text{time in minutes}$$

Using g to represent total gallons of water and t to represent time, you may abbreviate this relationship as $g=2.5t$, which looks very similar to the standard formula for proportional functions, $y=kx$.

Let’s make a table to chart the relationship of time versus the amount of water in the tub. After 1 minute, 2.5 gallons are in the tub. After 2 minutes, the total is 5 gallons, and so on. To find the total amount of water in the tub at any time, you can multiply the time by 2.5 gallons per minute. Six minutes should give you enough points to make a useful graph.

Time	Total Gallons
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0

Now you can graph those points.



This time, use a line to connect the points, because both time and water increase continuously. The points lie on a straight line that begins at the origin and rises at a steady angle, just like the last graph. In the egg example, it didn't make sense to purchase 2.5 cartons of eggs, so you did not draw a line to connect the points. But with the water example, you can measure the total number of gallons after 2.5 minutes, so connecting the points makes sense.

When the variables in a function change at a constant rate like this, they have a proportional relationship. This steady rate of change is called the constant of variation.

✓ Example

Solve for k, the constant of variation, in a direct variation problem where $y=300$ and $x=10$.

Solution

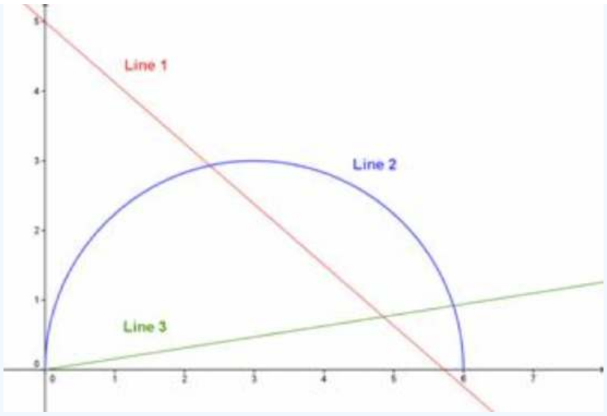
$y=kx$	Write the formula for a direct variation relationship.
$300=k\cdot 10$	Substitute known values into the equation.
$\frac{300}{10}=\frac{10k}{10}$ $30=k$	Solve for k by dividing both sides of the equation by 10.

The constant of variation, k, is 30.

? Exercise

Which line below is a graph showing direct variation?

A. Line 1
B. Line 2
C. Line 3



Answer

A. Incorrect. Line 1 is a line, but it does not go through the origin. This means the relationship does not vary directly. The correct answer is Line 3.
B. Incorrect. Line 2 is a curve, not a line, so it cannot be an example of direct variation. The correct answer is Line 3.
C. Correct. Line 3 is a line that includes the origin, which makes it an example of direct variation.

Inverse Variation

Another kind of variation is called **inverse variation**. In these equations, the dependent variable equals a constant divided by the independent variable. In symbolic form, this is the equation $y=\frac{k}{x}$, where y is the dependent variable, k is the constant, and x is the independent variable. Compare this with the equation for a function that has direct variation between the variables, such as the direct variation formula of $y=kx$. The only difference is that the independent variable is now in the denominator.

One example of an inverse variation is the speed required to travel between two cities in a given amount of time.

Let’s say you need to drive from Boston to Chicago, which is about 1,000 miles. The more time you have, the slower you can go. If you want to get there in 20 hours, you need to go 50 miles per hour (assuming you don’t stop driving!), because $\frac{1{,}000}{20}=50$. But if you can take 40 hours to get there, you only have to average 25 miles per hour, since $\frac{1{,}000}{40}=25$.

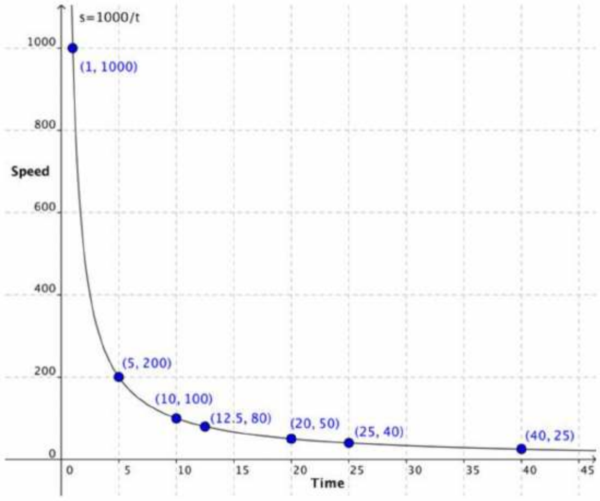
The equation for figuring out how fast to travel from the amount of time you have is $\text{speed}=\frac{\text{miles}}{\text{time}}$. This equation should remind you of the distance formula $d=rt$. If you solve $d=rt$ for r, you get $r=\frac{d}{t}$, or $\text{speed}=\frac{\text{miles}}{\text{time}}$.

In the case of the Boston to Chicago trip, you can write $s=\frac{1{,}000}{t}$. Notice that this is the same form as the inverse variation function formula, $y=\frac{k}{x}$.

Here’s a table that shows several times and speeds that satisfy the equation:

Time	**Speed (miles per hour)**
1	1,000
5	200
10	100
12.5	80
20	50
25	40
40	25

Now if you plot those points, you’ll see that the graph is definitely a curve, not a line.



✓ Example

Solve for k, the constant of variation, in an inverse variation problem where $x=5$ and $y=25$.

Solution

$y=\frac{k}{x}$	Write the formula for an inverse variation relationship.
$25=\frac{k}{5}$	Substitute known values into the equation.
$5\cdot 25=\frac{k}{5}\cdot 5$ $125=\frac{5k}{5}$ $125=k$	Solve for k by multiplying both sides of the equation by 5.

The constant of variation, k, is 125.

✓ Example

The water temperature in the ocean varies inversely with the depth of the water. The deeper a person dives, the colder the water becomes. At a depth of 1,000 meters, the temperature of the water is 5^{o} Celsius. What is the water temperature at a depth of 500 meters?

Solution

$y=\frac{k}{x}$ $\text{temp}=\frac{k}{\text{depth}}$	You are told that this is an inverse relationship, and that the water temperature (y) varies inversely with the depth of the water (x).
$5=\frac{k}{1{,}000}$ $1,000\cdot 5=\frac{k}{1{,}000}\cdot 1,000$	Substitute known values into the equation.
$5,000=\frac{1{,}000k}{1{,}000}$ $5,000=k$	Solve for k.
$\text{temp}=\frac{k}{\text{depth}}$ $\text{temp}=\frac{5{,}000}{500}$ $\text{temp}=10$	Now that k, the constant of variation is known, use that information to solve the problem: find the water temperature at 500 meters.

At 500 meters, the water temperature is 10^{o} Celsius.

? Exercise

In an inverse variation function, what happens to the output as the input gets smaller?

A. It gets larger.
B. It gets smaller.

Answer

A. Correct. Since the output is a constant divided by the input, as the input decreases, the constant is divided by a smaller and smaller number. This means that in an inverse variation function, the output continues to increase as the input gets smaller.
B. Incorrect. In a direct variation function, the output will shrink as the input shrinks, but just the opposite happens in an inverse variation function. The correct answer is that it gets larger.

Joint Variation

A third type of variation is called **joint variation**. Joint variation is the same as direct variation except there are two or more quantities. For example, the area of a rectangle can be found using the formula $A=lw$, where l is the length of the rectangle and w is the width of the rectangle. If you change the width of the rectangle, then the area changes. Similarly, if you change the length of the rectangle, then the area will also change. You can say that the area of the rectangle “varies jointly with the length and the width of the rectangle.”

The formula for the volume of a cylinder, $V=\pi r^2 h$ is another example of joint variation. The volume of the cylinder varies jointly with the square of the radius and the height of the cylinder. The constant of variation is π.

✓ Example

The area of a triangle varies jointly with the lengths of its base and height. If the area of a triangle is 30 inches ^2 when the base is 10 inches and the height is 6 inches, find the variation constant and the area of a triangle whose base is 15 inches and height is 20 inches.

Solution

$y=kxz$	You are told that this is a joint variation relationship, and that the area of a triangle (A) varies jointly with the lengths of the base (b) and height (h).
$30=k\cdot 10\cdot 6$ $30=60k$ $\frac{30}{60}=\frac{60k}{60}$ $\frac{1}{2}=k$	Substitute known values into the equation, and solve for k.
$\text{Area }=k\cdot\text{ base }\cdot\text{ height}$ $\text{Area }=(15)(20)\left(\frac{1}{2}\right)$ $\text{Area }=\frac{300}{2}$ $\text{Area }=150\text{ square inches}$	Now that k is known, solve for the area of a triangle whose base is 15 inches and height is 20 inches.

The constant of variation, k, is $\frac{1}{2}$, and the area of the triangle is 150 square inches.

Finding k to be $\frac{1}{2}$ shouldn’t be surprising. You know that the area of a triangle is one-half base times height, $A=\frac{1}{2}bh$. The $\frac{1}{2}$ in this formula is exactly the same $\frac{1}{2}$ that you calculated in this example!

Direct, Joint, and Inverse Variation

k is the constant of variation. In all cases, $k\neq 0$.

- Direct variation: $y=kx$
- Inverse variation: $y=\frac{k}{x}$

- Joint variation: $y = kxz$

It is important to be able to distinguish if an application is varying directly, inversely, or jointly.

Exercise

A rock that weighs 120lbs on Earth weighs 20lbs on the surface of the moon. Similarly, an astronaut who weighs 240lbs on Earth will weigh 40lbs on the moon. Is this an example of direct, inverse, or joint variation?

A. Direct variation
B. Inverse variation
C. Joint variation

Answer

A. Correct. These weights reveal direct variation. As an item's weight rises on Earth, so too will its weight rise on the surface of the moon.
B. Incorrect. If this were an inverse relationship, then as an item's weight on Earth rose, its weight on the surface of the moon would fall. This is not the case. The correct answer is direct variation.
C. Incorrect. The problem only gives two variables: an input (weight on Earth) and an output (weight on the moon). If this were an example of joint variation, more variables would have been presented. The correct answer is direct variation.

Summary

Rational formulas can be used to solve a variety of problems that involve rates, times, and work. Direct, inverse, and joint variation equations are examples of rational formulas. In direct variation, the variables have a direct relationship: as one quantity increases, the other quantity will also increase. As one quantity decreases, the other quantity decreases. In inverse variation, the variables have an inverse relationship: as one variable increases, the other variable decreases, and vice versa. Joint variation is the same as direct variation except there are two or more variables.

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CHAPTER OVERVIEW

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SECTION OVERVIEW

16.1: Introduction to Roots and Rational Exponents

Topic hierarchy

16.1.1: Roots

16.1.2: Squares, Cubes, and Beyond

16.1.3: Rational Exponents

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16.1.1: Roots

Learning Objectives

- Find principal square roots and their opposites.
- Approximate square roots and find exact roots with a calculator.

Introduction

You have probably dealt with the roots of plants and trees when gardening, but did you know that there are roots in mathematics, too?

Yes, roots do exist in math. The most common root is the **square root**. Let's take a look at what roots are, how they relate to exponents, and how you find the square root of a number.

Squares and Roots

To help understand square roots, let's review some facts about exponents. Look at the table below.

Exponent	**Name**	**Expanded Form**
3^2	"Three squared" "Three to the second power"	$3\cdot3$
4^5	"Four to the fifth power" "Four to the fifth" "Four raised to the power of five"	$4\cdot4\cdot4\cdot4\cdot4$
x^3	" x cubed" " x to the third power" " $x$to the third"	$x\cdot x\cdot x$
x^n	" x to the n th power" " x to the n th" " x to the power of n"	$x\cdot x\cdot x_{\ldots}\cdot x$ n times

You can think of exponential numbers as "repeated multiplication."

Just as dividing is the inverse of multiplying, the inverse of raising a number to a power is taking the root of a number. The most common root (and the one you will be studying here) is called the square root. When you are trying to find the square root of a number (say, 25), you are trying to find a number that can be multiplied by itself to create that original number. In the case of 25, you can find that $5\cdot5=25$, so 5 must be the square root.

The symbol for the square root is called a **radical symbol** and looks like this: $\sqrt{\ }$. The expression $\sqrt{25}$ is read "the square root of twenty-five" or "radical twenty-five." The number that is written under the radical symbol is called the **radicand**. Take a look at the following table.

Radical	**Name**	**Simplified Form**
$\sqrt{36}$	"Square root of thirty-six" "Radical thirty-six"	$\sqrt{36}=\sqrt{6\cdot6}=6$
$\sqrt{100}$	"Square root of one hundred" "Radical one hundred"	$\sqrt{100}=\sqrt{10\cdot10}=10$
$\sqrt{225}$	"Square root of two hundred twenty-five" "Radical two hundred twenty-five"	$\sqrt{225}=\sqrt{15\cdot15}=15$

Look at $\sqrt{25}$ again. You may realize that there is another value that, when multiplied by itself, also results in 25. That number is -5.

$5 \cdot 5=25$
$-5 \cdot -5=25$

By definition, the square root symbol always means to find the positive root, called the **principal root**. So while $5 \cdot 5$ and $-5 \cdot -5$ both equal 25, only 5 is the principal root. You should also know that zero is special because it has only one square root: itself (since $0 \cdot 0=0$).

If you know the principal square root, you can also find its opposite. (Recall that any number plus its opposite will equal 0. So, for instance, $a+(-a)=0$.) In the table below, notice that while $\sqrt{x}$ will give the principal root, $\sqrt{-x}$ will give its opposite. For example, $\sqrt{36}$ equals the principal square root, 6, and $-\sqrt{36}$ equals its opposite, -6.

Radical	**Principal Root**	**Opposite Radical**	**Opposite Root**
$\sqrt{36}$	$\sqrt{6 \cdot 6}=6$	$-\sqrt{36}$	$-\sqrt{6 \cdot 6}=-6$
$\sqrt{100}$	$\sqrt{10 \cdot 10}=10$	$-\sqrt{100}$	$-\sqrt{10 \cdot 10}=-10$
$\sqrt{225}$	$\sqrt{15 \cdot 15}=15$	$-\sqrt{225}$	$-\sqrt{15 \cdot 15}=-15$

There you have it: putting a negative sign in front of a radical has the effect of turning the principal root into its opposite (the negative square root of the radicand).

So now you have reviewed exponents, and you have been introduced to square roots. How does knowing about one help you understand the other?

Exponents and roots are connected because roots can be expressed as fractional exponents. For now, let’s look at the connection between the exponent “ $\frac{1}{2}$” and square roots; you will learn about other fractional exponents and other roots later on. The square root of a number can be displayed by using the radical symbol or by raising the number to the $\frac{1}{2}$ power. This is illustrated in the table below.

Exponent Form	**Root Form**	**Root of a Square**	**Simplified**
$25^{\frac{1}{2}}$	$\sqrt{25}$	$\sqrt{5^2}$	5
$16^{\frac{1}{2}}$	$\sqrt{16}$	$\sqrt{4^2}$	4
$100^{\frac{1}{2}}$	$\sqrt{100}$	$\sqrt{10^2}$	10

The square of 4 is 16 because 4 times 4 is equal to 16. Recall from your work with exponents that this can also be written as $4^2=16$.

Thinking this way, you can identify that the square root of 9 is 3 because $3 \cdot 3=9$. Similarly, the square root of 25 is 5 because $5 \cdot 5=25$, and the square root of x^2 is x since $x \cdot x=x^2$. For example, $\sqrt{7^2}=7$. (You will often see this type of notation, where you take the square root of a squared number, when you simplify, multiply, and divide radicals.)

Simplifying Square Roots

Square roots and exponents are connected. Keep that in mind as you begin simplifying some square roots.

This first example, “Simplify $\sqrt{144}$,” can be read “Simplify the square root of 144.” Think about a number that, when multiplied by itself, has a product of 144.

✓ Example

Simplify. $\sqrt{144}$

Solution

$\sqrt{144}$ $\sqrt{12 \cdot 12}$	Determine what number multiplied by itself has a product of 144.
12	The square root of 144 is 12.

$\sqrt{144}=12$

✓ Example

Simplify. $-\sqrt{81}$

Solution

$-\sqrt{81}$	The radical symbol acts like a grouping sign.
$-\sqrt{9\cdot 9}$	The negative in front means to take the opposite of the value after you simplify the radical.
$-(9)$	The square root of 81 is 9. Then, take the opposite of 9.

$-\sqrt{81}=-9$

However, if the negative sign is *under* the radical as in $\sqrt{-49}$, there is no way to simplify it using real numbers. That is because there is no number that you could multiply by itself to get -49. Remember, a negative number multiplied by a negative number results in a positive number: $-7\cdot -7=49$.

If finding the square root by trial and error is difficult, you can use what you know about factoring to help you determine the principal root.

✓ Example

Simplify. $\sqrt{144}$

Solution

$\sqrt{144}$ $\sqrt{2\cdot 72}$ $\sqrt{2\cdot 2\cdot 36}$ $\sqrt{2\cdot 2\cdot 2\cdot 18}$ $\sqrt{2\cdot 2\cdot 2\cdot 2\cdot 9}$ $\sqrt{2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3}$	Determine the prime factors of 144.
$\sqrt{(2\cdot 2\cdot 3)(2\cdot 2\cdot 3)}$ $\sqrt{12\cdot 12}$ $\sqrt{12^2}$	Regroup these factors into two identical groups. Recall that the square root of a squared number is the number itself. Here, $\sqrt{12^2}=12$.

$\sqrt{144}=12$

Notice something that happened in the final step of this example: the expression $\sqrt{2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3}$ was rewritten as $\sqrt{12\cdot 12}$, and then $\sqrt{12^2}$. You split the factors into identical groupings, multiplied them, and arrived at a squared number.

Many times, though, it is easier to identify factor pairs after you have gone through the process of factoring the original radical. For example, look at $\sqrt{2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3}$ again. How many pairs of $(2\cdot 2)$ do you see? What about $(3\cdot 3)$? If you could somehow identify smaller squared numbers underneath the radical instead of recombining all the factors (as you did when you found that $\sqrt{2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3}=\sqrt{12\cdot 12}$)), you might be able to simplify radicals more quickly.

This is where it helps to think of roots as fractional exponents. Recall the **Product Raised to a Power Rule** from when you studied exponents. This rule states that the product of two or more non-zero numbers raised to a power is equal to the product of each number raised to the same power. In math terms, it is written $(ab)^x=a^x\cdot b^x$. So, for example, you can use the rule to rewrite $(3x)^2$ as $3^2\cdot x^2=9\cdot x^2=9x^2$.

Now instead of using the exponent 2, let's use the exponent $\frac{1}{2}$. The exponent is distributed in the same way.

$(3x)^{\frac{1}{2}}=(3)^{\frac{1}{2}}\cdot(x)^{\frac{1}{2}}$

And since you know that raising a number to the $\frac{1}{2}$ power is the same as taking the square root of that number, you can also write it this way.

$\sqrt{3x} = \sqrt{3} \cdot \sqrt{x}$

You can think of any number underneath a radical as the *product of separate factors*, each underneath its own radical. Using this idea helps you identify smaller squared numbers, which often lets you simplify radicals more quickly.

A Product Raised to a Power Rule

or sometimes called

The Square Root of a Product Rule

For any numbers a and b, $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$

For example: $\sqrt{100} = \sqrt{10} \cdot \sqrt{10}$, and $\sqrt{75} = \sqrt{25} \cdot \sqrt{3}$

This rule is important because it helps you think of one radical as the product of multiple radicals. If you can identify perfect squares within a radical, as with $\sqrt{(2\cdot 2)(2\cdot 2)(3\cdot 3)}$, you can rewrite the expression as the product of multiple perfect squares: $\sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{3^2}$. Let's take another look at $\sqrt{144}$ using this new idea.

Example

Simplify. $\sqrt{144}$

Solution

$\sqrt{2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3}$	Determine the prime factors of 144.
$\sqrt{(2\cdot 2)\cdot(2\cdot 2)\cdot(3\cdot 3)}$	Group like factors into pairs.
$\sqrt{2^2\cdot 2^2\cdot 3^2}$	Rewrite as squares.
$\sqrt{2^2}\cdot\sqrt{2^2}\cdot\sqrt{3^2}$	Using the Product Raised to a Power rule, rewrite as a product of individual radicals.
$2\cdot 2\cdot 3$	Simplify each radical, then multiply.

$\sqrt{144} = 12$

You get the same solution in both cases, but it is often easier to pull out smaller factor pairs and then multiply them together (as shown here) than recombining all the factors to find the full root (as shown in the first example).

Exercise

Simplify. $\sqrt{324}$

A. 16
B. 18
C. 21
D. 162

Answer

A. Incorrect. $16^2 = 256$. To find the square root of 324, factor 324 and look for pairs of common factors. If you factor $\sqrt{324}$, you will find that $\sqrt{324} = \sqrt{9\cdot 9\cdot 2\cdot 2} = \sqrt{18\cdot 18}$. 324 can also be written as 18^2. The correct answer is 18.
B. Correct. $\sqrt{324} = 18$. If you factor $\sqrt{324}$, you will find that $\sqrt{324} = \sqrt{9\cdot 9\cdot 2\cdot 2} = \sqrt{18\cdot 18}$. 324 can also be written as 18^2.
C. Incorrect. $21^2 = 441$. To find the square root of 324, factor 324 and look for pairs of common factors. If you factor $\sqrt{324}$, you will find that $\sqrt{324} = \sqrt{9\cdot 9\cdot 2\cdot 2} = \sqrt{18\cdot 18}$. 324 can also be written as 18^2. The correct answer is 18.

D. Incorrect. $162^2 = 26,244$. Dividing 324 by 2 will not result in the number’s square root; try factoring 324 and looking for pairs of common factors. If you factor $\sqrt{324}$, you will find that $\sqrt{324} = \sqrt{9 \cdot 9 \cdot 2 \cdot 2} = \sqrt{18 \cdot 18}$. 324 can also be written as 18^2. The correct answer is 18.

Simplifying Square Roots by Factoring

So far, you have seen examples that are perfect squares. That is, each is a number whose square root is an integer. But many radical expressions are not perfect squares. Some of these radicals can still be simplified by finding perfect square factors. The example below illustrates how to factor the radicand, looking for pairs of factors that can be expressed as a power.

✓ Example

Simplify. $\sqrt{63}$

Solution

$\sqrt{7 \cdot 9}$	Factor 63 into 7 and 9.
$\sqrt{7 \cdot 3 \cdot 3}$	Factor 9 further into 3 and 3.
$\sqrt{7 \cdot 3^2}$	Rewrite $3 \cdot 3$ as 3^2.
$\sqrt{7} \cdot \sqrt{3^2}$	Using the Product Raised to a Power rule, separate the radical into the product of two factors, each under a radical.
$\sqrt{7} \cdot 3$	Take the square root of 3^2.
$3 \cdot \sqrt{7}$	Rearrange factors so the integer appears before the radical, and then multiply. (This is done so that it is clear that only the 7 is under the radical, not the 3.)

$\sqrt{63} = 3\sqrt{7}$

The final answer $3\sqrt{7}$ may look a bit odd, but it is in simplified form. You can read this as “three radical seven” or “three times the square root of seven.”

✓ Example

Simplify. $\sqrt{2,000}$

Solution

$\sqrt{100 \cdot 20}$ $\sqrt{100 \cdot 4 \cdot 5}$	Factor 2,000 to find perfect squares. Continue factoring until all perfect squares are identified.
$\sqrt{10 \cdot 10 \cdot 2 \cdot 2 \cdot 5}$	Factor 100 as $10 \cdot 10$ and 4 as $2 \cdot 2$.
$\sqrt{10^2 \cdot 2^2 \cdot 5}$	Rewrite $10 \cdot 10$ as 10^2 and $2 \cdot 2$ as 2^2.
$\sqrt{10^2 \cdot 2^2 \cdot 5}$	Using the Product Raised to a Power rule, rewrite the radical as the product of three factors, each under a radical.
$10 \cdot 2 \cdot \sqrt{5}$	Take the square root of 10^2 and 2^2.
$20 \cdot \sqrt{5}$	Multiply.

$\sqrt{2,000} = 20\sqrt{5}$

Approximating and Calculating Square Roots

Another approach to handling square roots that are not perfect squares is to approximate them by comparing the values to perfect squares. Suppose you wanted to know the square root of 17. Let’s look at how you might approximate it.

Example

Simplify. $\sqrt{17}$

Solution

17 is in between the perfect squares 16 and 25. So, $\sqrt{17}$ must be in between $\sqrt{16}$ and $\sqrt{25}$. $\sqrt{16}=4$ and $\sqrt{25}=5$	Think of two perfect squares that surround 17.
Since 17 is closer to 16 than 25, $\sqrt{17}$ is probably about 4.1 or 4.2.	Determine whether $\sqrt{17}$ is closer to 4 or to 5 and make another estimate.
$4.1\cdot4.1=16.81$ $4.2\cdot4.2=17.64$	Use trial and error to get a better estimate of $\sqrt{17}$. Try squaring incrementally greater numbers, beginning with 4.1, to find a good approximation for $\sqrt{17}$.
	$(4.1)^2$ gives a closer estimate than $(4.2)^2$.
$4.12\cdot4.12=16.9744$ $4.13\cdot4.13=17.0569$	Continue to use trial and error to get an even better estimate.

$\sqrt{17}\approx4.12$

This approximation is pretty close. If you kept using this trial and error strategy, you could continue to find the square root to the thousandths, ten-thousandths, and hundred-thousandths places, but eventually it would become too tedious to do by hand.

For this reason, when you need to find a more precise approximation of a square root, you should use a calculator. Most calculators have a square root key $(\sqrt{\quad})$ that will give you the square root approximation quickly. On a simple 4-function calculator, you would likely key in the number that you want to take the square root of and then press the square root key.

Try to find $\sqrt{17}$ using your calculator. Note that you will not be able to get an “exact” answer because $\sqrt{17}$ is an irrational number, a number that cannot be expressed as a fraction, and the decimal never terminates or repeats. To nine decimal positions, $\sqrt{17}$ is approximated as 4.123105626. A calculator can save a lot of time and yield a more precise square root when you are dealing with numbers that aren’t perfect squares.

Example

Approximate $\sqrt{50}$ and also find its value using a calculator.

Solution

50 is in between the perfect squares 49 and 64. $\sqrt{49}=7$ and $\sqrt{64}=8$, so $\sqrt{50}$ is between 7 and 8.	Find the perfect squares that surround 50.
49 and 50 are close, so $\sqrt{50}$ is only a little greater than 7. $7.1\cdot7.1=50.41$ Since 50.41 is greater than 50, the estimate must be between 7 and 7.1.	Using number sense and trial and error, try squaring incrementally greater numbers, beginning with 7.1, to find a good approximation for $\sqrt{50}$.
Since 50 is closer to 50.41 than to 49, try 7.07. $7.07\cdot7.07=49.9849$	Use reasoning to get an estimate to the hundredths place.
$\sqrt{50}\approx7.071067812$	Use a calculator.

By approximation: $\sqrt{50}\approx7.07$

Using a calculator: $\sqrt{50}\approx7.071067812$

Summary

The square root of a number is the number which, when multiplied by itself, gives the original number. Principal square roots are always positive and the square root of 0 is 0. You can only take the square root of values that are greater than or equal to 0. The square root of a perfect square will be an integer. Other square roots can be simplified by identifying factors that are perfect squares and taking their square root. Square roots can be approximated using trial and error or a calculator.

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16.1.2: Squares, Cubes, and Beyond

Learning Objectives

- Simplify square roots.
- Find cube roots.
- Simplify expressions with odd and even roots.

Introduction

Radical expressions are expressions that contain radicals. Radical expressions come in many forms, from simple and familiar, such as $\sqrt{16}$, to quite complicated, as in $\sqrt[3]{250x^4y}$. In addition to square roots, there are radicals called cube roots, fourth roots, fifth roots, and so on. Using factoring, you can simplify these radical expressions, too.

Simplifying Square Roots

Radical expressions will sometimes include variables as well as numbers. Consider the expression $\sqrt{9x^6y^4}$. To simplify a radical expression such as this, you can use factoring, but you'll have to apply the rules of exponents, too. Let's try it.

Example

Simplify. $\sqrt{9x^6y^4}$

Solution

$\sqrt{3\cdot 3\cdot x^6y^4}$	Factor the coefficient 9 into $3\cdot 3$.
$\sqrt{3\cdot 3\cdot x^2\cdot x^2\cdot x^2\cdot y^2\cdot y^2}$	Factor variables into squares.
$\sqrt{3^2}\cdot\sqrt{x^2}\cdot\sqrt{x^2}\cdot\sqrt{x^2}\cdot\sqrt{y^2}\cdot\sqrt{y^2}$	Write $3\cdot 3$ as 3^2 and separate into individual radicals.
$3\cdot x\cdot x\cdot x\cdot y\cdot y$	Simplify, using the rule that $\sqrt{x^2}=x$.
$3x^3y^2$	Rewrite the expression with constants in front and using exponents for the variables.

$$\sqrt{9x^6y^4}=3x^3y^2$$

The goal is to find factors under the radical that are perfect squares so that you can take their square root. Let's repeat the example above and focus on finding identical pairs of factors.

Example

Simplify. $\sqrt{9x^6y^4}$

Solution

$\sqrt{3\cdot 3\cdot x^3\cdot x^3\cdot y^2\cdot y^2}$	Factor to find identical pairs.
$\sqrt{(3^2)\cdot(x^3)^2\cdot(y^2)^2}$	Rewrite the pairs as perfect squares.
$\sqrt{3^2}\cdot\sqrt{(x^3)^2}\cdot\sqrt{(y^2)^2}$	Separate into individual radicals.
$3x^3y^2$	Simplify, using the rule that $\sqrt{x^2}=x$.

$$\sqrt{9x^6y^4}=3x^3y^2$$

Variable factors with even exponents can be written as squares. In the example above, $x^6=x^3\cdot x^3=(x^3)^2$ and $y^4=y^2\cdot y^2=(y^2)^2$. Let's try to simplify another radical expression.

Example

Simplify. $\sqrt{49x^{10}y^8}$

Solution

$\sqrt{7\cdot 7\cdot x^5\cdot x^5\cdot y^4\cdot y^4}$	Look for squared numbers and variables. Factor 49 into 7.7, x^{10} into $x^5\cdot x^5$, and y^8 into $y^4\cdot y^4$.
$\sqrt{7^2\cdot(x^5)^2\cdot(y^4)^2}$	Rewrite the pairs as squares.
$\sqrt{7^2}\cdot\sqrt{(x^5)^2}\cdot\sqrt{(y^4)^2}$	Separate the squared factors into individual radicals.
$7\cdot x^5\cdot y^4$	Take the square root of each radical using the rule that $\sqrt{x^2}=x$.
$7x^5y^4$	Multiply.

$\sqrt{49x^{10}y^8}=7x^5y^4$

You find that the square root of $49x^{10}y^8$ is $7x^5y^4$. In order to check this calculation, you could square $7x^5y^4$, hoping to arrive at $49x^{10}y^8$. And, in fact, you would get this expression if you evaluated $(7x^5y^4)^2$.

Absolute Value

Take a moment to think about two radical expressions: $\sqrt{900}$ and $\sqrt{a^3b^5c^7}$. You would use the same techniques to simplify either one of these: find squares within the radical, rewrite the expression as the product of separate radicals, simplify and multiply.

There is an added issue when you are taking the root of a radical expression that contains variables. Recall that the root of an integer, such as $\sqrt{900}$, is defined to be nonnegative. This means that although both 30^2 and $(-30)^2$ are equal to 900, $\sqrt{900}$ is defined *only* as 30. This is the idea behind 30 being the **principal root** of 900.

But it is not as straightforward with radical expressions that contain variables. Consider the expression $\sqrt{x^2}$. This looks like it should be equal to x, right? Let's test some values for x and see what happens.

In the chart below, look along each row and determine whether the value of x is the same as the value of $\sqrt{x^2}$. Where are they equal? Where are they not equal?

After doing that for each row, look again and determine whether the value of $\sqrt{x^2}$ is the same as the value of $|x|$.

x	x^2	$\sqrt{x^2}$	$\|x\|$
-5	25	5	5
-2	4	2	2
0	0	0	0
6	36	6	6
10	100	10	10

Notice in cases where x is a negative number, $\sqrt{x^2}\neq x$! (This happens because the process of squaring the number loses the negative sign, since a negative times a negative is a positive.) However, in all cases $\sqrt{x^2}=|x|$. You need to consider this fact when simplifying radicals that contain variables, because by definition $\sqrt{x^2}$ is always nonnegative.

Taking the Square Root of a Radical Expression

When finding the square root of an expression that contains variables raised to a power, consider that $\sqrt{x^2}=|x|$.

Examples: $\sqrt{9x^2}=3|x|$, and $\sqrt{16x^2y^2}=4|xy|$

Look at how this idea is applied in this next example.

Example

Simplify. $\sqrt{a^3b^5c^2}$

Solution

$\sqrt{a^2\cdot a\cdot b^4\cdot b\cdot c^2}$	Factor to find variables with even exponents.
$\sqrt{a^2\cdot a\cdot (b^2)^2\cdot b\cdot c^2}$	Rewrite b^4 as $(b^2)^2$
$\sqrt{a^2}\cdot\sqrt{(b^2)^2}\cdot\sqrt{c^2}\cdot\sqrt{a\cdot b}$	Separate the squared factors into individual radicals.
$\lvert a\rvert\cdot b^2\cdot\lvert c\rvert\cdot\sqrt{a\cdot b}$	Take the square root of each radical. Remember that $\sqrt{a^2}=\lvert a\rvert$.
$\lvert ab^2c\rvert\sqrt{ab}$	Simplify and multiply. The entire quantity ab^2c can be enclosed in the absolute value sign because b^2 will be positive, so its inclusion has no effect.

$\sqrt{a^3b^5c^2}=|ab^2c|\sqrt{ab}$

Exercise

Simplify. $\sqrt{72y^2z^3}$

A. $36yz\sqrt{z}$
B. $6\sqrt{2y^2z^3}$
C. $6yz\sqrt{2z}$
D. $6|yz|\sqrt{2z}$

Answer

A. Incorrect. The square root is not half of the number; $\sqrt{72}$ is the number that, when multiplied by itself, gives 72. 36 times 36 is much greater than 72. The answer is $6|yz|\sqrt{2z}$.
B. Incorrect. The square root of 72 was simplified correctly, but the square root of the variable part was not simplified. The answer is $6|yz|\sqrt{2z}$.
C. Incorrect. You didn't apply the rule that $\sqrt{x^2}=|x|$. The answer is $6|yz|\sqrt{2z}$.
D. Correct. To simplify the expression, factor 72 into $36\cdot 2$ then take the square root of 36 leaving a coefficient of 6 in front of the radical and the simplified factor $\sqrt{2}$. Next, simplify $\sqrt{y^2}=|y|$ and $\sqrt{z^3}=|z|\sqrt{z}$.

Finding Cube Roots

While square roots are probably the most common radical, you can also find the third root, the fifth root, the 10^{th} root, or really any other n^{th} root of a number. Just as the square root is a number that, when squared, gives the radicand, the **cube root** is a number that, when cubed, gives the radicand. Cubing a number is the same as taking it to the third power: 2^3 is 2 cubed, so the cube root of 2^3 is 2.

The cube root of a number is written with a small number 3, called the **index**, just outside and above the radical symbol. It looks like $\sqrt[3]{}$. This little 3 distinguishes cube roots from square roots, which are written without a small number outside and above the radical symbol.

Be careful to distinguish between $\sqrt[3]{x}$, the cube root of x, and $3\sqrt{x}$, three *times* the *square* root of x. They may look similar at first, but they lead you to much different expressions!

Example

Simplify. $\sqrt[3]{8}$

Solution

$2 \cdot 2 \cdot 2$	Ask yourself, “What number can I multiply by itself, and then by itself again, to get 8?”

$\sqrt[3]{8}=2$

Another approach to simplifying a cube root is to use factoring. Let’s explore factoring with the expression $\sqrt[3]{125}$. You can read this as “the third root of 125” or “the cube root of 125.” To simplify this expression, look for a number that, when multiplied by itself two times (for a total of three identical factors), equals 125. Let’s factor 125 and find that number.

✓ Example

Simplify. $\sqrt[3]{125}$

Solution

$\sqrt[3]{5 \cdot 25}$	125 ends in 5, so you know that 5 is a factor. Expand 125 into $5 \cdot 25$.
$\sqrt[3]{5 \cdot 5 \cdot 5}$	Factor 25 into 5 and 5.
$\sqrt[3]{5^3}$	The factors are $5 \cdot 5 \cdot 5$, or 5^3.

$\sqrt[3]{125}=5$

The prime factors of 125 are $5 \cdot 5 \cdot 5$, which can be rewritten as 5^3. The cube root of a cubed number is the number itself, so $\sqrt[3]{5^3}=5$. You have found the cube root, the three identical factors that when multiplied together give 125. 125 is known as a **perfect cube** because its cube root is an integer.

Here’s an example of how to simplify a radical that is not a perfect cube.

✓ Example

Simplify. $\sqrt[3]{32m^5}$

Solution

$\sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot m^5}$	Factor 32 into prime factors.
$\sqrt[3]{2^3 \cdot 2 \cdot 2 \cdot m^5}$	Since you are looking for the cube root, you need to find factors that appear 3 times under the radical. Rewrite $2 \cdot 2 \cdot 2$ as 2^3.
$\sqrt[3]{2^3 \cdot 2 \cdot 2 \cdot m^3 \cdot m^2}$	Rewrite m^5 as $m^3 \cdot m^2$.
$\sqrt[3]{2^3} \cdot \sqrt[3]{2 \cdot 2} \cdot \sqrt[3]{m^3} \cdot \sqrt[3]{m^2}$	Rewrite the expression as a product of multiple radicals.
$2 \cdot \sqrt[3]{4} \cdot m \cdot \sqrt[3]{m^2}$	Simplify and multiply.

$\sqrt[3]{32m^5}=2m\sqrt[3]{4m^2}$

? Exercise

Simplify. $\sqrt[3]{64h^6}$

A. $8h^3$
B. $8\sqrt{h^6}$
C. $4+h^2$
D. $4h^2$

Answer

A. Incorrect. $8h^3$ is the square root of $64h^6$, not the cube root. The answer is $4h^2$.

B. Incorrect. To find the cube root of $64h^6$, find the cube root of 64, and then the cube root of h^6. The answer is $4h^2$.
C. Incorrect. 4 is the cube root of 64 and h^2 is the cube root of h^6, but they are factors and should be multiplied. The answer is $4h^2$.
D. Correct. To simplify the expression, find the cube root of 64 and the cube root of \h^{6}. $\sqrt[3]{64h^6}=\sqrt[3]{4\cdot 4\cdot 4\cdot (h^2)^3}$ $=\sqrt[3]{4^3\cdot (h^2)^3}=\sqrt[3]{4^3}\cdot\sqrt[3]{(h^2)^3}=4h^2$

Simplifying Expressions with Odd and Even Roots

There is one interesting fact about cube roots that is not true of square roots. Negative numbers can't have real number square roots, but negative numbers can have real number cube roots! What is the cube root of -8? $\sqrt[3]{-8}=-2$ because $-2\cdot -2\cdot -2=-8$. Remember, when you are multiplying an odd number of negative numbers, the result is negative! In the example below, notice how $\sqrt[3]{(-1)^3}=-1$ is used to simplify the radical.

Example

Simplify. $\sqrt[3]{-27x^4y^3}$

Solution

$\sqrt[3]{-1\cdot 27\cdot x^4\cdot y^3}$ $\sqrt[3]{(-1)^3\cdot (3)^3\cdot x^3\cdot x\cdot y^3}$	Factor the expression into cubes.
$\sqrt[3]{-1^3}\cdot\sqrt[3]{(3)^3}\cdot\sqrt[3]{x^3}\cdot\sqrt[3]{x}\cdot\sqrt[3]{y^3}$	Separate the cubed factors into individual radicals.
$-1\cdot 3\cdot x\cdot y\cdot\sqrt[3]{x}$	Simplify the cube roots.

$\sqrt[3]{-27x^4y^3}=-3xy\sqrt[3]{x}$

You could check your answer by performing the inverse operation. If you are right, when you cube $-3xy\sqrt[3]{x}$ you should get $-27x^4y^3$.

$$(-3xy\sqrt[3]{x})(-3xy\sqrt[3]{x})(-3xy\sqrt[3]{x})$$
$$-3\cdot -3\cdot -3\cdot x\cdot x\cdot x\cdot y\cdot y\cdot y\cdot\sqrt[3]{x}\cdot\sqrt[3]{x}\cdot\sqrt[3]{x}$$
$$-27\cdot x^3\cdot y^3\cdot\sqrt[3]{x^3}$$
$$-27x^3y^3\cdot x$$
$$-27x^4y^3$$

So, you can find the odd root of a negative number, but you cannot find the even root of a negative number. This means you can simplify the radicals $\sqrt[3]{-81}$, $\sqrt[5]{-64}$, and $\sqrt[7]{-2187}$, but you cannot simplify the radicals $\sqrt{-100}$, $\sqrt[4]{-16}$, or $\sqrt[6]{-2,500}$.

Let's look at another example.

Example

Simplify. $\sqrt[3]{-24a^5}$

Solution

$\sqrt[3]{-1\cdot 8\cdot 3\cdot a^5}$	Factor -24 to find perfect cubes. Here, -1 and 8 are the perfect cubes.
$\sqrt[3]{(-1)^3\cdot 2^3\cdot 3\cdot a^3\cdot a^2}$	Factor variables. You are looking for cube exponents, so you factor a^5 into a^3 and a^2.
$\sqrt[3]{-1^3}\cdot\sqrt[3]{2^3}\cdot\sqrt[3]{a^3}\cdot\sqrt[3]{3\cdot a^2}$	Separate the factors into individual radicals.
$-1\cdot 2\cdot a\cdot\sqrt[3]{3\cdot a^2}$	Simplify, using the property $\sqrt[3]{x^3}=x$.
$-2a\sqrt[3]{3a^2}$	This is the simplest form of this expression; all cubes have been pulled out of the radical expression.

$\sqrt[3]{-24a^5} = -2a\sqrt[3]{3a^2}$

The steps to consider when simplifying a radical are outlined below.

Simplifying a radical

When working with exponents and radicals:

- If n is odd, $\sqrt[n]{x^n} = x$.
- If n is even, $\sqrt[n]{x^n} = |x|$. (The absolute value accounts for the fact that if x is negative and raised to an even power, that number will be positive, as will the n th principal root of that number.)

Example

Simplify. $\sqrt{100x^2y^4}$

Solution

$\sqrt{10 \cdot 10 \cdot x^2 \cdot y^4}$	Separate factors; look for squared numbers and variables. Factor 100 into $10 \cdot 10$.
$\sqrt{10 \cdot 10 \cdot x^2 \cdot (y^2)^2}$	Factor y^4 into $(y^2)^2$.
$\sqrt{10^2} \cdot \sqrt{x^2} \cdot \sqrt{(y^2)^2}$	Separate the squared factors into individual radicals.
$10 \cdot \|x\| \cdot y^2$	Take the square root of each radical. Since you do not know whether x is positive or negative, use $\|x\|$ to account for both possibilities, thereby guaranteeing that your answer will be positive.
$10\|x\|y^2$	Simplify and multiply.

$\sqrt{100x^2y^4} = 10|x|y^2$

You can check your answer by squaring it to be sure it equals $100x^2y^4$.

Summary

A radical expression is a mathematical way of representing the n th root of a number. Square roots and cube roots are the most common radicals, but a root can be any number. To simplify radical expressions, look for exponential factors within the radical, and then use the property $\sqrt[n]{x^n} = x$ if n is odd, and $\sqrt[n]{x^n} = |x|$ if n is even to pull out quantities. All rules of integer operations and exponents apply when simplifying radical expressions.

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16.1.3: Rational Exponents

Learning Objectives

- Convert radicals to expressions with rational exponents.
- Convert expressions with rational exponents to their radical equivalent.
- Use the laws of exponents to simplify expressions with rational exponents.
- Use rational exponents to simplify radical expressions.

Introduction

Square roots are most often written using a radical sign, like this, $\sqrt{4}$. But there is another way to represent the taking of a root. You can use rational exponents instead of a radical. A **rational exponent** is an exponent that is a fraction. For example, $\sqrt{4}$ can be written as $4^{\frac{1}{2}}$.

Can't imagine raising a number to a rational exponent? They may be hard to get used to, but rational exponents can actually help simplify some problems. Let's explore the relationship between rational (fractional) exponents and radicals.

Rewriting Radical Expressions Using Rational Exponents

Radicals and fractional exponents are alternate ways of expressing the same thing. You have already seen how square roots can be expressed as an exponent to the power of one-half.

Radical Form	Exponent Form	Integer
$\sqrt{16}$	$16^{\frac{1}{2}}$	4
$\sqrt{25}$	$25^{\frac{1}{2}}$	5
$\sqrt{100}$	$100^{\frac{1}{2}}$	10

Let's look at some more examples, but this time with cube roots. Remember, cubing a number raises it to the power of three. Notice that in these examples, the denominator of the rational exponent is the number 3.

Radical Form	Exponent Form	Integer
$\sqrt[3]{8}$	$8^{\frac{1}{3}}$	2
$\sqrt[3]{125}$	$125^{\frac{1}{3}}$	5
$\sqrt[3]{1000}$	$1000^{\frac{1}{3}}$	10

These examples help us model a relationship between radicals and rational exponents: namely, that the n th root of a number can be written as either $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

Radical Form	Exponent Form
$\sqrt{x}$	$x^{\frac{1}{2}}$
$\sqrt[3]{x}$	$x^{\frac{1}{3}}$
$\sqrt[4]{x}$	$x^{\frac{1}{4}}$
…	…
$\sqrt[n]{x}$	$x^{\frac{1}{n}}$

When faced with an expression containing a rational exponent, you can rewrite it using a radical. In the table above, notice how the denominator of the rational exponent determines the index of the root. So, an exponent of $\frac{1}{2}$ translates to the square root, an exponent of $\frac{1}{5}$ translates to the fifth root or $\sqrt[5]{}$, and $\frac{1}{8}$ translates to the eighth root or $\sqrt[8]{}$.

✓ Example

Write $\sqrt[4]{81}$ as an expression with a rational exponent.

Solution

$81^{\frac{1}{4}}$	The radical form $\sqrt[4]{}$ can be rewritten as the exponent $\frac{1}{4}$. Remove the radical and place the exponent next to the base.

$\sqrt[4]{81}=81^{\frac{1}{4}}$

✓ Example

Express $(2x)^{\frac{1}{3}}$ in radical form.

Solution

$\sqrt[3]{2x}$	Rewrite the expression with the fractional exponent as a radical. The denominator of the fraction determines the root, in this case the cube root.
	The parentheses in $(2x)^{\frac{1}{3}}$ indicate that the exponent refers to everything within the parentheses.

$(2x)^{\frac{1}{3}}=\sqrt[3]{2x}$

Remember that exponents only refer to the quantity immediately to their left unless a grouping symbol is used. The example below looks very similar to the previous example with one important difference: there are no parentheses! Look what happens.

✓ Example

Express $2x^{\frac{1}{3}}$ in radical form.

Solution

$2\sqrt[3]{x}$	Rewrite the expression with the fractional exponent as a radical. The denominator of the fraction determines the root, in this case the cube root.
	The exponent refers only to the part of the expression immediately to the left of the exponent, in this case x, but not the 2.

$2x^{\frac{1}{3}}=2\sqrt[3]{x}$

Rewriting Expressions with Rational Exponents Using Radicals

Just as you can rewrite an expression with a rational exponent as a radical expression, you can express a radical expression using a rational exponent.

✓ Example

Express $4\sqrt[3]{xy}$ with rational exponents.

Solution

$4(xy)^{\frac{1}{3}}$	Rewrite the radical using a rational exponent. The root determines the fraction. In this case, the index of the radical is 3, so the rational exponent will be $\frac{1}{3}$.

	Since 4 is outside the radical, it is not included in the grouping symbol and the exponent does not refer to it.

$4\sqrt[3]{xy} = 4(xy)^{\frac{1}{3}}$

Example

Simplify. $(36x^4)^{\frac{1}{2}}$

Solution

$\sqrt{36x^4}$	Rewrite the expression with the fractional exponent as a radical.
$\sqrt{(6^2)\cdot x^4}$ $\sqrt{6^2}\cdot\sqrt{x^4}$ $\sqrt{6^2}\cdot\sqrt{(x^2)^2}$ $6\cdot x^2$	Find the square root of both the coefficient and the variable.

$(36x^4)^{\frac{1}{2}} = 6x^2$

Rational Exponents with Numerators Other than One

All of the numerators for the fractional exponents in the examples above were 1. You can use fractional exponents that have numerators other than 1 to express roots, as shown below. Notice any patterns within this table?

Radical	Exponent
$\sqrt{9}$	$9^{\frac{1}{2}}$
$\sqrt[3]{9^2}$	$9^{\frac{2}{3}}$
$\sqrt[4]{(9^3)}$	$9^{\frac{3}{4}}$
$\sqrt[5]{9^2}$	$9^{\frac{2}{5}}$
...	...
$\sqrt[n]{9^x}$	$9^{\frac{x}{n}}$

To rewrite a radical using a fractional exponent, the power to which the radicand is raised becomes the numerator and the root becomes the denominator.

Writing Fractional Exponents

Any radical in the form $\sqrt[n]{a^x}$ can be written using a fractional exponent in the form $a^{\frac{x}{n}}$.

The relationship between $\sqrt[n]{a^x}$ and $a^{\frac{x}{n}}$ works for rational exponents that have a numerator of 1 as well. For example, the radical $\sqrt[3]{8}$ can also be written as $\sqrt[3]{8^1}$, since any number remains the same value if it is raised to the first power. You can now see where the numerator of 1 comes from in the equivalent form of $8^{\frac{1}{3}}$.

Simplifying Radical Expressions Using Rational Exponents and the Laws of Exponents

Let's explore some radical expressions now and see how to simplify them. Here's a radical expression that needs simplifying, $\sqrt[3]{a^6}$.

One method of simplifying this expression is to factor and pull out groups of a^3, as shown below in this example.

✓ Example

Simplify. $\sqrt[3]{a^6}$

Solution

$\sqrt[3]{a^3 \cdot a^3}$	Rewrite by factoring out cubes.
$\sqrt[3]{a^3} \cdot \sqrt[3]{a^3}$ $a \cdot a$	Write each factor under its own radical and simplify.

$\sqrt[3]{a^6} = a^2$

You can also simplify this expression by thinking about the radical as an expression with a rational exponent, and using the principle that any radical in the form $\sqrt[n]{a^x}$ can be written using a fractional exponent in the form $a^{\frac{x}{n}}$.

✓ Example

Simplify. $\sqrt[3]{a^6}$

Solution

$a^{\frac{6}{3}}$	Rewrite the radical using a rational exponent.
a^2	Simplify the exponent.

$\sqrt[3]{a^6} = a^2$

Note that rational exponents are subject to all of the same rules as other exponents when they appear in algebraic expressions.

Both simplification methods gave the same result, a^2. Depending on the context of the problem, it may be easier to use one method or the other, but for now, you'll note that you were able to simplify this expression more quickly using rational exponents than when using the "pull-out" method.

Let's try another example.

✓ Example

Simplify. $\sqrt[4]{81x^8y^3}$

Solution

$(81x^8y^3)^{\frac{1}{4}}$ $81^{\frac{1}{4}} \cdot x^{\frac{8}{4}} \cdot y^{\frac{3}{4}}$	Rewrite the radical using rational exponents. Use the rules of exponents to simplify the expression.
$(3 \cdot 3 \cdot 3 \cdot 3)^{\frac{1}{4}}x^2y^{\frac{3}{4}}$ $(3^4)^{\frac{1}{4}}x^2y^{\frac{3}{4}}$ $3x^2y^{\frac{3}{4}}$ $3x^2\sqrt[4]{y^3}$	Change the expression with the rational exponent back to radical form.

$\sqrt[4]{81x^8y^3} = 3x^2\sqrt[4]{y^3}$

Again, the alternative method is to work on simplifying under the radical by using factoring. For the example you just solved, it looks like this.

✓ Example

Simplify. $\sqrt[4]{81x^8y^3}$

Solution

$\sqrt[4]{81}\cdot\sqrt[4]{x^8}\sqrt[4]{y^3}$	Rewrite the expression.
$\sqrt[4]{3\cdot 3\cdot 3\cdot 3}\cdot\sqrt[4]{x^2\cdot x^2\cdot x^2\cdot x^2}\cdot\sqrt[4]{y^3}$	Factor each radicand.
$\sqrt[4]{3^4}\cdot\sqrt[4]{\left(x^2\right)^4}\cdot\sqrt[4]{y^3}$ $3\cdot x^2\cdot\sqrt[4]{y^3}$	Simplify.

$\sqrt[4]{81x^8y^3}=3x^2\sqrt[4]{y^3}$

Exercise

Which of the expressions below is equal to the expression $\sqrt{25a^4b^5}$ when written using a rational exponent?

A. $5a^2b^2\sqrt{b}$
B. $\left(25a^4b^5\right)^{\frac{1}{2}}$
C. $\left(25a^4b^5\right)$
D. $25\left(a^4b^5\right)^{\frac{1}{2}}$

Answer

A. Incorrect. The problem asks for an expression with rational, or fractional, exponents. This answer is for simplifying the radical expression. The correct answer is $\left(25a^4b^5\right)^{\frac{1}{2}}$.
B. Correct. Taking the square root is the same as raising the quantity under the radical to a power of $\frac{1}{2}$.
C. Incorrect. This quantity is the radicand—the expression under the radical sign. Taking the square root is the same as raising the quantity under the radical to a power of $\frac{1}{2}$. The correct answer is $\left(25a^4b^5\right)^{\frac{1}{2}}$.
D. Incorrect. Since 25 is also under the radical symbol, you must raise it to the power of $\frac{1}{2}$. The correct answer is $\left(25a^4b^5\right)^{\frac{1}{2}}$.

Let's try a more complicated expression, $\frac{10b^2c^2}{c\sqrt[3]{8b^4}}$. This expression has two variables, a fraction, and a radical. Let's take it step-by-step and see if using fractional exponents can help us simplify it.

Let's start by simplifying the denominator, since this is where the radical sign is located.

Example

Simplify. $\frac{10b^2c^2}{c\sqrt[3]{8b^4}}$

Solution

$\frac{10b^2c^2}{c\cdot\sqrt[3]{8}\cdot\sqrt[3]{b^4}}$	Separate the factors in the denominator.
$\frac{10b^2c^2}{c\cdot 2\cdot\sqrt[3]{b^4}}$	Take the cube root of 8, which is 2.
$\frac{10b^2c^2}{c\cdot 2\cdot b^{\frac{4}{3}}}$	Rewrite the radical using a fractional exponent.
$\frac{10}{2}\cdot\frac{c^2}{c}\cdot\frac{b^2}{b^{\frac{4}{3}}}$	Rewrite the fraction as a series of factors in order to cancel factors (see next step).
$5\cdot c\cdot\frac{b^2}{b^{\frac{4}{3}}}$	Simplify the constant and c factors.
$5cb^2b^{-\frac{4}{3}}$	Use the rule of negative exponents, $n^{-x}=\frac{1}{n^x}$ to rewrite $\frac{1}{b^{\frac{4}{3}}}$ as $b^{-\frac{4}{3}}$.
$5cb^{\frac{2}{3}}$	Combine the b factors by adding the exponents.
$5c\sqrt[3]{b^2}$	Change the expression with the fractional exponent back to radical form. By convention, an expression is not usually considered simplified if it has a fractional exponent or a radical in the denominator.

$\frac{10b^2c^2}{c\sqrt[3]{8b^4}} = 5c\sqrt[3]{b^2}$

Well, that took a while, but you did it. You applied what you know about fractional exponents, negative exponents, and the rules of exponents to simplify the expression.

Summary

A radical can be expressed as an expression with a fractional exponent by following the convention $\sqrt[n]{a^m} = a^{\frac{m}{n}}$. Rewriting radicals using fractional exponents can be useful in simplifying some radical expressions. When working with fractional exponents, remember that fractional exponents are subject to all of the same rules as other exponents when they appear in algebraic expressions.

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SECTION OVERVIEW

16.2: Operations with Radicals

Topic hierarchy

16.2.1: Multiplying and Dividing Radical Expressions

16.2.2: Adding and Subtracting Radicals

16.2.3: Multiplication of Multiple Term Radicals

16.2.4: Rationalizing Denominators

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16.2.1: Multiplying and Dividing Radical Expressions

Learning Objectives

- Multiply and simplify radical expressions that contain a single term.
- Divide and simplify radical expressions that contain a single term.

Introduction

You can do more than just simplify **radical expressions**. You can multiply and divide them, too. You can use your knowledge of exponents to help you when you have to operate on radical expressions this way.

Multiplying Radical Expressions

Let’s start with a quantity that you have seen before, $\sqrt{64}$. You can simplify this square root by thinking of it as $\sqrt{16 \cdot 4}$.

$$\begin{aligned} \sqrt{64} &= \sqrt{16 \cdot 4} \\ &= \sqrt{16} \cdot \sqrt{4} \\ &= 4 \cdot 2 \\ &= 8 \end{aligned}$$

If you think of the **radicand** as a product of two factors (here, thinking about 64 as the product of 16 and 4), you can take the square root of *each* factor and then multiply the roots. The end result is the same, $\sqrt{64} = 8$.

This is an example of the **Product Raised to a Power Rule**. This rule states that the product of two or more numbers raised to a power is equal to the product of each number raised to the same power.

This should be a familiar idea. You have applied this rule when expanding expressions such as $(ab)^x$ to $a^x \cdot b^x$; now you are going to amend it to include radicals as well. Imagine that the exponent x is not an integer but is a unit fraction, like $\frac{1}{3}$, so that you have the expression $(ab)^{\frac{1}{3}}$. According to the Product Raised to a Power Rule, this can also be written $a^{\frac{1}{3}} \cdot b^{\frac{1}{3}}$, which is the same as $\sqrt[3]{a} \cdot \sqrt[3]{b}$, since fractional exponents can be rewritten as roots. So, for the same reason that $(ab)^{\frac{1}{x}} = a^{\frac{1}{x}} \cdot b^{\frac{1}{x}}$, you find that $\sqrt[x]{ab} = \sqrt[x]{a} \cdot \sqrt[x]{b}$.

A Product Raised to a Power Rule

For any numbers a and b and any integer x: $(ab)^x = a^x \cdot b^x$

For any numbers x and b and any positive integer x: $(ab)^{\frac{1}{x}} = a^{\frac{1}{x}} \cdot b^{\frac{1}{x}}$

For any numbers a and b and any positive integer x: $\sqrt[x]{ab} = \sqrt[x]{a} \cdot \sqrt[x]{b}$

The Product Raised to a Power Rule is important because you can use it to multiply radical expressions. Note that the roots are the same: you can combine square roots with square roots, or cube roots with cube roots, for example. But you can’t multiply a square root and a cube root using this rule.

Let’s look at another example.

Example

Simplify. $\sqrt{18} \cdot \sqrt{16}$

Solution

$\sqrt{18} \cdot \sqrt{16}$ $\sqrt{288}$	Use the rule $\sqrt[x]{a} \cdot \sqrt[x]{b} = \sqrt[x]{ab}$ to multiply the radicands.
$\sqrt{144 \cdot 2}$	Look for perfect squares in the radicand, and rewrite the radicand as the product of two factors.
$\sqrt{(12)^2 \cdot 2}$	Identify perfect squares.
$\sqrt{(12)^2} \cdot \sqrt{2}$	Rewrite as the product of two radicals.

$\lvert 12\rvert \cdot \sqrt{2}$ $12 \cdot \sqrt{2}$	Simplify, using $\sqrt{x^2} = \lvert x\rvert$.

$\sqrt{18} \cdot \sqrt{16} = 12\sqrt{2}$

Using the Product Raised to a Power Rule, you can take a seemingly complicated expression, $\sqrt{18} \cdot \sqrt{16}$, and turn it into something more manageable, $12\sqrt{2}$.

You may have also noticed that both $\sqrt{18}$ and $\sqrt{16}$ can be written as products involving perfect square factors. How would the expression change if you simplified each radical first, *before* multiplying?

✓ Example

Simplify. $\sqrt{18} \cdot \sqrt{16}$

Solution

$\sqrt{9 \cdot 2} \cdot \sqrt{4 \cdot 4}$ $\sqrt{3 \cdot 3 \cdot 2} \cdot \sqrt{4 \cdot 4}$	Look for perfect squares in each radicand, and rewrite as the product of two factors.
$\sqrt{(3)^2 \cdot 2} \cdot \sqrt{(4)^2}$	Identify perfect squares.
$\sqrt{(3)^2} \cdot \sqrt{2} \cdot \sqrt{(4)^2}$	Rewrite as the product of radicals.
$\lvert 3\rvert \cdot \sqrt{2} \cdot \lvert 4\rvert$ $3 \cdot \sqrt{2} \cdot 4$	Simplify, using $\sqrt{x^2} = \lvert x\rvert$.
12	Multiply.

$\sqrt{18} \cdot \sqrt{16} = 12\sqrt{2}$

In both cases, you arrive at the same product, $12\sqrt{2}$. It does not matter whether you multiply the radicands or simplify each radical first.

You multiply radical expressions that contain variables in the same manner. As long as the roots of the radical expressions are the same, you can use the Product Raised to a Power Rule to multiply and simplify. Look at the two examples that follow. In both problems, the Product Raised to a Power Rule is used right away and then the expression is simplified.

✓ Example

Simplify. $\sqrt{12x^3} \cdot \sqrt{3x}, x \geq 0$

Solution

$\sqrt{12x^3 \cdot 3x}$	Use the rule $\sqrt[x]{a} \cdot \sqrt[x]{b} = \sqrt[x]{ab}$ to multiply the radicands.
$\sqrt{12 \cdot 3 \cdot x^3 \cdot x}$ $\sqrt{36 \cdot x^{3+1}}$ $\sqrt{36 \cdot x^4}$	Recall that $x^3 \cdot x = x^{3+1}$.
$\sqrt{(6)^2 \cdot (x^2)^2}$	Look for perfect squares in the radicand.
$\sqrt{(6)^2} \cdot \sqrt{(x^2)^2}$ $6 \cdot x^2$	Rewrite as the product of radicals.

$\sqrt{12x^3} \cdot \sqrt{3x} = 6x^2$

Example

Simplify. $\sqrt[3]{x^5y^2}\cdot 5\sqrt[3]{(8x^2y^4)}$

Solution

$5\sqrt[3]{x^5y^2\cdot 8x^2y^4}$	Notice that *both* radicals are cube roots, so you can use the rule $\sqrt[x]{a}\cdot\sqrt[x]{b}=\sqrt[x]{ab}$ to multiply the radicands.
$5\sqrt[3]{8\cdot x^5\cdot x^2\cdot y^2\cdot y^4}$ $5\sqrt[3]{8\cdot x^{5+2}\cdot y^{2+4}}$ $5\sqrt[3]{8\cdot x^7\cdot y^6}$	
$5\sqrt[3]{(2)^3\cdot\left(x^2\right)^3\cdot x\cdot\left(y^2\right)^3}$	Look for perfect cubes in the radicand. Since x^7 is not a perfect cube, it has to be rewritten as $x^{6+1}=\left(x^2\right)^3\cdot x$.
$5\sqrt[3]{(2)^3}\cdot\sqrt[3]{\left(x^2\right)^3}\cdot\sqrt[3]{\left(y^2\right)^3}\cdot\sqrt[3]{x}$ $5\cdot 2\cdot x^2\cdot y^2\cdot\sqrt[3]{x}$	Rewrite as the product of radicals.

$$\sqrt[3]{x^5y^2}\cdot 5\sqrt[3]{(8x^2y^4)}=10x^2y^2\sqrt[3]{x}$$

This next example is slightly more complicated because there are more than two radicals being multiplied. In this case, notice how the radicals are simplified before multiplication takes place. (Remember that the order you choose to use is up to you: you will find that sometimes it is easier to multiply before simplifying, and other times it is easier to simplify before multiplying. With some practice, you may be able to tell which is which before you approach the problem, but either order will work for all problems.)

Example

Simplify. $2\sqrt[4]{16x^9}\cdot\sqrt[4]{y^3}\cdot\sqrt[4]{81x^3y}, x\geq 0, y\geq 0$

Solution

$2\sqrt[4]{(2)^4\cdot\left(x^2\right)^4\cdot x}\cdot\sqrt[4]{y^3}\cdot\sqrt[4]{(3)^4\cdot x^3y}$	Notice this expression is multiplying three radicals with the same (fourth) root. Simplify each radical, if possible, before multiplying. Be looking for powers of 4 in each radicand.
$2\sqrt[4]{(2)^4}\cdot\sqrt[4]{\left(x^2\right)^4}\cdot\sqrt[4]{x}\cdot\sqrt[4]{y^3}\cdot\sqrt[4]{(3)^4}\cdot\sqrt[4]{x^3y}$	Rewrite as the product of radicals.
$2\cdot\lvert 2\rvert\cdot\left\lvert x^2\right\rvert\cdot\sqrt[4]{x}\cdot\sqrt[4]{y^3}\cdot\lvert 3\rvert\cdot\sqrt[4]{x^3y}$ $2\cdot 2\cdot x^2\cdot\sqrt[4]{x}\cdot\sqrt[4]{y^3}\cdot 3\cdot\sqrt[4]{x^3y}$	Identify and pull out powers of 4, using the fact that $\sqrt[4]{x^4}=\lvert x\rvert$.
$2\cdot 2\cdot 3\cdot x^2\cdot\sqrt[4]{x\cdot y^3\cdot x^3y}$	Since all the radicals are fourth roots, you can use the rule $\sqrt[x]{ab}=\sqrt[x]{a}\cdot\sqrt[x]{b}$ to multiply the radicands.
$12x^2\sqrt[4]{x^{1+3}\cdot y^{3+1}}$	
$12x^2\sqrt[4]{x^4\cdot y^4}$ $12x^2\sqrt[4]{x^4}\cdot\sqrt[4]{y^4}$ $12x^2\cdot\lvert x\rvert\cdot\lvert y\rvert$	Now that the radicands have been multiplied, look again for powers of 4, and pull them out. We can drop the absolute value signs in our final answer because at the start of the problem we were told $x\geq 0$, $y\geq 0$.

$$2\sqrt[4]{16x^9}\cdot\sqrt[4]{y^3}\cdot\sqrt[4]{81x^3y}=12x^3y$$
$$x\geq 0, y\geq 0$$

? Exercise

Which one of the following problem-and-answer pairs is incorrect?

A. Problem: $\sqrt{16}\cdot\sqrt{25}$ Answer: 20
B. Problem: $\sqrt{16}\cdot\sqrt{x^2}$ Answer: $4\lvert x\rvert$

C. Problem: $\sqrt[3]{x} \cdot \sqrt[3]{y^2}$ Answer: $\sqrt[3]{xy^2}$
D. Problem: $\sqrt{20} \cdot \sqrt[3]{y}$ Answer: $\sqrt[3]{20y}$

Answer

A. Incorrect. This problem does not contain any errors; $\sqrt{16} \cdot \sqrt{25} = 4 \cdot 5 = 20$. Answer D contains a problem-and-answer pair that is incorrect.
B. Incorrect. This problem does not contain any errors; $\sqrt{16} \cdot \sqrt{x^2} = 4 \cdot |x| = 4|x|$. Answer D contains a problem-and-answer pair that is incorrect.
C. Incorrect. This problem does not contain any errors. The two radicals that are being multiplied have the same root (3), so they can be multiplied together underneath the same radical sign. Answer D contains a problem-and-answer pair that is incorrect.
D. Correct. The two radicals have different roots, so you cannot multiply the product of the radicands and put it under the same radical sign. So, this problem-and-answer pair is incorrect.

Dividing Radical Expressions

You can use the same ideas to help you figure out how to simplify and divide radical expressions. Recall that the Product Raised to a Power Rule states that $\sqrt[x]{ab} = \sqrt[x]{a} \cdot \sqrt[x]{b}$. Well, what if you are dealing with a quotient instead of a product?

There is a rule for that, too. The **Quotient Raised to a Power Rule** states that $\left(\frac{a}{b}\right)^{\frac{1}{x}} = \frac{a^{\frac{1}{x}}}{b^{\frac{1}{x}}}$. Again, if you imagine that the exponent is a rational number, then you can make this rule applicable for square roots as well: $\left(\frac{a}{b}\right)^{\frac{1}{x}} = \frac{a^{\frac{1}{x}}}{b^{\frac{1}{x}}}$, so $\sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$.

A Quotient Raised to a Power Rule

For any real numbers a and b $(b \neq 0)$ and any positive integer x: $\left(\frac{a}{b}\right)^{\frac{1}{x}} = \frac{a^{\frac{1}{x}}}{b^{\frac{1}{x}}}$.

For any real numbers a and b $(b \neq 0)$ and any positive integer x: $\sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$

As you did with multiplication, you will start with some examples featuring integers before moving on to more complex expressions like $\frac{\sqrt[3]{24xy^4}}{\sqrt[3]{8y}}$.

Example

Simplify. $\sqrt{\frac{48}{25}}$

Solution

$\frac{\sqrt{48}}{\sqrt{25}}$	Use the rule $\sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$ to create two radicals; one in the numerator and one in the denominator.
$\frac{\sqrt{16 \cdot 3}}{\sqrt{25}}$ Or $\frac{\sqrt{4 \cdot 4 \cdot 3}}{\sqrt{5 \cdot 5}}$	Simplify each radical. Look for perfect square factors in the radicand, and rewrite the radicand as a product of factors.
$\frac{\sqrt{(4)^2 \cdot 3}}{\sqrt{(5)^2}}$ $\frac{\sqrt{(4)^2} \cdot \sqrt{3}}{\sqrt{(5)^2}}$	Identify and pull out perfect squares.
$\frac{4 \cdot \sqrt{3}}{5}$	Simplify.

$$\sqrt{\frac{48}{25}} = \frac{4\sqrt{3}}{5}$$

Example

Simplify. $\sqrt[3]{\frac{640}{40}}$

Solution

$\frac{\sqrt[3]{640}}{\sqrt[3]{40}}$	Rewrite using the Quotient Raised to a Power Rule.
$\frac{\sqrt[3]{64 \cdot 10}}{\sqrt[3]{8 \cdot 5}}$	Simplify each radical. Look for perfect cubes in the radicand, and rewrite the radicand as a product of factors.
$\frac{\sqrt[3]{(4)^3 \cdot 10}}{\sqrt[3]{(2)^3 \cdot 5}}$ $\frac{\sqrt[3]{(4)^3} \cdot \sqrt[3]{10}}{\sqrt[3]{(2)^3} \cdot \sqrt[3]{5}}$ $\frac{4 \cdot \sqrt[3]{10}}{2 \cdot \sqrt[3]{5}}$	Identify and pull out perfect cubes.
$\frac{4\sqrt[3]{10}}{2\sqrt[3]{5}}$	You can simplify this expression even further by looking for common factors in the numerator and denominator.
$\frac{2 \cdot 2\sqrt[3]{5} \cdot \sqrt[3]{2}}{2\sqrt[3]{5}}$	Rewrite the numerator as a product of factors.
$2 \cdot \frac{2\sqrt[3]{5}}{2\sqrt[3]{5}} \cdot \sqrt[3]{2}$ $2 \cdot 1 \cdot \sqrt[3]{2}$	Identify factors of 1, and simplify.

$$\sqrt[3]{\frac{640}{40}} = 2\sqrt[3]{2}$$

That was a lot of effort, but you were able to simplify using the Quotient Raised to a Power Rule. What if you found the quotient of this expression by dividing within the radical first, and then took the cube root of the quotient?

Let's take another look at that problem.

Example

Simplify. $\frac{\sqrt[3]{640}}{\sqrt[3]{40}}$

Solution

$\sqrt[3]{\frac{640}{40}}$	Since both radicals are cube roots, you can use the rule $\frac{\sqrt[x]{a}}{\sqrt[x]{b}} = \sqrt[x]{\frac{a}{b}}$ to create a single rational expression underneath the radical.
$640 \div 40 = 16$ $\sqrt[3]{16}$	Within the radical, divide 640 by 40.
$\sqrt[3]{8 \cdot 2}$	Look for perfect cubes in the radicand, and rewrite the radicand as a product of factors.
$\sqrt[3]{(2)^3 \cdot 2}$	Identify perfect cubes and pull them out.
$\sqrt[3]{(2)^3} \cdot \sqrt[3]{2}$ $2 \cdot \sqrt[3]{2}$	Simplify.

$$\frac{\sqrt[3]{640}}{\sqrt[3]{40}} = 2\sqrt[3]{2}$$

That was a more straightforward approach, wasn't it?

As with multiplication, the main idea here is that sometimes it makes sense to divide and then simplify, and other times it makes sense to simplify and then divide. Whichever order you choose, though, you should arrive at the same final expression.

Now let's turn to some radical expressions containing variables. Notice that the process for dividing these is the same as it is for dividing integers.

Example

Simplify. $\frac{\sqrt{30x}}{\sqrt{10x}}, x > 0$

Solution

$\frac{\sqrt{30x}}{\sqrt{10x}}$	Use the Quotient Raised to a Power Rule to rewrite this expression.
$\sqrt{\frac{3\cdot 10x}{10x}}$ $\sqrt{3\cdot\frac{10x}{10x}}$ $\sqrt{3\cdot 1}$	Simplify $\frac{\sqrt{30x}}{\sqrt{10x}}$ by identifying similar factors in the numerator and denominator and then identifying factors of 1.

$$\frac{\sqrt{30x}}{\sqrt{10x}}=\sqrt{3}$$

Example

Simplify. $\frac{\sqrt[3]{24xy^4}}{\sqrt[3]{8y}}, y \neq 0$

Solution

$\sqrt[3]{\frac{24xy^4}{8y}}$	Use the Quotient Raised to a Power Rule to rewrite this expression.
$\sqrt[3]{\frac{8\cdot 3\cdot x\cdot y^3\cdot y}{8\cdot y}}$ $\sqrt[3]{\frac{3\cdot x\cdot y^3}{1}\cdot\frac{8y}{8y}}$ $\sqrt[3]{\frac{3\cdot x\cdot y^3}{1}\cdot 1}$	Simplify $\sqrt[3]{\frac{24xy^4}{8y}}$ by identifying similar factors in the numerator and denominator and then identifying factors of 1.
$\sqrt[3]{3xy^3}$	Identify perfect cubes and pull them out of the radical.
$\sqrt[3]{(y)^3\cdot 3x}$ $\sqrt[3]{(y)^3}\cdot\sqrt[3]{3x}$	Simplify.

$$\frac{\sqrt[3]{24xy^4}}{\sqrt[3]{8y}}=y\sqrt[3]{3x}$$

As you become more familiar with dividing and simplifying radical expressions, make sure you continue to pay attention to the roots of the radicals that you are dividing. For example, while you can think of $\frac{\sqrt{8y^2}}{\sqrt{225y^4}}$ as equivalent to $\sqrt{\frac{8y^2}{225y^4}}$ since both the numerator and the denominator are square roots, notice that you cannot express $\frac{\sqrt{8y^2}}{\sqrt[4]{225y^4}}$ as $\sqrt[4]{\frac{8y^2}{225y^4}}$. In this second case, the numerator is a square root and the denominator is a fourth root.

Exercise

Divide and simplify. $\frac{\sqrt{27x^9}}{\sqrt{x^4}}, x > 0$

A. $3x^2\sqrt{3x}$
B. $\frac{3x^4\sqrt{3x}}{x^2}$
C. $x^2\sqrt{x}$
D. $\sqrt{27x^5}$

Answer

A. Correct. Using what you know about quotients, you can rewrite the expression as $\sqrt{\frac{27x^9}{x^4}}$, simplify it to $\sqrt{27x^5}$, and then pull out perfect squares. The simplified form is $3x^2\sqrt{3x}$.
B. Incorrect. You correctly took the square roots of $\sqrt{27x^9}$ and $\sqrt{x^4}$, but you can simplify this expression further. The correct answer is $3x^2\sqrt{3x}$.
C. Incorrect. You simplified $\frac{\sqrt{x^9}}{\sqrt{x^4}}$, not $\frac{\sqrt{27x^9}}{\sqrt{x^4}}$. The correct answer is $3x^2\sqrt{3x}$.

D. Incorrect. The expression $\frac{\sqrt{27x^9}}{\sqrt{x^4}}$ is the same as $\sqrt{27x^5}$, but it can also be simplified further. The correct answer is $3x^2\sqrt{3x}$.

Summary

The Product Raised to a Power Rule and the Quotient Raised to a Power Rule can be used to simplify radical expressions as long as the roots of the radicals are the same. The Product Rule states that the product of two or more numbers raised to a power is equal to the product of each number raised to the same power. The same is true of roots: $\sqrt[x]{ab} = \sqrt[x]{a} \cdot \sqrt[x]{b}$. When dividing radical expressions, the rules governing quotients are similar: $\sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$.

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16.2.2: Adding and Subtracting Radicals

Learning Objectives

- Add radicals and simplify.
- Subtract radicals and simplify.

Introduction

There are two keys to combining radicals by addition or subtraction: look at the **index**, and look at the **radicand**. If these are the same, then addition and subtraction are possible. If not, then you cannot combine the two radicals.

Making sense of a string of radicals may be difficult. One helpful tip is to think of radicals as variables, and treat them the same way. Let’s start there.

Thinking about Radicals as Variables

Radicals can look confusing when presented in a long string, as in $3+\sqrt{5}+\sqrt{7}+2+6\sqrt{5}$. How do you simplify this expression? (It is worth noting that you will not often see radicals presented this way, but it is a helpful way to introduce adding and subtracting radicals!)

Treating radicals the same way that you treat variables is often a helpful place to start. For example, you would have no problem simplifying the expression below.

$3+a+b+2+6a$

Combining like terms, you can quickly find that $3+2=5$ and $a+6a=7a$. The expression can be simplified to $5+7a+b$.

The same is true of radicals. As long as radicals have the same radicand (expression under the radical sign) and index (root), they can be combined. Below, the two expressions are evaluated.

$3+\sqrt{5}+\sqrt{7}+2+6\sqrt{5}$
$3+2+\sqrt{5}+6\sqrt{5}+\sqrt{7}$
$5+7\sqrt{5}+\sqrt{7}$

$3+a+b+2+6a$
$3+2+a+6a+b$
$5+7a+b$

Here’s another way to think about it. Recall that radicals are just an alternative way of writing fractional exponents. So, for example, $8^{\frac{1}{2}}=\sqrt{8}$, and $y^{\frac{1}{4}}=\sqrt[4]{y}$. If you think of radicals in terms of exponents, then all the regular rules of exponents apply.

Look at the expressions below. On the top, the expression is written in terms of radicals. On the bottom, the expression is written in terms of exponents.

$3+\sqrt{5}+\sqrt{7}+2+6\sqrt{5}$
$3+2+\sqrt{5}+6\sqrt{5}+\sqrt{7}$
$5+7\sqrt{5}+\sqrt{7}$

$3+5^{\frac{1}{2}}+7^{\frac{1}{2}}+2+6\left(5^{\frac{1}{2}}\right)$
$3+2+5^{\frac{1}{2}}+6\left(5^{\frac{1}{2}}\right)+7^{\frac{1}{2}}$
$5+7\left(5^{\frac{1}{2}}\right)+7^{\frac{1}{2}}$

So what does all this mean? Well, the bottom line is that if you need to combine radicals by adding or subtracting, make sure they have the same radicand and root. And if things get confusing, or if you just want to verify that you are combining them correctly, you can always use what you know about variables and the rules of exponents to help you.

Adding Radicals

Let’s look at some examples. In this first example, both radicals have the same root and index.

✓ Example

Add. $3\sqrt{11}+7\sqrt{11}$

Solution

$3\sqrt{11}+7\sqrt{11}$	The two radicals are the same, $\sqrt{11}$. This means you can combine them as you would combine the terms $3a+7a$.

$3\sqrt{11}+7\sqrt{11}=10\sqrt{11}$

This next example contains more addends. Notice how you can combine *like* terms (radicals that have the same root and index) but you cannot combine *unlike* terms.

✓ Example

Add. $5\sqrt{2}+\sqrt{3}+4\sqrt{3}+2\sqrt{2}$

Solution

$5\sqrt{2}+2\sqrt{2}+\sqrt{3}+4\sqrt{3}$	Rearrange terms so that like radicals are next to each other. Then add.

$5\sqrt{2}+\sqrt{3}+4\sqrt{3}+2\sqrt{2}=7\sqrt{2}+5\sqrt{3}$

Notice that the expression in the previous example is simplified even though it has two terms: $7\sqrt{2}$ and $5\sqrt{3}$. It would be a mistake to try to combine them further! (Some people make the mistake that $7\sqrt{2}+5\sqrt{3}=12\sqrt{5}$. This is incorrect because $\sqrt{2}$ and $\sqrt{3}$ are not like radicals so they cannot be added.)

✓ Example

Add. $3\sqrt{x}+12\sqrt[3]{xy}+\sqrt{x}$

Solution

$3\sqrt{x}+\sqrt{x}+12\sqrt[3]{xy}$	Rearrange terms so that like radicals are next to each other. Then add.

$3\sqrt{x}+12\sqrt[3]{xy}+\sqrt{x}=4\sqrt{x}+12\sqrt[3]{xy}$

Sometimes you may need to add *and* simplify the radical. If the radicals are different, try simplifying first. You may end up being able to combine the radicals at the end, as shown in these next two examples.

✓ Example

Add and simplify. $2\sqrt[3]{40}+\sqrt[3]{135}$

Solution

$2\sqrt[3]{8\cdot 5}+\sqrt[3]{27\cdot 5}$ $2\sqrt[3]{(2)^3\cdot 5}+\sqrt[3]{(3)^3\cdot 5}$ $2\sqrt[3]{(2)^3}\cdot\sqrt[3]{5}+\sqrt[3]{(3)^3}\cdot\sqrt[3]{5}$	Simplify each radical by identifying perfect cubes.

$2\cdot 2\cdot \sqrt[3]{5}+3\cdot \sqrt[3]{5}$	Simplify.
$4\sqrt[3]{5}+3\sqrt[3]{5}$	Add.

$2\sqrt[3]{40}+\sqrt[3]{135}=7\sqrt[3]{5}$

Example

Add and simplify. $x\sqrt[3]{xy^4}+y\sqrt[3]{x^4y}$

Solution

$x\sqrt[3]{x\cdot y^3\cdot y}+y\sqrt[3]{x^3\cdot x\cdot y}$	Simplify each radical by identifying perfect cubes.
$x\sqrt[3]{y^3}\cdot\sqrt[3]{xy}+y\sqrt[3]{x^3}\cdot\sqrt[3]{xy}$ $xy\cdot\sqrt[3]{xy}+xy\cdot\sqrt[3]{xy}$ $xy\sqrt[3]{xy}+xy\sqrt[3]{xy}$	Add like radicals.

$x\sqrt[3]{xy^4}+y\sqrt[3]{x^4y}=2xy\sqrt[3]{xy}$

Exercise

Add. $10\sqrt[3]{4}+4\sqrt[4]{3}+\sqrt[4]{3}+4\sqrt[3]{4}$

A. $14\sqrt[3]{4}+5\sqrt[4]{3}$
B. $5\sqrt[3]{4}+14\sqrt[4]{3}$
C. $19\sqrt[3]{7}$
D. $19\sqrt[4]{3}$

Answer

A. Correct. When adding radical expressions, you can combine like radicals just as you would add like variables. $10\sqrt[3]{4}+4\sqrt[4]{3}+\sqrt[4]{3}+4\sqrt[3]{4}=14\sqrt[3]{4}+5\sqrt[4]{3}$.
B. Incorrect. You reversed the coefficients and the radicals. The correct answer is $14\sqrt[3]{4}+5\sqrt[4]{3}$.
C. Incorrect. Remember that you cannot add radicals that have different index numbers or radicands. Identify like radicals in the expression and try adding again. The correct answer is $14\sqrt[3]{4}+5\sqrt[4]{3}$.
D. Incorrect. Remember that you cannot add two radicals that have different index numbers or radicands. Identify like radicals in the expression and try adding again. The correct answer is $14\sqrt[3]{4}+5\sqrt[4]{3}$.

Subtracting Radicals

Subtraction of radicals follows the same set of rules and approaches as addition: the radicands and the indices (plural of index) must be the same for two (or more) radicals to be subtracted. In the three examples that follow, subtraction has been rewritten as addition of the opposite.

Example

Subtract. $5\sqrt{13}-3\sqrt{13}$

Solution

$5\sqrt{13}-3\sqrt{13}$	The radicands and indices are the same, so these two radicals can be combined.

$5\sqrt{13}-3\sqrt{13}=2\sqrt{13}$

✓ Example

Subtract. $4\sqrt[3]{5a}-\sqrt[3]{3a}-2\sqrt[3]{5a}$

Solution

$4\sqrt[3]{5a}+(-\sqrt[3]{3a})+(-2\sqrt[3]{5a})$ $4\sqrt[3]{5a}+(-2\sqrt[3]{5a})+(-\sqrt[3]{3a})$	Two of the radicals have the same index and radicand, so they can be combined. Rewrite the expression so that like radicals are next to each other.
$2\sqrt[3]{5a}+(-\sqrt[3]{3a})$	Combine. Although the indices of $2\sqrt[3]{5a}$ and $-\sqrt[3]{3a}$ are the same, the radicands are not, so they cannot be combined.

$$4\sqrt[3]{5a}-\sqrt[3]{3a}-2\sqrt[3]{5a}=2\sqrt[3]{5a}-\sqrt[3]{3a}$$

✓ Example

Subtract and simplify. $5\sqrt[4]{a^5b}-a\sqrt[4]{16ab}$**, where** $a\geq 0$ **and** $b\geq 0$**.**

Solution

$5\sqrt[4]{a^4\cdot a\cdot b}-a\sqrt[4]{(2)^4\cdot a\cdot b}$	Simplify each radical by identifying and pulling out powers of 4.
$5\cdot a\sqrt[4]{a\cdot b}-a\cdot 2\sqrt[4]{a\cdot b}$ $5a\sqrt[4]{ab}-2a\sqrt[4]{ab}$	

$$5\sqrt[4]{a^5b}-a\sqrt[4]{16ab}=3a\sqrt[4]{ab}$$

? Exercise

Subtract and simplify. $2\sqrt{50}-4\sqrt{8}$

A. $-2\sqrt{3}$
B. $-2\sqrt{42}$
C. $2\sqrt{2}$
D. $8\sqrt{2}$

Answer

A. Incorrect. To simplify, you can rewrite $2\sqrt{50}-4\sqrt{8}$ as $2\sqrt{25\cdot 2}-4\sqrt{4\cdot 2}$. Then pull out the square roots to get $10\sqrt{2}-8\sqrt{2}=2\sqrt{2}$. The correct answer is $2\sqrt{2}$.
B. Incorrect. Remember that you cannot combine two radicands unless they are the same. $50-8=42$, but $\sqrt{50}-\sqrt{8}\neq\sqrt{42}$. The correct answer is $2\sqrt{2}$.
C. Correct. Rewriting $2\sqrt{50}-4\sqrt{8}$ as $2\sqrt{25\cdot 2}-4\sqrt{4\cdot 2}$, you found that $10\sqrt{2}-8\sqrt{2}=2\sqrt{2}$.
D. Incorrect. To simplify, you can rewrite $2\sqrt{50}-4\sqrt{8}$ as $2\sqrt{25\cdot 2}-4\sqrt{4\cdot 2}$. Then pull out the square roots to get $10\sqrt{2}-8\sqrt{2}=2\sqrt{2}$. The correct answer is $2\sqrt{2}$.

Summary

Combining radicals is possible when the index and the radicand of two or more radicals are the same. Radicals with the same index and radicand are known as like radicals. It is often helpful to treat radicals just as you would treat variables: like radicals can be added and subtracted in the same way that like variables can be added and subtracted. Sometimes, you will need to simplify a radical expression before it is possible to add or subtract like terms.

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16.2.3: Multiplication of Multiple Term Radicals

Learning Objectives

- Multiply and simplify radical expressions that contain more than one term.

Introduction

When multiplying multiple term radical expressions, it is important to follow the **Distributive Property of Multiplication**, as when you are multiplying regular, non-radical expressions. Radicals follow the same mathematical rules that other real numbers do. So, although the expression $\sqrt{x}(3\sqrt{x}-5)$ may look different than $a(3a-5)$, you can treat them the same way.

Using the Distributive Property

Let's have a look at how to apply the Distributive Property. First let's do a problem with the variable a, and then solve the same problem replacing a with $\sqrt{x}$.

Example

Simplify. $a(3a-5)$

Solution

$a(3a)-a(5)$	Use the Distributive Property of Multiplication over Subtraction.

$a(3a-5)=3a^2-5a$

Example

Simplify. $\sqrt{x}(3\sqrt{x}-5)$

Solution

$\sqrt{x}(3\sqrt{x})-\sqrt{x}(5)$	Use the Distributive Property of Multiplication over Subtraction.
$3\sqrt{x^2}-5\sqrt{x}$	Apply the rules of multiplying radicals: $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$ to multiply $\sqrt{x}(3\sqrt{x})$.
	Be sure to simplify radicals when you can: $\sqrt{x^2}=\lvert x\rvert$, so $3\sqrt{x^2}$ $=3\lvert x\rvert$.

$\sqrt{x}(3\sqrt{x}-5)=3\lvert x\rvert-5\sqrt{x}$

The answers to the previous two problems should look similar to you. The only difference is that in the second problem, $\sqrt{x}$ has replaced the variable a (and so $\lvert x\rvert$ has replaced a^2). The process of multiplying is very much the same in both problems.

In these next two problems, each term contains a radical.

Example

Simplify. $7\sqrt{x}(2\sqrt{xy}+\sqrt{y})$

Solution

$7\sqrt{x}(2\sqrt{xy})+7\sqrt{x}(\sqrt{y})$	Use the Distributive Property of Multiplication over Addition to multiply each term within parentheses by $7\sqrt{x}$.
$7\cdot 2\sqrt{x^2y}+7\sqrt{xy}$	Apply the rules of multiplying radicals.
$14\sqrt{x^2\cdot y}+7\sqrt{xy}$	$\sqrt{x^2}=\lvert x\rvert$, so $\lvert x\rvert$ can be pulled out of the radical.

$7\sqrt{x}(2\sqrt{xy}+\sqrt{y})=14|x|\sqrt{y}+7\sqrt{xy}$

✓ Example

Simplify. $\sqrt[3]{a}\left(2\sqrt[3]{a^2}-4\sqrt[3]{a^5}+8\sqrt[3]{a^8}\right)$

Solution

$\sqrt[3]{a}\left(2\sqrt[3]{a^2}\right)-\sqrt[3]{a}\left(4\sqrt[3]{a^5}\right)+\sqrt[3]{a}\left(8\sqrt[3]{a^8}\right)$	Use the Distributive Property.
$2\sqrt[3]{a\cdot a^2}-4\sqrt[3]{a\cdot a^5}+8\sqrt[3]{a\cdot a^8}$	Apply the rules of multiplying radicals.
$2\sqrt[3]{a^3}-4\sqrt[3]{a^6}+8\sqrt[3]{a^9}$	
$2\sqrt[3]{a^3}-4\sqrt[3]{(a^2)^3}+8\sqrt[3]{(a^3)^3}$	Identify cubes in each of the radicals.

$\sqrt[3]{a}\left(2\sqrt[3]{a^2}-4\sqrt[3]{a^5}+8\sqrt[3]{a^8}\right)=2a-4a^2+8a^3$

In all of these examples, multiplication of radicals has been shown following the pattern $\sqrt{a}\cdot\sqrt{b}\cdot\sqrt{ab}$. Then, only after multiplying, some radicals have been simplified, like in the last problem. After you have worked with radical expressions a bit more, you may feel more comfortable identifying quantities such as $\sqrt{x}\cdot\sqrt{x}=x$ without going through the intermediate step of finding that $\sqrt{x}\cdot\sqrt{x}=\sqrt{x^2}$. In the rest of the examples that follow, though, each step is shown.

? Exercise

Multiply and simplify. $\sqrt{10}(\sqrt{10}-\sqrt{5})$

A. $10-5\sqrt{2}$
B. 10
C. $5\sqrt{2}$
D. $\sqrt{100}-\sqrt{50}$

Answer

A. Correct. Multiplying $\sqrt{10}$ by $\sqrt{10}$ and $-\sqrt{5}$, you find it is equal to $\sqrt{100}-\sqrt{50}$, or $10-5\sqrt{2}$.
B. Incorrect. $\sqrt{10}\cdot\sqrt{10}=10$, but what does $\sqrt{10}\cdot(-\sqrt{5})$ simplify to? The correct answer is $10-5\sqrt{2}$.
C. Incorrect. You subtracted $\sqrt{10}-\sqrt{5}=\sqrt{5}$, and then multiplied by $\sqrt{10}$. Remember that you cannot subtract radicals unless the indices and the radicands are the same. The correct answer is $10-5\sqrt{2}$.
D. Incorrect. This is the correct product, but it is not in simplest form. Look for squares that exist within $\sqrt{100}$ and $\sqrt{50}$. The correct answer is $10-5\sqrt{2}$.

Multiplying Radical Expressions as Binomials

Sometimes, radical expressions appear in binomials as well. In these cases, you still follow the rules of binomial multiplication, but it is very important that you be precise and structured when you are multiplying the different terms.

As a refresher, here is the process for multiplying two binomials. If you like using the expression “FOIL” (First, Outside, Inside, Last) to help you figure out the order in which the terms should be multiplied, you can use it here, too.

✓ Example

Multiply. $(2x+5)(3x-2)$

Solution

First: $2x\cdot 3x=6x^2$ Outside: $2x\cdot(-2)=-4x$ Inside: $5\cdot 3x=15x$ Last: $5\cdot(-2)=-10$	Use the Distributive Property.

$6x^2-4x+15x-10$	Record the terms, and then combine like terms.

$(2x+5)(3x-2)=6x^2+11x-10$

Here is the same problem, with $\sqrt{b}$ replacing the variable x.

✓ Example

Multiply. $(2\sqrt{b}+5)(3\sqrt{b}-2), b\geq 0$

Solution

First: $2\sqrt{b}\cdot 3\sqrt{b}=2\cdot 3\cdot\sqrt{b}\cdot\sqrt{b}=6b$ Outside: $2\sqrt{b}\cdot(-2)=-4\sqrt{b}$ Inside: $5\cdot 3\sqrt{b}=15\sqrt{b}$ Last: $5\cdot(-2)=-10$	Use the Distributive Property to multiply. Simplify using $\sqrt{x}\cdot\sqrt{x}=x$.
$6b-4\sqrt{b}+15\sqrt{b}-10$	Record the terms, and then combine like terms.

$(2\sqrt{b}+5)(3\sqrt{b}-2)=6b+11\sqrt{b}-10$

The multiplication works the same way in both problems; you just have to pay attention to the index of the radical (that is, whether the roots are square roots, cube roots, etc.) when multiplying radical expressions.

Multiplying Binomial Radical Expressions

To multiply radical expressions, use the same method as used to multiply polynomials.

- Use the Distributive Property (or, if you prefer, the shortcut FOIL method);
- Remember that $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$; and
- Combine like terms.

✓ Example

Multiply. $(4x^2+\sqrt[3]{x})\left(\sqrt[3]{x^2}+2\right)$

Solution

First: $4x^2\cdot\sqrt[3]{x^2}=4x^2\sqrt[3]{x^2}$ Outside: $4x^2\cdot 2=8x^2$ Inside: $\sqrt[3]{x}\cdot\sqrt[3]{x^2}=\sqrt[3]{x^2\cdot x}=\sqrt[3]{x^3}=x$ Last: $\sqrt[3]{x}\cdot 2=2\sqrt[3]{x}$	Use FOIL to multiply.
$4x^2\sqrt[3]{x^2}+8x^2+x+2\sqrt[3]{x}$	Record the terms, and then combine like terms (if possible). Here, there are no like terms to combine.

$(4x^2+\sqrt[3]{x})\left(\sqrt[3]{x^2}+2\right)=4x^2\sqrt[3]{x^2}+8x^2+x+2\sqrt[3]{x}$

? Exercise

Multiply and simplify. $(4\sqrt{x}+3)(2\sqrt{x}-1), x\geq 0$

A. 7
B. $8x+2\sqrt{x}-3$
C. $8x^2+2x+3$
D. $4x+8\sqrt{x}-3$

Answer

A. Incorrect. Using the FOIL method, the product is $8x-4\sqrt{x}+6\sqrt{x}-3$, which simplifies to $8x+2\sqrt{x}-3$. It looks like you forgot the variables and added the coefficients $8+2-3$ in order to arrive at 7. The correct answer is $8x+2\sqrt{x}-3$.
B. Correct. Using the FOIL method, you find that the product of the binomials is $8\cdot\sqrt{x}\cdot\sqrt{x}-4\sqrt{x}+6\sqrt{x}-3$, which simplifies to $8x+2\sqrt{x}-3$.
C. Incorrect. Remember that $4\sqrt{x}\cdot 2\sqrt{x}=4\cdot 2\cdot\sqrt{x}\cdot\sqrt{x}=8x$, not $8x^2$. The correct answer is $8x+2\sqrt{x}-3$.
D. Incorrect. You have multiplied the variable terms correctly, but you have the incorrect coefficients. Using the FOIL method, the product is $8x-4\sqrt{x}+6\sqrt{x}-3$, which simplifies to $8x+2\sqrt{x}-3$. The correct answer is $8x+2\sqrt{x}-3$.

Summary

To multiply radical expressions that contain more than one term, use the same method that you use to multiply polynomials. First, use the Distributive Property (or, if you prefer, the shortcut FOIL method) to multiply the terms. Then, apply the rules $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$, and $\sqrt{x}\cdot\sqrt{x}=x$ to multiply and simplify. Finally, combine like terms.

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16.2.4: Rationalizing Denominators

Learning Objectives

- Rationalize a denominator with a monomial containing a square root.
- Rationalize a denominator that contains two terms.

Introduction

Although radicals follow the same rules that integers do, it is often difficult to figure out the value of an expression containing radicals. For example, you probably have a good sense of how much $\frac{4}{8}$, 0.75, and $\frac{6}{9}$ are, but what about the quantities $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{5}}$? These are much harder to visualize.

That said, sometimes you have to work with expressions that contain many radicals. Often the value of these expressions is not immediately clear. In cases where you have a fraction with a radical in the denominator, you can use a technique called **rationalizing a denominator** to eliminate the radical. The point of rationalizing a denominator is to make it easier to understand what the quantity really is by removing radicals from the denominators.

What is Rationalizing a Denominator?

The idea of rationalizing a denominator makes a bit more sense if you consider the definition of "rationalize." Recall that the numbers 5, $\frac{1}{2}$, and 0.75 are all known as rational numbers because they can each be expressed as a ratio of two integers ($\frac{5}{1}$, $\frac{1}{2}$, and $\frac{3}{4}$ respectively). Some radicals are irrational numbers because they cannot be represented as a ratio of two integers. As a result, the point of rationalizing a denominator is to change the expression so that the denominator becomes a rational number.

Here are some examples of irrational and rational denominators.

Irrational		**Rational**
$\frac{1}{\sqrt{2}}$	$=$	$\frac{\sqrt{2}}{2}$
$\frac{2+\sqrt{3}}{\sqrt{3}}$	$=$	$\frac{2\sqrt{3}+3}{3}$

Now let's examine how to get from irrational to rational denominators.

Rationalizing Denominators with One Term

Let's start with the fraction $\frac{1}{\sqrt{2}}$. Its denominator is $\sqrt{2}$, an irrational number. This makes it difficult to figure out the value of $\frac{1}{\sqrt{2}}$.

You can rename this fraction without changing its value, if you multiply it by 1. In this case, set 1 equal to $\frac{\sqrt{2}}{\sqrt{2}}$. Watch what happens.

$$\frac{1}{\sqrt{2}}\cdot 1=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{\sqrt{2\cdot 2}}=\frac{\sqrt{2}}{\sqrt{4}}=\frac{\sqrt{2}}{2}$$

The denominator of the new fraction is no longer a radical (notice, however, that the numerator is).

So why choose to multiply $\frac{1}{\sqrt{2}}$ by $\frac{\sqrt{2}}{\sqrt{2}}$? You knew that the square root of a number times itself will be a whole number. In algebraic terms, this idea is represented by $\sqrt{x}\cdot\sqrt{x}=x$. Look back to the denominators in the multiplication of $\frac{1}{\sqrt{2}}\cdot 1$. Do you see where $\sqrt{2}\cdot\sqrt{2}=\sqrt{4}=2$?

Here are some more examples. Notice how the value of the fraction is not changed at all; it is simply being multiplied by another name for 1.

Example

Rationalize the denominator. $\frac{2+\sqrt{3}}{\sqrt{3}}$

Solution

$\frac{2+\sqrt{3}}{\sqrt{3}}$	The denominator of this fraction is $\sqrt{3}$. To make it into a rational number, multiply it by $\sqrt{3}$, since $\sqrt{3}\cdot\sqrt{3}=3$.

$\frac{2+\sqrt{3}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$	Multiply the entire fraction by another name for 1, $\frac{\sqrt{3}}{\sqrt{3}}$.
$\frac{\sqrt{3}(2+\sqrt{3})}{\sqrt{3}\cdot\sqrt{3}}$	
$\frac{2\sqrt{3}+\sqrt{3}\cdot\sqrt{3}}{\sqrt{9}}$	Use the Distributive Property to multiply $\sqrt{3}(2+\sqrt{3})$.
$\frac{2\sqrt{3}+\sqrt{9}}{\sqrt{9}}$	Simplify the radicals, where possible. $\sqrt{9}=3$.

$$\frac{2+\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}+3}{3}$$

You can use the same method to rationalize denominators to simplify fractions with radicals that contain a variable. As long as you multiply the original expression by another name for 1, you can eliminate a radical in the denominator without changing the value of the expression itself.

✓ Example

Rationalize the denominator. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}$**, where** $x\neq 0$

Solution

$\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}}$	The denominator is $\sqrt{x}$, so the entire expression can be multiplied by $\frac{\sqrt{x}}{\sqrt{x}}$ to get rid of the radical in the denominator.
$\frac{\sqrt{x}(\sqrt{x}+\sqrt{y})}{\sqrt{x}\cdot\sqrt{x}}$	
$\frac{\sqrt{x}\cdot\sqrt{x}+\sqrt{x}\cdot\sqrt{y}}{\sqrt{x}\cdot\sqrt{x}}$	Use the Distributive Property. Simplify the radicals, where possible. Remember that $\sqrt{x}\cdot\sqrt{x}=x$.

$$\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}=\frac{x+\sqrt{xy}}{x}$$

✓ Example

Rationalize the denominator and simplify. $\sqrt{\frac{100x}{11y}}$**, where** $y\neq 0$

Solution

$\frac{\sqrt{100x}}{\sqrt{11y}}$	Rewrite $\sqrt{\frac{a}{b}}$ as $\frac{\sqrt{a}}{\sqrt{b}}$.
$\frac{\sqrt{100x}}{\sqrt{11y}}\cdot\frac{\sqrt{11y}}{\sqrt{11y}}$	The denominator is $\sqrt{11y}$, so multiplying the entire expression by $\frac{\sqrt{11y}}{\sqrt{11y}}$ will rationalize the denominator.
$\frac{\sqrt{100\cdot 11xy}}{\sqrt{11y}\cdot\sqrt{11y}}$	Multiply and simplify the radicals, where possible.
$\frac{\sqrt{100}\cdot\sqrt{11xy}}{\sqrt{11y}\cdot\sqrt{11y}}$	100 is a perfect square. Remember that $\sqrt{100}=10$ and $\sqrt{x}\cdot\sqrt{x}=x$.

$$\sqrt{\frac{100x}{11y}}=\frac{10\sqrt{11xy}}{11y}$$

? Exercise

Rationalize the denominator and simplify. $\frac{3}{\sqrt{7}}$

A. $\frac{3\sqrt{7}}{\sqrt{7}}$
B. 3
C. $\frac{7\sqrt{3}}{3}$
D. $\frac{3\sqrt{7}}{7}$

Answer

A. Incorrect. After you rationalize a denominator, the denominator should not be in radical form. Try multiplying the fraction by $\frac{\sqrt{7}}{\sqrt{7}}$. The correct answer is $\frac{3\sqrt{7}}{7}$.
B. Incorrect. The process of rationalizing does not remove the denominator, it just changes the fraction so that the denominator is no longer a radical. Try multiplying the numerator and denominator by $\frac{\sqrt{7}}{\sqrt{7}}$. The correct answer is $\frac{3\sqrt{7}}{7}$.
C. Incorrect. You may have rationalized the fraction $\frac{7}{\sqrt{3}}$ instead of $\frac{3}{\sqrt{7}}$. Try multiplying the numerator and denominator by $\frac{\sqrt{7}}{\sqrt{7}}$. The correct answer is $\frac{3\sqrt{7}}{7}$.
D. Correct. To rationalize the denominator, multiply $\frac{3}{\sqrt{7}}$ by $\frac{\sqrt{7}}{\sqrt{7}}$. This will create an integer in the denominator, so the fraction will have been rationalized.

Rationalizing Denominators with Two Terms

Denominators do not always contain just one term, as shown in the previous examples. Sometimes, you will see expressions like $\frac{3}{\sqrt{2}+3}$ where the denominator is composed of two terms, $\sqrt{2}$ and $+3$.

Unfortunately, you cannot rationalize these denominators the same way you rationalize single-term denominators. If you multiply $\sqrt{2}+3$ by $\sqrt{2}$, you get $2+3\sqrt{2}$. The original $\sqrt{2}$ is gone, but now the quantity $3\sqrt{2}$ has appeared, but this is no better!

In order to rationalize this denominator, you want to square the radical term and somehow prevent the integer term from being multiplied by a radical. Is this possible?

It is possible, and you have already seen how to do it!

Recall that when binomials of the form $(a+b)(a-b)$ are multiplied, the product is a^2-b^2. So, for example, $(x+3)(x-3)=x^2-3x+3x-9=x^2-9$; notice that the terms $-3x$ and $+3x$ combine to 0. Now for the connection to rationalizing denominators: what if you replaced x with $\sqrt{2}$?

Look at the examples below. Just as $-3x+3x$ combines to 0 in the top example, $-3\sqrt{2}+3\sqrt{2}$ combines to 0 in the bottom example.

$$\begin{aligned}&(x+3)(x-3)\\&=x^2-3x+3x-9\\&=x^2-9\end{aligned}$$

$$\begin{aligned}&(\sqrt{2}+3)(\sqrt{2}-3)\\&=(\sqrt{2})^2-3\sqrt{2}+3\sqrt{2}-9\\&=(\sqrt{2})^2-9\\&=2-9\\&=-7\end{aligned}$$

There you have it! Multiplying $\sqrt{2}+3$ by $\sqrt{2}-3$ removed one radical without adding another.

In this example, $\sqrt{2}-3$ is known as a **conjugate**, and $\sqrt{2}+3$ and $\sqrt{2}-3$ are known as a **conjugate pair**. To find the conjugate of a binomial that includes radicals, change the sign of the second term to its opposite as shown in the table below.

Term	**Conjugate**	**Product**
$\sqrt{2}+3$	$\sqrt{2}-3$	$(\sqrt{2}+3)(\sqrt{2}-3)=(\sqrt{2})^2-(3)^2=2-9=-7$
$\sqrt{x}-5$	$\sqrt{x}+5$	$(\sqrt{x}-5)(\sqrt{x}+5)=(\sqrt{x})^2-(5)^2=x-25$
$8-2\sqrt{x}$	$8+2\sqrt{x}$	$(8-2\sqrt{x})(8+2\sqrt{x})=(8)^2-(2\sqrt{x})^2=64-4x$
$1+\sqrt{xy}$	$1-\sqrt{xy}$	$(1+\sqrt{xy})(1-\sqrt{xy})=(1)^2-(\sqrt{xy})^2=1-xy$

Example

Rationalize the denominator and simplify. $\frac{5-\sqrt{7}}{3+\sqrt{5}}$

Solution

$\frac{5-\sqrt{7}}{3+\sqrt{5}} \cdot \frac{3-\sqrt{5}}{3-\sqrt{5}}$	Find the conjugate of $3+\sqrt{5}$. Then multiply the entire expression by $\frac{3-\sqrt{5}}{3-\sqrt{5}}$.
$\frac{(5-\sqrt{7})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})}$	
$\frac{5\cdot 3-5\sqrt{5}-3\sqrt{7}+\sqrt{7}\cdot\sqrt{5}}{3\cdot 3-3\sqrt{5}+3\sqrt{5}-\sqrt{5}\cdot\sqrt{5}}$	Use the Distributive Property to multiply the binomials in the numerator and denominator.
$\frac{15-5\sqrt{5}-3\sqrt{7}+\sqrt{35}}{9-3\sqrt{5}+3\sqrt{5}-\sqrt{25}}$	Since you multiplied by the conjugate of the denominator, the radical terms in the denominator will combine to 0.
$\frac{15-5\sqrt{5}-3\sqrt{7}+\sqrt{35}}{9-\sqrt{25}}$ $\frac{15-5\sqrt{5}-3\sqrt{7}+\sqrt{35}}{9-5}$	Simplify radicals where possible.

$$\frac{5-\sqrt{7}}{3+\sqrt{5}} = \frac{15-5\sqrt{5}-3\sqrt{7}+\sqrt{35}}{4}$$

Example

Rationalize the denominator and simplify. $\frac{\sqrt{x}}{\sqrt{x}+2}$

Solution

$\frac{\sqrt{x}}{\sqrt{x}+2} \cdot \frac{\sqrt{x}-2}{\sqrt{x}-2}$	Find the conjugate of $\sqrt{x}+2$. Then multiply the numerator and denominator by $\frac{\sqrt{x}-2}{\sqrt{x}-2}$.
$\frac{\sqrt{x}(\sqrt{x}-2)}{(\sqrt{x}+2)(\sqrt{x}-2)}$	
$\frac{\sqrt{x}\cdot\sqrt{x}-2\sqrt{x}}{\sqrt{x}\cdot\sqrt{x}-2\sqrt{x}+2\sqrt{x}-2\cdot 2}$	Use the Distributive Property to multiply the binomials in the numerator and denominator.
$\frac{\sqrt{x}\cdot\sqrt{x}-2\sqrt{x}}{\sqrt{x}\cdot\sqrt{x}-2\sqrt{x}+2\sqrt{x}-4}$	Simplify. Remember that $\sqrt{x}\cdot\sqrt{x}=x$. Since you multiplied by the conjugate of the denominator, the radical terms in the denominator will combine to 0.

$$\frac{\sqrt{x}}{\sqrt{x}+2} = \frac{x-2\sqrt{x}}{x-4}$$

One word of caution: this method will work for binomials that include a square root, but not for binomials with roots greater than 2. This is because squaring a root that has an index greater than 2 does not remove the root, as shown below.

$$\begin{aligned}
&(\sqrt[3]{10}+5)(\sqrt[3]{10}-5)\\
&=(\sqrt[3]{10})^2-5\sqrt[3]{10}+5\sqrt[3]{10}-25\\
&=(\sqrt[3]{10})^2-25\\
&=\sqrt[3]{100}-25
\end{aligned}$$

$\sqrt[3]{100}$ cannot be simplified any further, since its prime factors are $2\cdot 2\cdot 5\cdot 5$. There are no cubed numbers to pull out! Multiplying $\sqrt[3]{10}+5$ by its conjugate does not result in a radical-free expression.

Exercise

Identify the conjugate of the denominator. $\frac{x+2}{\sqrt{2x}-7}$

A. $\sqrt{2x}+7$
B. $\sqrt{2x}-7$
C. $2x+\sqrt{7}$

D. $(2x)^2+7$

Answer

A. Correct. The conjugate will be the binomial that, when multiplied by the denominator, eliminates the radical. The conjugate is $\sqrt{2x}+7$.

B. Incorrect. The denominator is not the conjugate. Look for the binomial that, when multiplied by the denominator, eliminates the radical. The correct answer is $\sqrt{2x}+7$.

C. Incorrect. Multiplying the denominator by $2x+\sqrt{7}$ will not remove the radical. Look for the binomial that follows the pattern $(a+b)(a-b)=a^2-b^2$. The correct answer is $\sqrt{2x}+7$.

D. Incorrect. Multiplying the denominator by $(2x)^2+7$ will not remove the radical. Look for the binomial that follows the pattern $(a+b)(a-b)=a^2-b^2$. The correct answer is $\sqrt{2x}+7$.

Summary

When you encounter a fraction that contains a radical in the denominator, you can eliminate the radical by using a process called rationalizing the denominator. To rationalize a denominator, you need to find a quantity that, when multiplied by the denominator, will create a rational number (no radical terms) in the denominator. When the denominator contains a single term, as in $\frac{1}{\sqrt{5}}$, multiplying the fraction by $\frac{\sqrt{5}}{\sqrt{5}}$ will remove the radical from the denominator. When the denominator contains two terms, as in $\frac{2}{\sqrt{5}+3}$, identify the conjugate of the denominator, here $\sqrt{5}-3$, and multiply both numerator and denominator by the conjugate.

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SECTION OVERVIEW

16.3: Radical Equations

Topic hierarchy

16.3.1: Solving Radical Equations

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16.3.1: Solving Radical Equations

Learning Objectives

- Solve equations containing radicals.
- Recognize extraneous solutions.
- Solve application problems that involve radical equations as part of the solution.

Introduction

An equation that contains a **radical expression** is called a **radical equation**. Solving radical equations requires applying the rules of exponents and following some basic algebraic principles. In some cases, it also requires looking out for errors generated by raising unknown quantities to an even power.

Squaring Both Sides

A basic strategy for solving radical equations is to isolate the radical term first, and then raise both sides of the equation to a power to remove the radical. (The reason for using powers will become clear in a moment.) This is the same type of strategy you used to solve other, non-radical equations: rearrange the expression to isolate the variable you want to know, and then solve the resulting equation.

There are two key ideas that you will be using to solve radical equations. The first is that if $a=b$, then $a^2=b^2$. (This property allows you to square both sides of an equation and remain certain that the two sides are still equal.) The second is that if the square root of any nonnegative number x is squared, then you get x: $(\sqrt{x})^2=x$. (This property allows you to “remove” the radicals from your equations.)

Let’s start with a radical equation that you can solve in a few steps: $\sqrt{x}-3=5$.

Example

Solve. $\sqrt{x}-3=5$

Solution

$\begin{array}{r}\sqrt{x}-3 \quad =5\\ \underline{\quad +3 \quad +3}\end{array}$	Add 3 to both sides to isolate the variable term on the left side of the equation.
$\sqrt{x}=8$	Collect like terms.
$\begin{aligned}(\sqrt{x})^2&=8^2\\ x&=64\end{aligned}$	Square both sides to remove the radical, since $(\sqrt{x})^2=x$. Make sure to square the 8 also! Then simplify.

$x=64$ is the solution to $\sqrt{x}-3=5$.

To check your solution, you can substitute 64 in for x in the original equation. Does $\sqrt{64}-3=5$? Yes—the square root of 64 is 8, and $8-3=5$.

Notice how you combined like terms and then squared both *sides* of the equation in this problem. This is a standard method for removing a radical from an equation. It is important to isolate a radical on one side of the equation and simplify as much as possible *before* squaring. The fewer terms there are before squaring, the fewer additional terms will be generated by the process of squaring.

In the example above, only the variable x was underneath the radical. Sometimes you will need to solve an equation that contains multiple terms underneath a radical. Follow the same steps to solve these, but pay attention to a critical point: square both *sides* of an equation, not individual *terms*. Watch how the next two problems are solved.

Example

Solve. $\sqrt{x+8}=3$

Solution

$(\sqrt{x+8})^2=(3)^2$	Notice how the radical contains a binomial: $x+8$. Square both sides to remove the radical.
$x+8=9$ $x=1$	$(\sqrt{x+8})^2=x+8$. Now simplify the equation and solve for x.
$\sqrt{1+8}=3$ $\sqrt{9}=3$ $3=3$	Check your answer. Substituting 1 for x in the original equation yields a true statement, so the solution is correct.

$x=1$ is the solution to $\sqrt{x+8}=3$.

Example

Solve. $1+\sqrt{2x+3}=6$

Solution

$1+\sqrt{2x+3}-1=6-1$ $\sqrt{2x+3}=5$ $(\sqrt{2x+3})^2=(5)^2$	Begin by subtracting 1 from both sides in order to isolate the radical term. Then square both sides to remove the binomial from the radical.
$2x+3=25$ $2x=22$ $x=11$	Simplify the equation and solve for x.
$1+\sqrt{2(11)+3}=6$ $1+\sqrt{22+3}=6$ $1+\sqrt{25}=6$ $1+5=6$ $6=6$	Check your answer. Substituting 11 for x in the original equation yields a true statement, so the solution is correct.

$x=11$ is the solution for $1+\sqrt{2x+3}=6$.

Solving Radical Equations

Follow the four steps to solve radical equations.

1. Isolate the radical expression.
2. Square both sides of the equation: If $x=y$ then $x^2=y^2$.
3. Once the radical is removed, solve for the unknown.
4. Check all answers.

Exercise

Solve. $\sqrt{3x+22}=4$

A. $x=2$
B. $x=\frac{16}{3}$
C. $x=-2$
D. $x=-6$

Answer

A. Incorrect. Check your answer. If you substitute $x=2$ into the equation, you get $\sqrt{3(2)+22}=4$, or $\sqrt{28}=4$. This is not correct. Remember to square both sides and then solve for x. The correct answer is $x=-2$.

B. Incorrect. It looks like you squared both sides but ignored the $+22$ underneath the radical. Remember to include the entire binomial when you square both sides; then solve for x. The correct answer is $x=-2$.
C. Correct. Squaring both sides, you find $(\sqrt{3x+22})^2=(4)^2$ becomes $3x+22=16$, so $3x=-6$ and $x=-2$.
D. Incorrect. It looks like you only squared the left side of the equation. Remember to square both sides: $(\sqrt{3x+22})^2$ $=(4)^2$, which becomes $3x+22=16$. Now solve for x. The correct answer is $x=-2$.

Extraneous Solutions

Following rules is important, but so is paying attention to the math in front of you, especially when solving radical equations. Take a look at this next problem that demonstrates a potential pitfall of squaring both sides to remove the radical.

✓ Example

Solve. $\sqrt{a-5}=-2$

Solution

$(\sqrt{a-5})^2=(-2)^2$	Square both sides to remove the term $a-5$ from the radical.
$a-5=4$ $a=9$	Write the simplified equation, and solve for a.
$\sqrt{9-5}=-2$ $\sqrt{4}=-2$ $2\neq-2$	Now check the solution by substituting $a=9$ into the original equation.t does not check!

No solution.

The answer $a=9$ does not produce a true statement when substituted back into the original equation. What happened?

Check the original problem: $\sqrt{a-5}=-2$. Notice that the radical is set equal to -2, and recall that the principal square root of a number can only be *positive*. This means that no value for a will result in a radical expression whose positive square root is -2! You might have noticed that right away and concluded that there were no solutions for a. But why did the process of squaring create an answer, $a=9$, that proved to be incorrect?

The answer lies in the process of *squaring* itself. When you raise a number to an even power, whether it is the second, fourth, or 50th power, you can introduce a false solution because the result of an even power is always a positive number. Think about it: 3^2 and $(-3)^2$ are both 9, and 2^4 and $(-2)^4$ are both 16. So when you squared -2 and got 4 in this problem, you artificially turned the quantity positive. This is why you were still able to find a value for a; you solved the problem as if you were solving $\sqrt{a-5}=2$! (The correct solution to $\sqrt{a-5}=-2$ is actually “no solution.”)

Incorrect values of the variable, such as those that are introduced as a result of the squaring process are called **extraneous solutions**. Extraneous solutions may look like the real solution, but you can identify them because they will not create a true statement when substituted back into the original equation. This is one of the reasons why checking your work is so important. If you do not check your answers by substituting them back into the original equation, you may be introducing extraneous solutions into the problem.

Have a look at the following problem. Notice how the original problem is $x+4=\sqrt{x+10}$, but after both sides are squared, it becomes $x^2+8x+16=x+10$. Squaring both sides may have introduced an extraneous solution.

✓ Example

Solve. $x+4=\sqrt{x+10}$

Solution

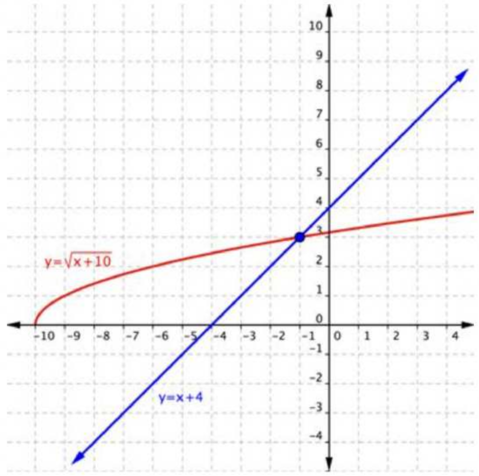
$(x+4)^2=(\sqrt{x+10})^2$	Square both sides to remove the term $x+10$ from the radical.

$(x+4)(x+4)=x+10$ $x^2+8x+16=x+10$ $x^2+8x-x+16-10=0$ $x^2+7x+6=0$ $(x+6)(x+1)=0$	Now simplify and solve the equation. Combine like terms, and then factor.
$(x+6)=0 \quad \text{or} \quad (x+1)=0$ $x=-6 \quad \text{or} \quad x=-1$	Set each factor equal to zero and solve for x.
$-6+4=\sqrt{-6+10} \quad -1+4=\sqrt{-1+10}$ $-2=\sqrt{4} \quad 3=\sqrt{9}$ $-2=2 \quad 3=3$ FALSE! TRUE!	Now check both solutions by substituting them into the original equation. Since $x=-6$ produces a false statement, it is an extraneous solution.

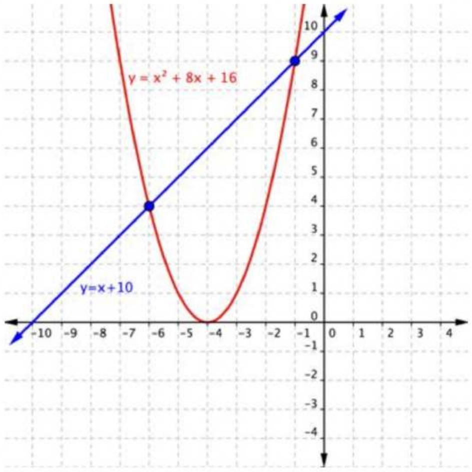
$x=-1$ is the only solution

It may be difficult to understand why extraneous solutions exist at all. Thinking about extraneous solutions by graphing the equation may help you make sense of what is going on.

You can graph $x+4=\sqrt{x+10}$ on a coordinate plane by breaking it into a system of two equations: $y=x+4$ and $y=\sqrt{x+10}$. The graph is shown below. Notice how the two graphs intersect at one point, when the value of x is -1. This is the value of x that satisfies both equations, so it is the solution to the system.



Now, following the work we did in the example problem, let’s square both of the expressions to remove the variable from the radical. Instead of solving the equation $x+4=\sqrt{x+10}$ we are now solving the equation $(x+4)^2=(\sqrt{x+10})^2$, or $x^2+8x+16=x+10$. The graphs of $y=x^2+8x+16$ and $y=x+10$ are plotted below. Notice how the two graphs intersect at two points, when the values of x are -1 and -6.



Although $x=-1$ is shown as a solution in both graphs, squaring both sides of the equation had the effect of adding an extraneous solution, $x=-6$. Again, this is why it is so important to check your answers when solving radical equations!

✓ Example

Solve. $4+\sqrt{x+2}=x$

Solution

$\sqrt{x+2}=x-4$	Isolate the radical term.
$(\sqrt{x+2})^2=(x-4)^2$	Square both sides to remove the term $x+2$ from the radical.
$x+2=x^2-8x+16$ $0=x^2-8x-x+16-2$ $0=x^2-9x+14$ $0=(x-7)(x-2)$	Now simplify and solve the equation. Combine like terms, and then factor.
$(x-7)=0$ or $(x-2)=0$ $x=7$ or $x=2$	Set each factor equal to zero and solve for x.
$4+\sqrt{7+2}=7$ $4+\sqrt{2+2}=2$ $4+\sqrt{9}=7$ $4+\sqrt{4}=2$ $4+3=7$ $4+2=2$ $7=7$ $6=2$ TRUE! FALSE!	Now check both solutions by substituting them into the original equation. Since $x=2$ produces a false statement, it is an extraneous solution.

$x=7$ is the only solution.

? Exercise

Solve. $x-3=\sqrt{4x+9}$

A. $x=3,0$
B. $x=0,10$
C. $x=0$
D. $x=10$

Answer

A. Incorrect. To solve the equation, square both sides and then solve the resulting equation $x^2-6x+9=4x+9$. The correct answer is: $x=10$.

B. Incorrect. You solved the equation $x^2-6x+9=4x+9$ correctly, but you forgot to evaluate both values for x in the original equation. $x=0$ is an extraneous solution since it doesn't make the original equation true! The correct answer is: $x=10$

C. Incorrect. When you square both sides and then solve the resulting equation, $x^2-6x+9=4x+9$, you do get $x=0$ as a possible solution. However, $x=0$ is an extraneous solution since it does not make the original equation true! The correct answer is: $x=10$

D. Correct. Solving the equation, you find that squaring both sides results in $x^2-6x+9=4x+9$, which simplifies to $x^2-10x=0$. Although this equation produces x values of 0 or 10, 0 is extraneous since it does not make the original equation true.

Solving Application Problems with Radical Equations

Radical equations play a significant role in science, engineering, and even music. Sometimes you may need to use what you know about radical equations to solve for different variables in these types of problems.

✓ Example

One way to measure the amount of energy that a moving object (such as a car) possesses is by finding its Kinetic Energy. The Kinetic Energy (E_k, measured in Joules) of an object depends on the object's mass (m, measured in kg) and velocity (v, measured in meters per second) and can be written as $v=\sqrt{\frac{2E_k}{m}}$.

What is the Kinetic Energy of an object with a mass of 1,000 kilograms that is traveling at 30 meters per second?

Solution

$E_k=$ unknown $m=1000$ $v=30$	Identify variables and known values.
$30=\sqrt{\frac{2E_k}{1{,}000}}$	Substitute values into the formula.
$(30)^2=\left(\frac{\sqrt{2E_k}}{1,000}\right)^2$ $900=\frac{2E_k}{1,000}$ $900\cdot 1,000=\frac{2E_k}{1,000}\cdot 1,000$ $900,000=2E_k$ $\frac{900,000}{2}=\frac{2E_k}{2}$ $450,000=E_k$	Solve the radical equation for E_k.
$30=\sqrt{\frac{2\cdot 450{,}000}{1{,}000}}$ $30=\sqrt{\frac{900{,}000}{1{,}000}}$ $30=\sqrt{900}$ $30=30$	Now check the solution by substituting it into the original equation.

The Kinetic Energy is 450,000 Joules.

Summary

A common method for solving radical equations is to raise both sides of an equation to whatever power will eliminate the radical sign from the equation. But be careful: when both sides of an equation are raised to an *even* power, the possibility exists that extraneous solutions will be introduced. When solving a radical equation, it is important to always check your answer by substituting the value back into the original equation. If you get a true statement, then that value is a solution; if you get a false statement, then that value is not a solution.

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SECTION OVERVIEW

16.4: Complex Numbers

Topic hierarchy

16.4.1: Complex Numbers

16.4.2: Operations with Complex Numbers

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16.4.1: Complex Numbers

Learning Objectives

- Express roots of negative numbers in terms of i.
- Express imaginary numbers as bi and complex numbers as $a+bi$.

Introduction

Several times in your learning of mathematics, you have been introduced to new kinds of numbers. Each time, these numbers made possible something that seemed impossible! Before you learned about negative numbers, you couldn't subtract a greater number from a lesser one, but negative numbers give us a way to do it. When you were learning to divide, you initially weren't able to do a problem like 13 divided by 5 because 13 isn't a multiple of 5. You then learned how to do this problem writing the answer as 2 remainder 3. Eventually, you were able to express this answer as $2\frac{3}{5}$. Using fractions allowed you to make sense of this division.

Up to now, you've known it was impossible to take a square root of a negative number. This is true, using only the *real numbers*. But here you will learn about a new kind of number that lets you work with square roots of negative numbers! Like fractions and negative numbers, this new kind of number will let you do what was previously impossible.

Using i to Simplify Roots of Negative Numbers

You really need only one new number to start working with the square roots of negative numbers. That number is the square root of -1, $\sqrt{-1}$. The *real numbers* are those that can be shown on a number line; they seem pretty *real* to us! When something's not real, you often say it is *imaginary*. So let's call this new number i and use it to represent the square root of -1.

$$i=\sqrt{-1}$$

Because $\sqrt{x}\cdot\sqrt{x}=x$, we can also see that $\sqrt{-1}\cdot\sqrt{-1}=-1$ or $i\cdot i=-1$. We also know that $i\cdot i=i^2$, so we can conclude that $i^2=-1$.

$$i^2=-1$$

The number i allows us to work with roots of all negative numbers, not just $\sqrt{-1}$. There are two important rules to remember: $\sqrt{-1}=i$, and $\sqrt{ab}=\sqrt{a}\sqrt{b}$. You will use these rules to rewrite the square root of a negative number as the square root of a positive number times $\sqrt{-1}$. Next you will simplify the square root and rewrite $\sqrt{-1}$ as i. Let's try an example.

Example

Simplify. $\sqrt{-4}$

Solution

$\sqrt{-4}=\sqrt{(4)(-1)}=\sqrt{4}\sqrt{-1}$	Use the rule $\sqrt{ab}=\sqrt{a}\sqrt{b}$ to rewrite this as a product using $\sqrt{-1}$.
$\sqrt{4}\sqrt{-1}=2\sqrt{-1}$	Since 4 is a perfect square $(4=2^2)$, you can simplify the square root of 4.
$2\sqrt{-1}=2i$	Use the definition of i to rewrite $\sqrt{-1}$ as i.

$\sqrt{-4}=2i$

Example

Simplify. $\sqrt{-18}$

Solution

$\sqrt{-18}=\sqrt{18\cdot -1}=\sqrt{18}\sqrt{-1}$	Use the rule $\sqrt{ab}=\sqrt{a}\sqrt{b}$ to rewrite this as a product using $\sqrt{-1}$.

$\sqrt{18}\sqrt{-1} = \sqrt{9}\sqrt{2}\sqrt{-1} = 3\sqrt{2}\sqrt{-1}$	Since 18 is *not* a perfect square, use the same rule to rewrite it using factors that are perfect squares. In this case, 9 is the only perfect square factor, and the square root of 9 is 3.
$3\sqrt{2}\sqrt{-1} = 3\sqrt{2}i = 3i\sqrt{2}$	Use the definition of i to rewrite $\sqrt{-1}$ as i. Remember to write i in front of the radical.

$\sqrt{-18} = 3i\sqrt{2}$

Example

Simplify. $-\sqrt{-72}$

Solution

$-\sqrt{-72} = -\sqrt{72 \cdot -1} = -\sqrt{72}\sqrt{-1}$	Use the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ to rewrite this as a product using $\sqrt{-1}$
$-\sqrt{72}\sqrt{-1} = -\sqrt{36}\sqrt{2}\sqrt{-1} = -6\sqrt{2}\sqrt{-1}$	Since 72 is *not* a perfect square, use the same rule to rewrite it using factors that are perfect squares. Notice that 72 has three perfect squares as factors: 4, 9, and 36. It's easiest to use the largest factor that is a perfect square.
$-6\sqrt{2}\sqrt{-1} = -6\sqrt{2}i = -6i\sqrt{2}$	Use the definition of i to rewrite $\sqrt{-1}$ as i. Remember to write i in front of the radical.

$-\sqrt{-72} = -6i\sqrt{2}$

You may have wanted to simplify $-\sqrt{-72}$ using different factors. Some may have thought of rewriting this radical as $-\sqrt{-9}\sqrt{8}$, or $-\sqrt{-4}\sqrt{18}$, or $-\sqrt{-6}\sqrt{12}$ for instance. Each of these radicals would have eventually yielded the same answer of $-6i\sqrt{2}$.

Rewriting the Square Root of a Negative Number

- Find perfect squares within the radical.
- Rewrite the radical using the rule $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.
- Rewrite $\sqrt{-1}$ as i.

Example: $\sqrt{-18} = \sqrt{9}\sqrt{-2} = \sqrt{9}\sqrt{2}\sqrt{-1} = 3i\sqrt{2}$

Exercise

Simplify. $\sqrt{-50}$

A. 5
B. $-5\sqrt{2}$
C. $5i$
D. $5i\sqrt{2}$

Answer

A. 5

Incorrect. You may have noticed the perfect square 25 as a factor of 50, but forgot the rest of the number under the radical. The correct answer is: $\sqrt{-50} = \sqrt{25}\sqrt{2}\sqrt{-1} = 5\sqrt{2}i = 5i\sqrt{2}$.

B. $-5\sqrt{2}$

Incorrect. While $\sqrt{50} = 5\sqrt{2}$, the negative under the radical cannot be moved outside the radical. Remember, $\sqrt{-1} = i$. The correct answer is: $\sqrt{-50} = \sqrt{25}\sqrt{2}\sqrt{-1} = 5\sqrt{2}i = 5i\sqrt{2}$.

C. $5i$

Incorrect. You may have correctly noticed the perfect square 25 as a factor of 50, and correctly used $\sqrt{-1}=i$, but you forgot the remaining factor of -50, which is 2. The correct answer is: $\sqrt{-50}=\sqrt{25}\sqrt{2}\sqrt{-1}=5\sqrt{2i}=5i\sqrt{2}$.

D. $5i\sqrt{2}$

Correct. $\sqrt{-50}=\sqrt{25}\sqrt{2}\sqrt{-1}=5\sqrt{2i}=5i\sqrt{2}$.

Imaginary and Complex Numbers

You can create other numbers by multiplying i by a real number. An **imaginary number** is any number of the form bi, where b is real (but not 0) and i is the square root of -1. Look at the following examples, and notice that b can be any kind of real number (positive, negative, whole number, rational, or irrational), but not 0. (If b is 0, $0i$ would just be 0, a real number.)

Imaginary Numbers

3i $b=3$ $\frac{3}{5}$ i $b=\frac{3}{5}$	$-672i$ $b=-672$ $\frac{\sqrt{2}}{2}i$ $b=\frac{\sqrt{2}}{2}$

You can use the usual operations (addition, subtraction, multiplication, and so on) with imaginary numbers. You'll see more of that, later. When you *add* a real number to an imaginary number, however, you get a **complex number**. A complex number is any number in the form $a+bi$, where a is a real number and bi is an imaginary number. The number a is sometimes called the **real part** of the complex number, and bi is sometimes called the **imaginary part**.

Complex Number	**Real part**	**Imaginary part**
$3+7i$	3	$7i$
$18-32i$	18	$-32i$
$-\frac{3}{5}+i\sqrt{2}$	$-\frac{3}{5}$	$i\sqrt{2}$
$\frac{\sqrt{2}}{2}-\frac{1}{2}i$	$\frac{\sqrt{2}}{2}$	$-\frac{1}{2}i$

In a number with a radical as part of b, such as $-\frac{3}{5}+i\sqrt{2}$ above, the imaginary i should be written in front of the radical. Though writing this number as $-\frac{3}{5}+\sqrt{2}i$ is technically correct, it makes it much more difficult to tell whether i is inside or outside of the radical. Putting it before the radical, as in $-\frac{3}{5}+i\sqrt{2}$, clears up any confusion. Look at these last two examples.

Number	**Number in complex form:** $a+bi$	**Real part**	**Imaginary part**
17	$17+0i$	17	$0i$
$-3i$	$0-3i$	0	$-3i$

By making $b=0$, any real number can be expressed as a complex number. The real number a is written $a+0i$ in complex form. Similarly, any imaginary number can be expressed as a complex number. By making $a=0$, any imaginary number bi is written $0+bi$ in complex form.

✓ Example

Write 83.6 as a complex number.

Solution

$a+bi$	Remember that a complex number has the form $a+bi$. You need to figure out what a and b need to be.
$83.6+bi$	Since 83.6 is a real number, it is the real part (a) of the complex number $a+bi$. A real number does not contain any imaginary parts, so the value of b is 0.

$83.6+0i$

✓ Example

Write $-3i$ as a complex number.

Solution

$a+bi$	Remember that a complex number has the form $a+bi$. You need to figure out what a and b need to be.
$a-3i$	Since $-3i$ is an imaginary number, it is the imaginary part (bi) of the complex number $a+bi$. This imaginary number has no real parts, so the value of a is 0.

$0-3i$

? Exercise

Which is the real part of the complex number $-35+9i$?

A. 9
B. -35
C. 35
D. 9 and -35

Answer

A. 9

Incorrect. The number 9 is in the imaginary part $(9i)$ of this complex number. In a complex number, $a+bi$, a is the real part. In this case, $a=-35$, so the real part is -35.

B. -35

Correct. In a complex number, $a+bi$, a is the real part. In this case, $a=-35$, so the real part is -35.

C. 35

Incorrect. In a complex number, $a+bi$, a is the real part. In this case, $a=-35$, The real part can be any real number, including negative numbers.

D. 9 and -35

Incorrect. The number 9 is in the imaginary part $(9i)$ of this complex number. In a complex number, $a+bi$, a is the real part. In this case, $a=-35$, so the real part is just -35.

Summary

Complex numbers have the form $a+bi$, where a and b are real numbers and i is the square root of -1. All real numbers can be written as complex numbers by setting $b=0$. Imaginary numbers have the form bi and can also be written as complex numbers by setting $a=0$. Square roots of negative numbers can be simplified using $\sqrt{-1}=i$ and $\sqrt{ab}=\sqrt{a}\sqrt{b}$.

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16.4.2: Operations with Complex Numbers

Learning Objectives

- Add complex numbers.
- Subtract complex numbers.
- Multiply complex numbers.
- Find conjugates of complex numbers.
- Divide complex numbers.

Introduction

Any time new kinds of numbers are introduced, one of the first questions that needs to be addressed is, “How do you add them?” In this topic, you’ll learn how to add complex numbers and also how to subtract, multiply, and divide them.

Adding and Subtracting Complex Numbers

First, consider the following expression.

$(6x+8)+(4x+2)$

To simplify this expression, you combine the like terms, $6x$ and $4x$. These are like terms because they have the same variable with the same exponents. Similarly, 8 and 2 are like terms because they are both constants, with no variables.

$(6x+8)+(4x+2)=10x+10$

In the same way, you can simplify expressions with radicals.

$(6\sqrt{3}+8)+(4\sqrt{3}+2)=10\sqrt{3}+10$

You can add $6\sqrt{3}$ to $4\sqrt{3}$ because the two terms have the same radical, $\sqrt{3}$, just as $6x$ and $4x$ have the same variable and exponent.

The number i looks like a variable, but remember that it is equal to $\sqrt{-1}$. The great thing is you have no new rules to worry about. Whether you treat it as a variable or a radical, the exact same rules apply to adding and subtracting **complex numbers**. You combine the imaginary parts (the terms with i), and you combine the real parts.

Example

Add. $(-3+3i)+(7-2i)$

Solution

$-3+3i+7-2i=-3+7+3i-2i$	Rearrange the sums to put like terms together.
$-3+7=4$ and $3i-2i=(3-2)i=i$	Combine like terms.

$(-3+3i)+(7-2i)=4+i$

Example

Subtract. $(-3+3i)-(7-2i)$

Solution

$(-3+3i)-(7-2i)=-3+3i-7+2i$	Be sure to distribute the subtraction sign to all terms in the **subtrahend**.
$-3-7+3i+2i$	Rearrange the terms to put like terms together.
$-3-7=-10$ and $3i+2i=(3+2)i=5i$	Combine like terms.

$(-3+3i)-(7-2i)=10+5i$

? Exercise

Subtract. $(5+3i)-(3-i)$

A. $2+4i$
B. 6
C. $2+2i$
D. $8+2i$

Answer

A. $2+4i$

Correct. Distributing the subtraction to the second complex number gives $5+3i-3+i$. Rearranging to put like terms together gives $5-3+3i+i$, and combining like terms gives $2+4i$.

B. 6

Incorrect. You may have combined $5+3$ from the first number (ignoring the i) and $3-1$ from the second number (ignoring the i), giving that result of $8-2=6$. Instead, you should distribute the subtraction across the second complex number to get $5+3i-3+i$. Rearranging to put like terms together gives $5-3+3i+i$, and combining like terms gives the correct answer $2+4i$.

C. $2+2i$

Incorrect. You probably forgot to distribute the subtraction to the imaginary part of the second complex number, leaving it as $-i$ instead of $+i$. Distributing the subtraction to the second complex number gives $5+3i-3+i$. Rearranging to put like terms together gives $5-3+3i+i$, and combining like terms gives the correct answer $2+4i$.

D. $8+2i$

Incorrect. You may have added instead of subtracted. Distributing the subtraction to the second complex number gives $5+3i-3+i$. Rearranging to put like terms together gives $5-3+3i+i$, and combining like terms gives the correct answer $2+4i$.

Multiplying Complex Numbers

Again, consider the following expression. Before reading further, consider how you would simplify it.

$$(5x)(-3x)$$

You can simplify by multiplying the coefficients together, then the variables.

$$\begin{aligned}(5x)(-3x)&=(5)(-3)(x)(x)\\&=-15x^2\end{aligned}$$

Multiplying two imaginary (but not complex!) numbers together works in a similar way, but there is an additional step. Start with the same method to multiply $5\mathrm{i}$ and $-3i$.

$$\begin{aligned}(5i)(-3i)&=(5)(-3)(i)(i)\\&=-15i^2\end{aligned}$$

This seems fine so far, but the i^2 can be simplified further.

When you multiply a square root by itself, you get the number under the radical. This is what square root means.

$$\begin{aligned}(\sqrt{3})(\sqrt{3})&=3\\(\sqrt{15})(\sqrt{15})&=15\end{aligned}$$

Well, i is also a square root. It's equal to $\sqrt{-1}$.

$$i^2=(i)(i)$$

$$\begin{aligned}&=(\sqrt{-1})(\sqrt{-1})\\&=-1\end{aligned}$$

So, the final step to simplifying $(5i)(-3i)=-15i^2$ is to replace i^2 with -1.

$$\begin{aligned}(5i)(-3i)&=(5)(-3)(i)(i)\\&=-15i^2\\&=-15(-1)\\&=15\end{aligned}$$

Example

Multiply. $(3i)(2i)$

Solution

$(3i)(2i)=(3)(2)(i)(i)=6i^2$	Multiply the coefficients of i together, and then multiply i times i.
$6i^2=6(-1)$	i^2 with -1.
$6(-1)=-6$	Multiply.

$(3i)(2i)=-6$

Notice that the product of two imaginary numbers is a real number! We will see this again when we multiply two complex numbers.

Exercise

Multiply and simplify. $(3i)(-i)$

A. 3
B. -3
C. $3i$
D. $-3i^2$

Answer

A. 3

Correct. $(3i)(-i)=3(-1)(i)(i)=-3i^2=-3(-1)=3$

B. -3

Incorrect. You probably either missed the sign on the second factor, or you incorrectly replaced i^2. $(3i)(-i)$ $=3(-1)(i)(i)$ $=-3i^2=-3(-1)=3$

C. $3i$

Incorrect. You probably incorrectly multiplied $(i)(-i)$, or incorrectly replaced i^2. $(3i)(-i)$ $=3(-1)(i)(i)$ $=-3i^2$ $=-3(-1)=3$

D. $-3i^2$

Incorrect. This is a correct multiplication, but i^2 can be simplified further. $(3i)(-i)=3(-1)(i)(i)=-3i^2=-3(-1)=3$

Using the Distributive Property

The following expression is a little more complicated because two binomials are being multiplied. This means you have to use the Distributive Property of Multiplication. (Remember that multiplying using the FOIL method (First, Outside, Inside, Last) is an application of the Distributive Property of Multiplication.) Once the binomials have been multiplied, simplify the expression by combining like terms.

$$(6x+8)(4x+2)=6x(4x+2)+8(4x+2)$$

$$\begin{aligned}(6x+8)(4x+2) &= 6x(4x+2)+8(4x+2)\\ &= 6x\cdot 4x+6x\cdot 2+8\cdot 4x+8\cdot 2\\ &= 24x^2+12x+32x+16\\ &= 24x^2+44x+16\end{aligned}$$

Again, in the same way, you can multiply complex numbers. At the end, you will need to simplify i^2.

✓ Example

Multiply and simplify. $(6+8i)(4+2i)$

Solution

$(6+8i)(4+2i)$ $6(4+2i)+8i(4+2i)$ $6\cdot 4+6\cdot 2i+8i\cdot 4+8i\cdot 2i$ $24+12i+32i+16i^2$	Two binomials are being multiplied, so you need to use the Distributive Property of Multiplication. We could have used FOIL and gone directly to the line $6\cdot 4+6\cdot 2i+8i\cdot 4+8i\cdot 2i$.
$24+44i+16i^2$ $24+44i+16(-1)$ $24+44i-16$ $8+44i$	Combine like terms. Replace i^2 with -1 and simplify.

$(6+8i)(4+2i)=8+44i$

In this case, the product of two complex numbers is complex. But in the following example, the product is real, not complex. Can you figure out why?

✓ Example

Multiply and simplify. $(6+8i)(6-8i)$

Solution

$(6+8i)(6-8i)$ $6\cdot 6+6\cdot -8i+8i\cdot 6+8i\cdot -8i$ $36-48i+48i-64i^2$ $36-64i^2$ $36-64(-1)$ $36+64$ 100	Use FOIL to expand the product. Combine like terms. Replace i^2 with -1 and simplify.

$(6+8i)(6-8i)=100$

Just as $6+8\sqrt{3}$ and $6-8\sqrt{3}$ are *conjugates*, $6+8i$ and $6-8i$ are conjugates. (Again, i is a square root, so this isn't really a new idea.) When the numbers are complex, they are called **complex conjugates**. Because conjugates have terms that are the same except for the operation between them (one is addition and one is subtraction), the i terms in the product will add to 0. In the example above, $-48i$ was added to $48i$, and that sum is 0, so there was no i term in the final product. That means the product of complex conjugates will always be a real (not complex) number.

? Exercise

Multiply. $(9+i)(9-i)$

A. $82+18i$
B. $80-18i$
C. 80
D. 82

Answer

A. $82+18i$

Incorrect. Don’t forget to use the FOIL method so that every term in the first complex number is multiplied by every term in the other complex number. Since the factors are complex conjugates, their product should be a real number. $(9+i)(9-i)=81-9i+9i-i^2=81-i^2=81-(-1)=81+1=82$.

B. $80-18i$

Incorrect. Don’t forget to use the FOIL method so that every term in the first complex number is multiplied by every term in the other complex number. Since the factors are complex conjugates, their product should be a real number. $(9+i)(9-i)=81-9i+9i-i^2=81-i^2=81-(-1)=81+1=82$.

C. 80

Incorrect. You probably multiplied correctly but forgot to subtract i^2, or you forgot the negative when you multiplied i and $-i$. $(9+i)(9-i)=81-9i+9i-i^2=81-i^2=81-(-1)=81+1=82$

D. 82

Correct. $(9+i)(9-i)=81-9i+9i-i^2=81-i^2=81-(-1)=81+1=82$

Division of Complex Numbers

So far, each operation with complex numbers has worked just like the same operation with radical expressions. This should no longer be a surprise. The number i is a radical, after all, so complex numbers are radical expressions!

Let’s look at division in two parts, like we did multiplication. First, let’s look at a situation in which the divisor is a monomial.

✓ Example

Simplify. $-24i \div 6$

Solution

$-24i \div 6=\frac{-24i}{6}=\frac{-4i}{1}\cdot\frac{6}{6}$	Treat the division as a fraction. Simplify the fraction using a factor that the numerator and denominator have in common.
$-24i \div 6=-4i$	Since the result has no denominator, no more simplification is needed.

✓ Example

Simplify. $32i \div 6i$

Solution

$32i \div 6i=\frac{32i}{6i}=\frac{16}{3}\cdot\frac{2i}{2i}$	Treat the division as a fraction. Simplify the fraction using a factor that the numerator and denominator have in common. Note in this case, i is part of the common factor.
$32i \div 6i=\frac{16}{3}$	The fraction is in simplest form.

✓ Example

Simplify. $56 \div -7i$

Solution

$56 \div -7i=\frac{56}{-7i}=\frac{8}{-i}\cdot\frac{7}{7}$	Treat the division as a fraction. Simplify the fraction using a factor that the numerator and denominator have in common.
$\frac{8}{-i}$	In this case, the denominator still has i in it. Since i is a radical, you should simplify further by rationalizing the denominator.

$\begin{aligned}\frac{8}{-i}\cdot\frac{i}{i}&=\frac{8i}{-i^2}\\&=\frac{8i}{-(-1)}\\&=8i\end{aligned}$	Since the denominator is just one term, you don’t need to think about complex conjugates. Just multiply by 1 in the form $\frac{i}{i}$ and simplify. (Remember, the product of two imaginary numbers is real, so the denominator is real.)

$56\div -7i=8i$

Exercise

Simplify. $12\div 10i$

A. $\frac{6}{5}$
B. $-\frac{6}{5}i$
C. $\frac{6}{5}i$
D. $-\frac{6}{5}$

Answer

A. $\frac{6}{5}$

Incorrect. When you simplify the numerical parts of $\frac{12}{10i}$, you still have i in the denominator. Rationalize the denominator of $\frac{6}{5i}$ by multiplying by $\frac{i}{i}$: $\frac{6}{5i}=\frac{6}{5i}\cdot\frac{i}{i}=\frac{6i}{5i^2}=\frac{6i}{5(-1)}=-\frac{6}{5}i$

B. $-\frac{6}{5}i$

Correct. After simplifying the numerical parts of $\frac{12}{10}i$, you still have to rationalize the denominator because i is left in the denominator: $\frac{12}{10i}=\frac{6}{5i}\cdot\frac{2}{2}=\frac{6}{5i}=\frac{6}{5i}\cdot\frac{i}{i}=\frac{6i}{5i^2}=\frac{6i}{5(-1)}=-\frac{6}{5}i$

C. $\frac{6}{5}i$

Incorrect. When you rationalize the denominator, you have i^2 in the denominator. Since $i^2=-1$, the quotient is negative. $\frac{12}{10i}=\frac{6}{5i}\cdot\frac{2}{2}=\frac{6}{5i}=\frac{6}{5i}\cdot\frac{i}{i}=\frac{6i}{5i^2}=\frac{6i}{5(-1)}=-\frac{6}{5}i$

D. $-\frac{6}{5}$

Incorrect. You may have forgotten the radical i in the denominator, or you may have made a mistake when you rationalized the denominator. When you rationalize the denominator, you must also multiply the numerator by i. $\frac{12}{10i}=\frac{6}{5i}\cdot\frac{2}{2}=\frac{6}{5i}=\frac{6}{5i}\cdot\frac{i}{i}=\frac{6i}{5i^2}=\frac{6i}{5(-1)}=-\frac{6}{5}i$

When the divisor (that is, the denominator in the fraction) is a complex number with non-zero real and imaginary parts, you must rationalize the denominator using the complex conjugate. Remember that the product of a complex number with its complex conjugate is always a real number, so the denominator will be a real number. That means the result will be equivalent, but rationalized.

Example

Simplify. $(56-8i)\div(14+10i)$

Solution

$\begin{aligned}(56-8i)\div(14+10i)&=\frac{56-8i}{14+10i}\\&=\frac{(28-4i)(2)}{(7+5i)(2)}\\&=\frac{28-4i}{7+5i}\cdot\frac{2}{2}\\&=\frac{28-4i}{7+5i}\end{aligned}$	Treat the division as a fraction. Simplify the fraction using a factor that the numerator and denominator have in common, if any. Be careful to use the distributive property: the numbers must be a factor of *all* terms.
$\begin{aligned}\frac{28-4i}{7+5i}&=\frac{28-4i}{7+5i}\cdot\frac{7-5i}{7-5i}\\&=\frac{(28-4i)(7-5i)}{(7+5i)(7-5i)}\end{aligned}$	In this case, the denominator still has i in it. To rationalize the denominator, multiply by the complex conjugate of the denominator. In this case, the complex conjugate is $(7-5i)$.

$\begin{aligned}\frac{(28-4i)(7-5i)}{(7+5i)(7-5i)} &= \frac{28(7)+28(-5i)+(-4i)(7)+(-4i)(-5i)}{7(7)+7(-5i)+5i(7)+5i(-5i)} \\ &= \frac{196-140i-28i+20i^2}{49-35i+35i-25i^2} \\ &= \frac{196-168i+20i^2}{49-25i^2}\end{aligned}$	(For complex conjugates, the real parts are equal and the imaginary parts are additive inverses.) Expand the numerator and the denominator. Remember, the denominator should be a real number (no i term) if you chose the correct complex conjugate and performed the multiplication correctly.
$\begin{aligned}\frac{196-168i+20i^2}{49-25i^2} &= \frac{196-168i+20(-1)}{49-25(-1)} \\ &= \frac{196-168i-20}{49+25} \\ &= \frac{176-168i}{74}\end{aligned}$	Replace i^2 with -1 and simplify. Be sure to replace i^2 in both the numerator and the denominator!
$\frac{176-168i}{74} = \frac{176}{74} - \frac{168}{74}i$	The quotient can be written in the form $a+bi$ using fractions for both a and b.
$\begin{aligned}\frac{176}{74} - \frac{168i}{74} &= \frac{88}{37}\cdot\frac{2}{2} - \left(\frac{84}{37}\cdot\frac{2}{2}\right)i \\ &= \frac{88}{37} - \frac{84}{37}i\end{aligned}$	Always check the final product to see if you can simplify further. In this case, both fractions can be simplified.

$(56-8i) \div (14+10i) = \frac{88}{37} - \frac{84}{37}i$

? Exercise

Simplify. $(10+6i) \div (5-3i)$

A. $\frac{16}{17} + \frac{30}{17}i$
B. $2-2i$
C. $2+\frac{15}{4}i$
D. $\frac{16}{17} + 60i$

Answer

A. $\frac{16}{17} + \frac{30}{17}i$

Correct. Write as a rational expression and multiply both numerator and denominator by the complex conjugate of the divisor: $\frac{10+6i}{5-3i} \cdot \frac{5+3i}{5+3i} = \frac{50+30i+30i+18i^2}{25-9i^2} = \frac{50+60i-18}{25+9} = \frac{32+60i}{34}$. This simplifies to $\frac{16}{17} + \frac{30}{17}i$.

B. $2-2i$

Incorrect. You probably just divided the real parts, then divided the coefficients of the imaginary parts. This works with addition and subtraction, but not division. Try writing the division as a rational expression, and multiply both numerator and denominator by the complex conjugate of the divisor. The correct answer is $\frac{16}{17} + \frac{30}{17}i$.

C. $2+\frac{15}{4}i$

Incorrect. You probably correctly set up the rational expression and rationalized: $\frac{10+6i}{5-3i} \cdot \frac{5+3i}{5+3i} = \frac{50+30i+30i+18i^2}{25-9i^2}$. Remember at this point that $i^2 = -1$, so subtracting $9i^2$ is the same as *adding* 9. The correct answer is $\frac{16}{17} + \frac{30}{17}i$.

D. $\frac{16}{17} + 60i$

Incorrect. You probably correctly set up the rational expression and rationalized: $\frac{10+6i}{5-3i} \cdot \frac{5+3i}{5+3i} = \frac{50+30i+30i+18i^2}{25-9i^2} = \frac{50+60i-18}{25+9} = \frac{32+60i}{34}$. Remember at this point the denominator (34) divides *both* terms in the numerator, so this simplifies to $\frac{32}{34} + \frac{60}{34}i = \frac{16}{17} + \frac{30}{17}i$.

Operations with Complex Numbers

To add or subtract, combine like terms.

To multiply monomials, multiply the coefficients and then multiply the imaginary numbers i. If i^2 appears, replace it with -1.

> **To multiply complex numbers** that are binomials, use the Distributive Property of Multiplication, or the FOIL method. Multiply the resulting terms as monomials.
>
> **To divide,** treat the quotient as a fraction.
>
> - Simplify the numerical parts, and then rationalize the denominator, if needed.
> - Replace i^2 in both the numerator and the denominator with -1, as needed.
> - Write the answer in $a+bi$ form, which may require further simplification of a and b when they are fractions.

Summary

Complex numbers can be added, subtracted, multiplied, and divided using the same ideas you used for radicals and variables. With multiplication and division, you may need to replace i^2 with -1 and simplify further.

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SECTION OVERVIEW

16.5: Solving Quadratic Equations

Topic hierarchy

16.5.1: Square Roots and Completing the Square

16.5.2: The Quadratic Formula

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16.5.1: Square Roots and Completing the Square

Learning Objectives

- Solve quadratic equations by using the Square Root Property.
- Identify and complete perfect square trinomials.
- Solve quadratic equations by completing the square.

Introduction

Quadratic equations can be solved in many ways. You may already be familiar with factoring to solve some quadratic equations. However, not all quadratic equations can be factored. In this topic, you will use square roots to learn another way to solve quadratic equations, and this method will work with *all* quadratic equations.

Solving Quadratics Using Square Roots

One way to solve the quadratic equation $x^2=9$ is to subtract 9 from both sides to get one side equal to 0: $x^2-9=0$. The expression on the left can be factored:

$(x+3)(x-3)=0$ Using the zero factor property, you know this means $x+3=0$ or $x-3=0$, so $x=-3$ or 3.

Another property would let you solve that equation more easily.

The Square Root Property

If $x^2=a$, then $x=\sqrt{a}$ or $-\sqrt{a}$.

The property above says that you can take the square root of both sides of an equation, but you have to think about two cases: the positive square root of a and the negative square root of a.

A shortcut way to write "$\sqrt{a}$" or "$-\sqrt{a}$" is "$\pm\sqrt{a}$". The symbol ± is often read "positive or negative." If it is used as an operation (addition or subtraction), it is read "plus or minus."

Example

Solve using the Square Root Property. $x^2=9$

Solution

$x^2=9$ $x=\pm\sqrt{9}$ $x=\pm 3$	Since one side is simply x^2, you can take the square root of both sides to get x on one side. Don't forget to use both positive and negative square roots!

$x=\pm 3$ (that is, $x=3$ or -3)

Notice that there is a difference here in solving $x^2=9$ and finding $\sqrt{9}$. For $x^2=9$, you are looking for *all numbers* whose square is 9. For $\sqrt{9}$, you only want the *principal* (nonnegative) square root. The negative of the principal square root is $-\sqrt{9}$; both would be $\pm\sqrt{9}$. Unless there is a symbol in front of the radical sign, only the nonnegative value is wanted!

In the example above, you can take the square root of both sides easily because there is only one term on each side. In some equations, you may need to do some work to get the equation in this form. You will find that this involves isolating x^2.

Example

Solve. $10x^2+5=85$

Solution

$10x^2+5=85$	If you try taking the square root of both sides of the original equation, you will have $\sqrt{10x^2+5}$ on the left, and you can't simplify that. Subtract 5 from both sides to get the x^2 term by itself.
$10x^2=80$	You could now take the square root of both sides, but you would have $\sqrt{10}$ as a coefficient, and you would need to divide by that coefficient. Dividing by 10 before you take the square root will be a little easier.
$x^2=8$ $x=\pm\sqrt{8}$ $=\pm\sqrt{(4)(2)}$ $=\pm\sqrt{4}\sqrt{2}$ $=\pm 2\sqrt{2}$	Now you have only x^2 on the left, so you can use the Square Root Property easily. Be sure to simplify the radical if possible.

$x=\pm 2\sqrt{2}$

Sometimes more than just the x is being squared:

✓ Example

Solve. $(x-2)^2-50=0$

Solution

$(x-2)^2-50=0$	Again, taking the square root of both sides at this stage will leave something you can't work with on the left. Start by adding 50 to both sides.
$(x-2)^2=50$ $x-2=\pm\sqrt{50}$	Because $(x-2)^2$ is a squared quantity, you can take the square root of both sides.
$x=2\pm\sqrt{50}$ $=2\pm\sqrt{(25)(2)}$ $=2\pm\sqrt{25}\sqrt{2}$ $=2\pm 5\sqrt{2}$	To isolate x on the left, you need to add 2 to both sides. Be sure to simplify the radical if possible.

$x=2\pm 5\sqrt{2}$

This method can be helpful when solving real-world problems.

✓ Example

The formula for compounding interest annually is $A=P(1+r)^t$, where A is the balance after t years, when P is the principal (initial amount invested) and r is the interest rate.

Find the interest rate r if \$3,000 is invested and grows to \$3,307.50 after 2 years.

Solution

$A=P(1+r)^t$ $A=3,307.50$ $t=2$ $P=3,000$	First, identify what you know. The amount after 2 years is 3,307.50, so $A=3,307.50$. This also means $t=2$. The principal P is the original amount invested, so that is 3,000.
$3,307.50=3,000(1+r)^2$	Substitute the values for the variables you know. Only r is left, so try to isolate r.

$\frac{3,307.50}{3,000}=(1+r)^2$ $1.1025=(1+r)^2$ $\pm\sqrt{1.1025}=1+r$	Dividing both sides by 3,000 leaves only $(1+r)^2$ on the right. Because $(1+r)^2$ is a squared quantity, you can use the Square Root Property.
$\pm 1.05=1+r$	Don't forget the ±! Using a calculator, you can find that $\sqrt{1.1025}$ is 1.05.
$\pm 1.05-1=r$	Subtract 1 from both sides to isolate r on the right.
$1.05-1=r$ or $-1.05-1=r$	You now have two equations, one using 1.05 and one using -1.05.
$r=0.05$ or -2.05	Simplifying the two equations gives two solutions to the equation.
The interest rate is 0.05 or 5%	Notice that a negative interest rate doesn't make sense for this context, so only the positive value could be the interest rate. The -2.05 is an extraneous solution and must be discarded.

? Exercise

Solve. $(x-3)^2-2=16$

A. $x=3\pm 3\sqrt{2}$
B. $x=3+3\sqrt{2}$
C. $x=7$
D. $x=1$ or 9

Answer

A. Correct. Before taking the square root, add 2 to both sides: $(x-3)^2=18$. Applying the Square Root Property gives $x-3=\pm\sqrt{18}$, so $x=3\pm\sqrt{18}$. Simplifying the radical gives $3\pm 3\sqrt{2}$.
B. Incorrect. You forgot the negative square root when you took the square root of both sides. Before taking the square root, add 2 to both sides: $(x-3)^2=18$. Applying the Square Root Property gives $x-3=\pm\sqrt{18}$, so $x=3\pm\sqrt{18}$. Simplifying the radical gives $3\pm 3\sqrt{2}$.
C. Incorrect. There are two mistakes here: Knowing the square root of 16 might have made you forget that to solve this equation, the squared quantity needs to be isolated. Before taking the square root, add 2 to both sides: $(x-3)^2=18$. Applying the Square Root Property gives $x-3=\pm\sqrt{18}$. (Note that both the positive and negative square roots are included; this is the other probable mistake.) So, $x=3\pm\sqrt{18}$. Simplifying the radical gives $3\pm 3\sqrt{2}$.
D. Incorrect. Knowing the square root of 16 might have made you forget that to solve this equation, the squared quantity needs to be isolated. Before taking the square root, add 2 to both sides: $(x-3)^2=18$. Applying the Square Root Property gives $x-3=\pm\sqrt{18}$. So, $x=3\pm\sqrt{18}$. Simplifying the radical gives $3\pm 3\sqrt{2}$.

Perfect Square Trinomials

Of course, quadratic equations often will not come in the format of the examples above. Most of them will have x terms. However, you may be able to factor the expression into a squared binomial, and, if not, you can still use squared binomials to help you.

First, let's consider squared binomials. Some of the above examples have squared binomials: $(1+r)^2$ and $(x-2)^2$ are squared binomials. (They are binomials, two terms, that are squared.) If you expand these, you get a **perfect square trinomial**. For example, $(1+r)^2=(1+r)(1+r)=1+2r+r^2$, or r^2+2r+1. The trinomial r^2+2r+1 is a perfect square trinomial. Notice that the first and last terms are squares (r^2 and 1). The middle term is twice the product of the square roots of the first and last terms, the square roots are r and 1, and the middle term is $2(r)(1)$.

Perfect square trinomials have the form $r^2+2rs+s^2$ and can be factored as $(r+s)^2$, or they have the form $r^2-2rs+s^2$ and can be factored as $(r-s)^2$. Let's factor a perfect square trinomial into a squared binomial.

✓ Example

Factor $9x^2-24x+16$**.**

Solution

$9x^2=(3x)^2$ $16=4^2$	First, notice that the x^2 term and the constant term are both perfect squares.
$24x=2(3x)(4)$	Then notice that the middle term (ignoring the sign) is twice the product of the square roots of the other terms.
$r=3x$ $s=4$	A trinomial in the form $r^2-2rs+s^2$ can be factored as $(r-s)^2$.
$9x^2-24x+16=(3x-4)^2$	In this case, the middle term is subtracted, so subtract r and s and square it to get $(r-s)^2$.

$(3x-4)^2$

You can use the procedure in this next example to help you solve equations where you identify perfect square trinomials, even if the equation is not set equal to 0.

✓ Example

Solve. $4x^2+20x+25=8$

Solution

$4x^2+20x+25=8$	Since there's an x term, you can't use the Square Root Property immediately (or even after adding or dividing by a constant). Notice, however, that the x^2 and constant terms on the left are both perfect squares: $(2x)^2$ and 5^2. Check the middle term: is it $2(2x)(5)$? Yes!
$(2x+5)^2=8$	A trinomial in the form $r^2+2rs+s^2$ can be factored as $(r+s)^2$, so rewrite the left side as a squared binomial.
$2x+5=\pm\sqrt{8}$ $2x=-5\pm\sqrt{8}$ $x=-\frac{5}{2}\pm\frac{1}{2}\sqrt{8}$	Now you *can* use the Square Root Property. Some additional steps are needed to isolate.
$x=-\frac{5}{2}\pm\frac{1}{2}\sqrt{4}\sqrt{2}$ $x=-\frac{5}{2}\pm\frac{1}{2}(2)\sqrt{2}$ $x=-\frac{5}{2}\pm\sqrt{2}$	Simplify the radical when possible.

$x=-\frac{5}{2}\pm\sqrt{2}$

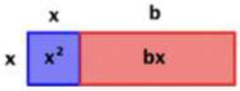
You may have looked at the problem above and thought, "Why not subtract 8 from both sides first, making the equation $4x^2+20x+17=0$?" This is one technique for solving for x. The problem is that, even if you did this, the equation $4x^2+20x+17=0$ cannot be factored with real numbers. (Try it. Can you think of two numbers whose product is 68 and whose sum is 20?)

Since the expression $4x^2+20x+25$ can be identified as a perfect square trinomial, it is best to factor it as $(2x+5)^2$ and then use the Square Root Property.

Completing the Square

One way to solve quadratic equations is by **completing the square**. When you don't have a perfect square trinomial, you can *create* one by adding a constant term that is a perfect square to both sides of the equation. Let's see how to find that constant term.

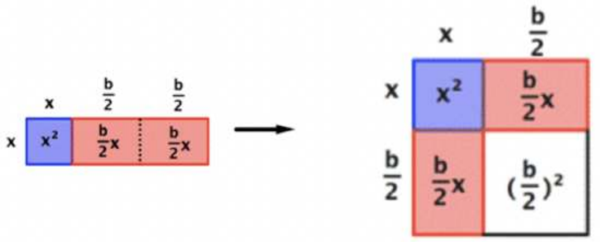
“Completing the square” does exactly what it says: it takes something that is not a square and makes it one. This idea can be illustrated using an area model of the binomial x^2+bx.



$x(x+b)=x^2+bx$

In this example, the area of the overall rectangle is given by $x(x+b)$.

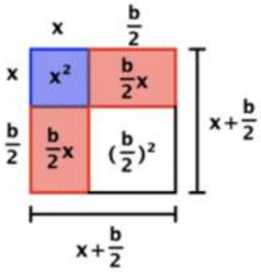
Now let's make this rectangle into a square. First, divide the red rectangle with area bx into two equal rectangles each with area $\frac{b}{2}x$. Then rotate and reposition one of them. You haven't changed the size of the red area; it still adds up to bx.



$x^2+2\left(\frac{b}{2}\right)x+\left(\frac{b}{2}\right)^2$

The red rectangles now make up two sides of a square, shown in white. The area of that square is the length of the red rectangles squared, or $\left(\frac{b}{2}\right)^2$.

Here comes the cool part: when the white square is added to the blue and red regions, the whole shape is also now a square. In other words, you've “completed the square!” By adding the quantity $\left(\frac{b}{2}\right)^2$ to the original binomial, you've made a square, a square with sides of $x+\frac{b}{2}$.



$x^2+bx+\left(\frac{b}{2}\right)^2=x^2+2\left(\frac{b}{2}\right)x+\left(\frac{b}{2}\right)^2=\left(x+\frac{b}{2}\right)^2$

Notice that the area of this square can be written as a squared binomial: $\left(x+\frac{b}{2}\right)^2$

Finding a Value that will Complete the Square in an Expression

To complete the square for an expression of the form x^2+bx:

- Identify the value of b;
- Calculate and add $\left(\frac{b}{2}\right)^2$.

The expression becomes $x^2+bx+\left(\frac{b}{2}\right)^2=\left(x+\frac{b}{2}\right)^2$.

✓ Example

Find the number to add to x^2+8x to make it a perfect square trinomial.

Solution

x^2+8x $b=8$	First, identify b if this has the form x^2+bx.
$x^2+8x+\left(\frac{8}{2}\right)^2$	To complete the square, add $\left(\frac{b}{2}\right)^2$. $b=8$, so $\left(\frac{b}{2}\right)^2=\left(\frac{8}{2}\right)^2$ Simplify.

$x^2+8x+(4)^2$ $x^2+8x+16$	Check that the result is a perfect square trinomial.
$x^2+8x+16=(x+4)^2$	$(x+4)^2=x^2+4x+4x+16=x^2+8x+16$ It is!

Adding $+16$ will make x^2+8x a perfect square trinomial.

Notice that $\left(\frac{b}{2}\right)^2$ is always positive, since it is the square of a number. When you complete the square, you are always adding a positive value.

? Exercise

Use completing the square to find the value to add that makes x^2-12x a perfect square trinomial. Then write the expression as the square of a binomial.

A. Add 12. $(x-6)^2$
B. Add 36. $(x+6)^2$
C. Add -12. $(x-12)^2$
D. Add 36. $(x-6)^2$

Answer

A. Incorrect. The value of b is -12, so the value to add is $\left(\frac{-12}{2}\right)^2=(-6)^2=36$, not 12. Add 36 to get $x^2-12x+36$. The correct answer is $(x-6)^2$.
B. Incorrect. The value to add has been calculated correctly: $\left(\frac{-12}{2}\right)^2=(-6)^2=36$. However, the resulting trinomial is $x^2-12x+36$, which factors to $(x-6)^2$, not $(x+6)^2$. The correct answer is $(x-6)^2$.
C. Incorrect. The value of b is -12, so the value to add is $\left(\frac{-12}{2}\right)^2=(-6)^2=36$, not -12. Note also, that the number you add will always be positive because it is the square of a number. The correct answer is $(x-6)^2$.
D. Correct. The value to add is $\left(\frac{-12}{2}\right)^2=(-6)^2=36$ and the resulting trinomial $x^2-12x+36$ factors to: $(x-6)^2$

Solving a Quadratic Equation using Completing the Square

You can use completing the square to help you solve a quadratic equation that cannot be solved by factoring.

Let's start by seeing what happens when you complete the square in an equation. In the example below, notice that completing the square will result in adding a number to *both* sides of the equation. You have to do this in order to keep both sides equal!

✓ Example

Rewrite $x^2+6x=8$ so that the left side is a perfect square trinomial.

Solution

$x^2+6x=8$ $b=6$	This equation has a constant of 8. Ignore it for now and focus on the x^2 and x terms on the left side of the equation. The left side has the form x^2+bx, so you can identify b. To complete the square, add $\left(\frac{b}{2}\right)^2$ to the left side.
$x^2+6x+9=8+9$	$b=6,\ \text{so}\ \left(\frac{b}{2}\right)^2=\left(\frac{6}{2}\right)^2=3^2=9$ This is an equation, though, so you must add the same number to the *right* side as well.
	Simplify.
$x^2+6x+9=17$	Check that the left side is a perfect square trinomial.
$x^2+6x+9=17$ $(x+3)^2=17$	$(x+3)^2=x^2+3x+3x+9=x^2+6x+9$ It is!

$x^2+6x+9=17$

Can you see that completing the square in an equation is very similar to completing the square in an expression? The main difference is that you have to add the new number (+9 in this case) to both sides of the equation to maintain equality.

Now let's look at an example where you are using completing the square to actually solve an equation, finding a value for the variable.

✓ Example

Solve. $x^2-12x-4=0$

Solution

$x^2-12x=4$ $b=-12$	Since you cannot factor the trinomial on the left side, you will use completing the square to solve the equation. Rewrite the equation with the left side in the form x^2+bx, to prepare to complete the square. Identify b.
$x^2-12x+36=4+36$ $x^2-12x+36=40$	Figure out what value to add to complete the square. Add $\left(\frac{b}{2}\right)^2$ to complete the square, so: $\left(\frac{b}{2}\right)^2=\left(\frac{-12}{2}\right)^2=(-6)^2=36$ Add the value to *both* sides of the equation and simplify.
$(x-6)^2=40$	Rewrite the left side as a squared binomial.
$x-6=\pm\sqrt{40}$	Use the Square Root Property. Remember to include both the positive and negative square root, or you'll miss one of the solutions.
$x=6\pm\sqrt{40}$ $=6\pm\sqrt{4}\sqrt{10}$ $=6\pm2\sqrt{10}$	Solve for x by adding 6 to both sides. Simplify as needed.

$x=6\pm2\sqrt{10}$

You may have noticed that because you have to use both square roots, all the examples have two solutions. Here is another example that's slightly different.

✓ Example

Solve. $x^2+16x+17=-47$

Solution

$x^2+16x=-64$ $b=16$	Rewrite the equation so the left side has the form x^2+bx. Identify b.
$x^2+16x+64=-64+64$ $x^2+16x+64=0$	Add $\left(\frac{b}{2}\right)^2$, which is $\left(\frac{16}{2}\right)^2=8^2=64$, to both sides.
$(x+8)^2=0$	Write the left side as a squared binomial.
$x+8=0$	Take the square roots of both sides. Normally both positive and negative square roots are needed, but 0 is neither positive nor negative. 0 has only one root.
$x=-8$	Solve for x.

$x=-8$

Take a closer look at this problem and you may see something familiar. Instead of completing the square, try adding 47 to both sides in the equation. The equation $x^2+16x+17=-47$ becomes $x^2+16x+64=0$. Can you factor this equation using grouping?

(Think of two numbers whose product is 64 and whose sum is 16).

It can be factored as $(x+8)(x+8)=0$, of course! Knowing how to complete the square is very helpful, but it is not always the only way to solve an equation.

Exercise

Solve. $x^2-16x=-1$

A. $x=4\pm\sqrt{15}$
B. $x=-8\pm3\sqrt{7}$
C. $x=8\pm i$
D. $x=8\pm3\sqrt{7}$

Answer

A. Incorrect. You appear to have added 16 and then factored the trinomial incorrectly. However, the value of b is -16, so to complete the square, add $\left(\frac{-16}{2}\right)^2=(-8)^2=64$ (not 16). Add 64 to both sides to get $x^2-16x+64=63$. This is $(x-8)^2=63$, so $x-8=\pm\sqrt{63}$. That means $x=8\pm\sqrt{63}=8\pm3\sqrt{7}$. The correct answer is $x=8\pm3\sqrt{7}$.
B. Incorrect. The value to add has been calculated correctly: $\left(\frac{-8}{2}\right)^2=(-8)^2=64$. However, the resulting trinomial is $x^2-16x+64$, which factors to $(x-8)^2$, not $(x+8)^2$. The equation becomes $(x-8)^2=63$, so $x-8=\pm\sqrt{63}$, which means that $x=8\pm\sqrt{63}=8\pm3\sqrt{7}$. The correct answer is $x=8\pm3\sqrt{7}$.
C. Incorrect. The value to add has been calculated correctly: $\left(\frac{-8}{2}\right)^2=(-8)^2=64$. However, you appear to have added 64 to the left side of the equation only, and not to the right side. The equation becomes $(x-8)^2=63$, so $x-8=\pm\sqrt{63}$, which means that $x=8\pm\sqrt{63}=8\pm3\sqrt{7}$. The correct answer is $x=8\pm3\sqrt{7}$.
D. Correct. The value of b is -16, so to complete the square, add $\left(\frac{-16}{2}\right)^2=(-8)^2=64$. Add 64 to both sides to get $x^2-16x+64=63$. This is $(x-8)^2=63$, so $x-8=\pm\sqrt{63}$. That means $x=8\pm\sqrt{63}=8\pm3\sqrt{7}$.

Summary

Completing the square is used to change a binomial of the form x^2+bx into a perfect square trinomial $x^2+bx+\left(\frac{b}{2}\right)^2$, which can be factored to $\left(x+\frac{b}{2}\right)^2$. When solving quadratic equations by completing the square, be careful to add $\left(\frac{b}{2}\right)^2$ to *both* sides of the equation to maintain equality. The Square Root Property can then be used to solve for x. With the Square Root Property, be careful to include both the principal square root and its opposite. Be sure to simplify as needed.

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16.5.2: The Quadratic Formula

Learning Objectives

- Write a quadratic equation in standard form and identify the values of a, b, and c in a standard form quadratic equation.
- Use the Quadratic Formula to find all real solutions.
- Use the Quadratic Formula to find all complex solutions.
- Compute the discriminant and state the number and type of solutions.
- Solve application problems requiring the use of the Quadratic Formula.

Introduction

You can solve any quadratic equation by **completing the square**, or rewriting part of the equation as a perfect square trinomial. If you complete the square on the generic equation $ax^2+bx+c=0$ and then solve for x, you find that $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. This equation is known as the Quadratic Formula.

This formula is very helpful for solving quadratic equations that are difficult or impossible to factor, and using it can be faster than completing the square. The Quadratic Formula can be used to solve any quadratic equation of the form $ax^2+bx+c=0$.

Standard Form

The form $ax^2+bx+c=0$ is called the standard form of a quadratic equation. Before solving a quadratic equation using the Quadratic Formula, it's *vital* that you be sure the equation is in this form. If you don't, you might use the wrong values for a, b, and c, and then the formula will give incorrect solutions.

Example

Rewrite the equation $3x+2x^2+4=5$ in standard form and identify a, b, and c.

Solution

$3x+2x^2+4=5$ $3x+2x^2+4-5=5-5$	First, be sure that the right side of the equation is 0. In this case, all you need to do is subtract 5 from both sides.
$3x+2x^2-1=0$ $2x^2+3x-1=0$	Simplify, and write the terms with the exponent on the variable in descending order.
$\begin{array}{lllll} 2x^2 & + & 3x & - & 1 \quad =0 \\ \downarrow & & \downarrow & & \downarrow \\ ax^2 & & bx & & c \end{array}$ $a=2, b=3, c=-1$	Now that the equation is in standard form, you can read the values of a, b, and c from the coefficients and constant. Note that since the constant 1 is subtracted, c must be negative.

$2x^2+3x-1=0; a=2, b=3, c=-1$

Example

Rewrite the equation $2(x+3)^2-5x=6$ in standard form and identify a, b, and c.

Solution

$2(x+3)^2-5x=6$ $2(x+3)^2-5x-6=6-6$	First, be sure that the right side of the equation is 0.

$2\left(x^2+6x+9\right)-5x-6=0$ $2x^2+12x+18-5x-6=0$ $2x^2+12x-5x+18-6=0$ $2x^2+7x+12=0$ $\begin{array}{ccccc} 2x^2 & + & 7x & + & 12=0 \\ \downarrow & & \downarrow & & \downarrow \\ a & & b & & c \end{array}$ $a=2, b=7, c=12$	Expand the squared binomial, then simplify by combining like terms. Be sure to write the terms with the exponent on the variable in descending order. Now that the equation is in standard form, you can read the values of a, b, and c from the coefficients and constant.

$2x^2+7x+12=0; a=2, b=7, c=12$

? Exercise

Identify the values of a, b, and c in the standard form of the equation $3x+x^2=6$.

A. $a=3, b=1, c=6$
B. $a=1, b=3, c=6$
C. $a=1, b=3, c=-6$
D. $a=3, b=1, c=-6$

Answer

A. Incorrect. Remember that in standard form, the equation is written in the form $ax^2+bx+c=0$. $3x+x^2=6$ becomes $3x+x^2-6=0$, so the standard form is $x^2+3x-6=0$. This means the correct answer is $a=1$, $b=3$, and $c=-6$.
B. Incorrect. You put the terms in the correct order, but the right side must be equal to 0. The c must be on the left side of the equation. $3x+x^2=6$ becomes $3x+x^2-6=0$, so the standard form is $x^2+3x-6=0$. This means the correct answer is $a=1$, $b=3$, and $c=-6$.
C. Correct. $3x+x^2=6$ becomes $3x+x^2-6=0$, so the standard form is $x^2+3x-6=0$. This means $a=1$, $b=3$, and $c=-6$.
D. Incorrect. Remember that in standard form, the equation is written in the form $ax^2+bx+c=0$. You correctly found that $3x+x^2=6$ becomes $3x+x^2-6=0$. Putting the terms in order gives the standard form $x^2+3x-6=0$. This means the correct answer is $a=1$, $b=3$, and $c=-6$.

Deriving the Quadratic Formula

Let's complete the square on the general equation and see exactly how that produces the Quadratic Formula. Recall the process of completing the square.

- Start with an equation of the form $x^2+bx+c=0$.
- Rewrite the equation so that x^2+bx is isolated on one side.
- Complete the square by adding $\left(\frac{b}{x}\right)^2$ to both sides.
- Rewrite the perfect square trinomial as a square of a binomial.
- Use the **Square Root Property** and solve for x.

Can you complete the square on the general quadratic equation $ax^2+bx+c=0$? Try it yourself before you continue to the example below. Hint: Notice that in the general equation, the coefficient of x^2 is not equal to 1. You can divide the equation by a, which makes some of the expressions a bit messy, but if you are careful, everything will work out, and at the end, you'll have the Quadratic Formula!

✓ Example

Complete the square of $ax^2+bx+c=0$ to arrive at the Quadratic Formula.

Solution

$x^2+\frac{b}{a}x+\frac{c}{a}=0$	Divide both sides of the equation by a, so that the coefficient of x^2 is 1.

$x^2+\frac{b}{a}x=-\frac{c}{a}$	Rewrite so the left side is in the form x^2+bx, (although in this case bx is actually $\frac{b}{a}x$).
$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$	The coefficient on x is $\frac{b}{a}$, so the value to add to both sides is $\left(\frac{b}{2a}\right)^2$.
$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$	Write the left side as a binomial squared.
$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$	Evaluate $\left(\frac{b}{2a}\right)^2$ as $\frac{b^2}{4a^2}$.
$\left(x+\frac{b}{2a}\right)^2=-\frac{4ac}{4a^2}+\frac{b^2}{4a^2}$	Write the fractions on the right side using a common denominator.
$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$	Add the fractions on the right.
$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$	Use the Square Root Property. Remember that you want both the positive and negative square roots!
$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}$	Subtract $\frac{b}{2a}$ from both sides to isolate x.
$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$	The denominator under the radical is a perfect square, so: $\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}=\frac{\sqrt{b^2-4ac}}{2a}$
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Add the fractions since they have a common denominator.

There you have it, the Quadratic Formula.

Solving a Quadratic Equation using the Quadratic Formula

The Quadratic Formula will work with *any* quadratic equation, but *only* if the equation is in standard form, $ax^2+bx+c=0$. To use it, follow these steps.

- Put the equation in standard form first.
- Identify the coefficients, a, b, and c. Be careful to include negative signs if the bx or c terms are subtracted.
- Substitute the values for the coefficients into the Quadratic Formula.
- Simplify as much as possible.
- Use the $\pm$ in front of the radical to separate the solution into two values: one in which the square root is added, and one in which it is subtracted.
- Simplify both values to get the possible solutions.

That's a lot of steps. Let's try using the Quadratic Formula to solve a relatively simple equation first; then you'll go back and solve it again using another factoring method.

✓ Example

Use the Quadratic Formula to solve the equation $x^2+4x=5$.

Solution

$x^2+4x=5$ $x^2+4x-5=0$	First, write the equation in standard form. $a=1, b=4, c=-5$
$\begin{array}{ccccc} x^2 & + & 4x & - & 5=0 \\ \downarrow & & \downarrow & & \downarrow \\ ax^2 & + & bx & + & c=0 \end{array}$ $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-4\pm\sqrt{(4)^2-4(1)(-5)}}{2(1)}$	Note that the subtraction sign means the constant c is negative. Substitute the values into the Quadratic Formula.
$x=\frac{-4\pm\sqrt{16+20}}{2}$	Simplify, being careful to get the signs correct.
$x=\frac{-4\pm\sqrt{36}}{2}$	Simplify some more.

$x=\frac{-4\pm 6}{2}$	Simplify the radical: $\sqrt{36}=6$.
$x=\frac{-4+6}{2}=\frac{2}{2}=1$ or $x=\frac{-4-6}{2}=\frac{-10}{2}=-5$	Separate and simplify to find the solutions to the quadratic equation. Note that in one, 6 is added and in the other, 6 is subtracted.

$x=1 \text{ or } -5$

You can check these solutions by substituting 1 and -5 into the original equation. The top table shows 1 being substituted into the original equation, and the bottom table shows -5 being substituted into the original equation.

$x=1$
$x^2+4x=5$
$(1)^2+4(1)=5$
$1+4=5$
$5=5$

$x=-5$
$x^2+4x=5$
$(-5)^2+4(-5)=5$
$25-20=5$
$5=5$

You get two true statements, so you know that both solutions work: $x=1$ or -5. You've solved the equation successfully using the Quadratic Formula!

However, upon looking at $x^2+4x=5$, you may have thought, "I already know how to do this; I can rewrite this equation as $x^2+4x-5=0$, and then factor it as $(x+5)(x-1)=0$, so $x=-5$ or 1." This is correct, and congratulations if you made this connection!

Sometimes, it may be easier to solve an equation using conventional factoring methods, like finding number pairs that sum to one number (in this example, 4) and that produce a specific product (in this example -5) when multiplied. The power of the Quadratic Formula is that it can be used to solve *any* quadratic equation, even those where finding number combinations will not work.

Most of the quadratic equations you've looked at have two solutions, like the one above. The following example is a little different.

✓ Example

Use the Quadratic Formula to solve the equation $x^2-2x=6x-16$.

Solution

$x^2-2x=6x-16$ $x^2-2x-6x+16=0$ $x^2-8x+16=0$	Subtract $6x$ from each side and add 16 to both sides to put the equation in standard form.
$\begin{array}{ccccc} x^2 & - & 8x & + & 16=0 \\ \downarrow & & \downarrow & & \downarrow \\ ax^2 & + & bx & + & c=0 \end{array}$	Identify the coefficients a, b, and c. $x^2=1x^2$, so $a=1$. Since $8x$ is subtracted, b is negative. $a=1, b=-8, c=16$
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-(-8)\pm\sqrt{(-8)^2-4(1)(16)}}{2(1)}$	Substitute the values into the Quadratic Formula.
$x=\frac{8\pm\sqrt{64-64}}{2}$	Simplify.

$x=\frac{8\pm\sqrt{0}}{2}=\frac{8}{2}=4$	Since the square root of 0 is 0, and both adding and subtracting 0 give the same result, there is only one possible value.

$x=4$

Again, check using the original equation.

$$\begin{aligned} x^2-2x&=6x-16\\ (4)^2-2(4)&=6(4)-16\\ 16-8&=24-16\\ 8&=8 \end{aligned}$$

Let's try one final example. This one also has a difference in the solution.

✓ Example

Use the Quadratic Formula to solve the equation $x^2+2x=-5$**.**

Solution

$x^2+2x=-5$ $x^2+2x+5=0$	First, write the equation in standard form.
$\begin{array}{lllll} x^2 & + & 2x & + & 5=0\\ \downarrow & & \downarrow & & \downarrow\\ ax^2 & + & bx & + & c=0 \end{array}$	$a=1,b=2,c=5$
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-2\pm\sqrt{(2)^2-4(1)(5)}}{2(1)}$	Substitute the values into the Quadratic Formula.
$x=\frac{-2\pm\sqrt{4-20}}{2}$	Simplify, being careful to get the signs correct.
$x=\frac{-2\pm\sqrt{-16}}{2}$	Simplify some more.
$x=\frac{-2\pm 4i}{2}$	Simplify the radical, but notice that the number under the radical symbol is negative! The square root of -16 is imaginary. $\sqrt{-16}=4i$
$x=\frac{-2+4i}{2}=\frac{-1+2i}{1}\cdot\frac{2}{2}=-1+2i$ or $x=\frac{-2-4i}{2}=\frac{-1-2i}{1}\cdot\frac{2}{2}=-1-2i$	Separate and simplify to find the solutions to the quadratic equation.

$x=-1+2i \text{ or } -1-2i$

Check these solutions in the original equation in the tables below. Be careful when expanding the squares and replacing i^2 with -1.

$$\begin{aligned} x&=-1+2i\\ x^2+2x&=-5\\ (-1+2i)^2+2(-1+2i)&=-5\\ 1-4i+4i^2-2+4i&=-5\\ 1-4i+4(-1)-2+4i&=-5\\ 1-4-2&=-5\\ -5&=-5 \end{aligned}$$

$$\begin{aligned} x&=-1-2i\\ x^2+2x&=-5\\ (-1-2i)^2+2(-1-2i)&=-5\\ 1+4i+4i^2-2-4i&=-5\\ 1+4i+4(-1)-2-4i&=-5\\ 1-4-2&=-5\\ -5&=-5 \end{aligned}$$

Exercise

Use the Quadratic Formula to solve the equation $x^2-2x-4=0$.

A. $x=2$
B. $x=11, x=-9$
C. $x=2+\sqrt{5}, x=2-\sqrt{5}$
D. $x=1+\sqrt{5}, x=1-\sqrt{5}$

Answer

A. Incorrect. You may have incorrectly factored the left side as $(x-2)^2$. Since $a=1$, $b=-2$, and $c=-4$, $x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-4)}}{2(1)}=\frac{2\pm\sqrt{4+16}}{2}=\frac{2\pm\sqrt{20}}{2}=\frac{2\pm2\sqrt{5}}{2}=\frac{2}{2}\pm\frac{2\sqrt{5}}{2}=1\pm\sqrt{5}$. The correct answer is $x=1+\sqrt{5}$ or $x=1-\sqrt{5}$.

B. Incorrect. $x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-4)}}{2(1)}=\frac{2\pm\sqrt{4+16}}{2}=\frac{2\pm\sqrt{20}}{2}$. If you ignore the square root, this becomes $x=11$ or $x=-9$, so you may have forgotten to take the square root of 20. $x=\frac{2\pm\sqrt{20}}{2}=\frac{2\pm2\sqrt{5}}{2}=\frac{2}{2}\pm\frac{2\sqrt{5}}{2}=1\pm\sqrt{5}$, which means the correct answer is $x=1+\sqrt{5}$ or $x=1-\sqrt{5}$.

C. Incorrect. Using the formula, $x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-4)}}{2(1)}=\frac{2\pm\sqrt{4+16}}{2}=\frac{2\pm\sqrt{20}}{2}=\frac{2\pm2\sqrt{5}}{2}$. If you forget that the denominator is under both terms in the numerator, you might get $x=2+\sqrt{5}$ or $x=2-\sqrt{5}$. However, the correct simplification is $x=\frac{2\pm2\sqrt{5}}{2}=\frac{2}{2}\pm\frac{2\sqrt{5}}{2}=1\pm\sqrt{5}$, so the answer is $x=1+\sqrt{5}$ or $x=1-\sqrt{5}$.

D. Correct. Using the formula, $x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-4)}}{2(1)}=\frac{2\pm\sqrt{4+16}}{2}=\frac{2\pm\sqrt{20}}{2}=\frac{2\pm2\sqrt{5}}{2}=\frac{2}{2}\pm\frac{2\sqrt{5}}{2}=1\pm\sqrt{5}$, so $x=1+\sqrt{5}$ or $x=1-\sqrt{5}$.

The Discriminant

These examples have shown that a quadratic equation may have two real solutions, one real solution, or two complex solutions.

In the Quadratic Formula, the expression underneath the radical symbol determines the number and type of solutions the formula will reveal. This expression, b^2-4ac, is called the **discriminant** of the equation $ax^2+bx+c=0$.

Let's think about how the discriminant affects the evaluation of $\sqrt{b^2-4ac}$, and how it helps to determine the solution set.

- If $b^2-4ac>0$, then the number underneath the radical will be a positive value. You can always find the square root of a positive, so evaluating the Quadratic Formula will result in two real solutions (one by adding the positive square root, and one by subtracting it).
- If $b^2-4ac=0$, then you will be taking the square root of 0, which is 0. Since adding and subtracting 0 both give the same result, the $\pm$ portion of the formula doesn't matter. There will be one real solution.
- If $b^2-4ac<0$, then the number underneath the radical will be a negative value. Since you cannot find the square root of a negative number using real numbers, there are no real solutions. However, you can use imaginary numbers. You will then have two complex solutions, one by adding the imaginary square root and one by subtracting it.

Example

Use the discriminant to determine how many and what kind of solutions the quadratic equation $x^2-4x+10=0$ has.

Solution

b^2-4ac $(-4)^2-4(1)(10)$	Evaluate b^2-4ac. First note that $a=1$, $b=-4$, and $c=10$.
$16-40=-24$	The result is a negative number. The discriminant is negative, so the quadratic equation has two complex solutions.

The quadratic equation $x^2-4x+10=0$ has two complex solutions.

Exercise

Suppose a quadratic equation has a discriminant that evaluates to zero. Which of the following statements is always true?

A. The equation has two solutions.
B. The equation has one solution.
C. The equation has zero solutions.

Answer

A. Incorrect. Since adding and subtracting 0 both give the same result, the $\pm$ portion of the Quadratic Formula doesn't matter. A discriminant of zero means the equation has one solution. The correct answer is “The equation has one solution.”
B. Correct. A discriminant of zero means the equation has one solution.
C. Incorrect. When the discriminant is zero, the equation will have one solution. The correct answer is “The equation has one solution.”

Applying the Quadratic Formula

Quadratic equations are widely used in science, business, and engineering. Quadratic equations are commonly used in situations where two things are multiplied together and they both depend on the same variable. For example, when working with area, if both dimensions are written in terms of the same variable, you use a quadratic equation. Because the quantity of a product sold often depends on the price, you sometimes use a quadratic equation to represent revenue as a product of the price and the quantity sold. Quadratic equations are also used when gravity is involved, such as the path of a ball or the shape of cables in a suspension bridge.

A very common and easy-to-understand application is the height of a ball thrown at the ground off a building. Because gravity will make the ball speed up as it falls, a quadratic equation can be used to estimate its height any time before it hits the ground. Note: The equation isn't completely accurate, because friction from the air will slow the ball down a little. For our purposes, this is close enough.

Example

A ball is thrown off a building from 200 feet above the ground. Its starting velocity (also called *initial velocity*) is -10 feet per second. (The negative value means it's heading toward the ground.)

The equation $h=-16t^2-10t+200$ can be used to model the height of the ball after t seconds. About how long does it take for the ball to hit the ground?

Solution

$h=-16t^2-10t+200$ $0=-16t^2-10t+200$ $-16t^2-10t+200=0$	When the ball hits the ground, the height is 0. Substitute 0 for h.
$t=\frac{-(-10)\pm\sqrt{(-10)^2-4(-16)(200)}}{2(-16)}$	This equation is difficult to solve by factoring or by completing the square, so solve it by applying the Quadratic Formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. In this case, the variable is t rather than x. $a=-16$, $b=-10$, and $c=200$.
$t=\dfrac{10\pm\sqrt{100+12,800}}{-32}$ $=\dfrac{10\pm\sqrt{12,900}}{-32}$	Simplify. Be very careful with the signs. Use a calculator to find both roots.
t is approximately -3.86 or 3.24.	Consider the roots logically. One solution, -3.86, cannot be the time because it is a negative number. The other solution, 3.24 seconds, must be when the ball hits the ground.

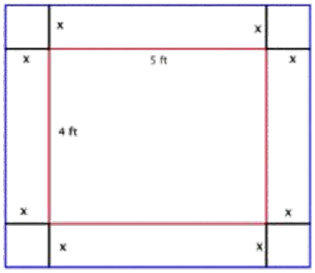
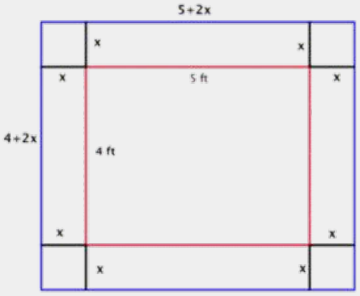
The ball hits the ground approximately 3.24 seconds after being thrown.

The area problem below does not look like it includes a Quadratic Formula of any type, and the problem seems to be something you have solved many times before by simply multiplying. But in order to solve it, you will need to use a quadratic equation.

✓ Example

Bob made a quilt that is 4 feet by 5 feet. He has 10 square feet of fabric he can use to add a border around the quilt. How wide should he make the border to use all the fabric? (The border must be the same width on all four sides.)

Solution

	Sketch the problem. Since you don't know the width of the border, you will let the variable x represent the width. In the diagram, the original quilt is indicated by the red rectangle. The border is the area between the red and blue lines.
 5+2x x x x 5 ft x 4+2x 4 ft x x x x	Since each side of the original 4 by 5 quilt has the border of width x added, the length of the quilt with the border will be $5+2x$, and the width will be $4+2x$. (Both dimensions are written in terms of the same variable, and you will multiply them to get an area! This is where you might start to think that a quadratic equation might be used to solve this problem.)
Area of border = Area of the blue rectangle minus the area of the red rectangle Area of border $=(4+2x)(5+2x)-(4)(5)$	You are only interested in the area of the border strips. Write an expression for the area of the border.
$10=(4+2x)(5+2x)-20$	There are 10 sq ft of fabric for the border, so set the area of border to be 10.
$10=20+8x+10x+4x^2-20$	Multiply $(4+2x)(5+2x)$.
$10=18x+4x^2$	Simplify.
$0=18x+4x^2-10$ or $4x^2+18x-10=0$	Subtract 10 from both sides so that you have a quadratic equation in standard form and can apply the Quadratic Formula to find the roots of the equation.
$2\left(2x^2+9x-5\right)=0$ $\frac{2\left(2x^2+9x-5\right)}{2}=\frac{0}{2}$ $2x^2+9x-5=0$	Factor out the greatest common factor, 2, so that you can work with the simpler equivalent equation, $2x^2+9x-5=0$.
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-9\pm\sqrt{9^2-4(2)(-5)}}{2(2)}$	Use the Quadratic Formula. In this case, $a=2$, $b=9$, and $c=-5$.
$x=\frac{-9\pm\sqrt{121}}{4}=\frac{-9\pm11}{4}$	Simplify.
$x=\frac{-9+11}{4}=\frac{2}{4}=\frac{1}{2}=0.5$ or $x=\frac{-9-11}{4}=-\frac{20}{4}=-5$	Find the solutions, making sure that the $\pm$ is evaluated for both values.
The width of the border should be 0.5 ft.	Ignore the solution $x=-5$, since the width could not be negative.

Exercise

A ball is launched upward at 48 feet/second from a platform that is 100 feet high. The equation giving its height t seconds after launch is $h=-16t^2+48t+100$. The ball will shoot up to 136 feet high, then begin to come back down. About how long will the ball take to get to that maximum height?

A. 1.5 seconds
B. 3.6 seconds
C. 4.4 seconds
D. This problem cannot be solved.

Answer

A. Correct. When the height is 136 feet, the equation becomes $136=-16t^2+48t+100$. Subtracting 136 from both sides gives $0=-16t^2+48t-36$. Factoring out -4 results in the simpler equation $0=4t^2-12t+9$ and using the Quadratic Formula gives $t=\frac{-(-12)\pm\sqrt{(-12)^2-4(4)(9)}}{2(4)}$. This simplifies to $t=\frac{12\pm\sqrt{0}}{8}=1.5$. (Since the discriminant is 0, there is only one solution.) The ball will reach its maximum height in 1.5 seconds.
B. Incorrect. You probably set up the equation correctly, $136=-16t^2+48t+100$. However, you probably either forgot a negative sign for the values of a or c, or you incorrectly evaluated $-4ac$ in the discriminant. Subtracting 136 from both sides of the equation gives $0=-16t^2+48t-36$. Factoring out -4 results in the simpler equation $0=4t^2-12t+9$, so $a=4$, $b=-12$, and $c=9$. Using the Quadratic Formula gives $t=\frac{-(-12)\pm\sqrt{(-12)^2-4(4)(9)}}{2(4)}$. This simplifies to $t=\frac{12\pm\sqrt{0}}{8}=1.5$. (Since the discriminant is 0, there is only one solution.) The correct answer is 1.5 seconds.
C. Incorrect. You probably correctly solved the equation $0=-16t^2+48t+100$, but you are looking for the time when the height is 136, not 0. The equation to solve is $136=-16t^2+48t+100$. Subtracting 136 from both sides gives $0=-16t^2+48t-36$. Factoring out -4 results in the simpler equation $0=4t^2-12t+9$ and using the Quadratic Formula gives $t=\frac{-(-12)\pm\sqrt{(-12)^2-4(4)(9)}}{2(4)}$. This simplifies to $t=\frac{12\pm\sqrt{0}}{8}=1.5$. (Since the discriminant is 0, there is only one solution.) The correct answer is 1.5 seconds.
D. Incorrect. This problem can be solved if you substitute 136 in for h in the original equation. This gives you: $136=-16t^2+48t+100$. Subtracting 136 from both sides gives $0=-16t^2+48t-36$. Factoring out -4 results in the simpler equation $0=4t^2-12t+9$ and using the Quadratic Formula gives $t=\frac{-(-12)\pm\sqrt{(-12)^2-4(4)(9)}}{2(4)}$. This simplifies to $t=\frac{12\pm\sqrt{0}}{8}=1.5$. (Since the discriminant is 0, there is only one solution.) The correct answer is 1.5 seconds.

Summary

Quadratic equations can appear in different applications. The Quadratic Formula is a useful way to solve these equations, or any other quadratic equation! The Quadratic Formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, is found by completing the square of the quadratic equation $ax^2+bx+c=0$.

The discriminant of the Quadratic Formula is the quantity under the radical, b^2-4ac. It determines the number and the type of solutions that a quadratic equation has. If the discriminant is positive, there are 2 real solutions. If it is 0, there is 1 real solution. If the discriminant is negative, there are 2 complex solutions (but no real solutions).

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CHAPTER OVERVIEW

17: Functions

- 17.1: Introduction to Functions
  - 17.1.1: Identifying Functions
- 17.2: Using Functions
  - 17.2.1: Evaluating Functions
  - 17.2.2: Graphing Types of Functions
  - 17.2.3: Finding Domain and Range
- 17.3: Operations with Functions
  - 17.3.1: Arithmetic Operations with Functions

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SECTION OVERVIEW

17.1: Introduction to Functions

Topic hierarchy

17.1.1: Identifying Functions

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17.1.1: Identifying Functions

Learning Objectives

- Determine whether a relation is a function.
- Determine the domain of a function and the range of a function.
- Determine whether a graph is that of a function by using a vertical line test.

Introduction

Algebra gives us a way to explore and describe relationships. Imagine tossing a ball straight up in the air and watching it rise to reach its highest point before dropping back down into your hands. As time passes, the height of the ball changes. There is a relationship between the amount of time that has elapsed since the toss and the height of the ball. In mathematics, a correspondence between variables that change together (such as time and height) is called a **relation**. Some, but not all, relations can also be described as functions.

Defining Function

There are many kinds of relations. Relations are simply correspondences between sets of values or information. Think about members of your family and their ages. The pairing of each member of your family and their age is a relation. Each family member can be paired with an age in the set of ages of your family members. Another example of a relation is the pairing of a state with its United States' senators. Each state can be matched with two individuals who have been elected to serve as senator. In turn, each senator can be matched with one specific state that he or she represents. Both of these are real-life examples of relations.

The first value of a relation is an input value and the second value is the output value. A **function** is a specific type of relation in which each input value has one and only one output value. An input is the *independent* value, and the output value is the *dependent* value, as it depends on the value of the input.

Notice in the first table below, where the input is "name" and the output is "age", each input matches with exactly one output. This is an example of a function.

(Input) **Family Member's Name**	**(Output)** **Family Member's Age**
Nellie	13
Marcos	11
Esther	46
Samuel	47
Nina	47
Paul	47
Katrina	21
Andrew	16
Maria	13
Ana	81

Compare this with the next table, where the input is "age" and the output is "name." Some of the inputs result in more than one output. This is an example of a correspondence that is *not* a function.

Starting Information (Input): Family Member's Age	**Related Information (Output): Family Member's Name**
11	Marcos

13	Nellie Maria
16	Andrew
21	Katrina
46	Esther
47	Samuel Nina Paul
81	Ana

Let’s look back at our examples to determine whether the relations are functions or not and under what circumstances. Remember that a relation is a function if there is only ***one*** output for each input.

Input	**Output**	**Function?**	**Why or why not?**
Name of senator	Name of state	Yes	For each input, there will only be one output because a senator only represents one state.
Name of state	Name of senator	No	For each state that is an input, 2 names of senators would result because each state has two senators.
Time elapsed	Height of a tossed ball	Yes	At a specific time, the ball has one specific height.
Height of a tossed ball	Time elapsed	No	Remember that the ball was tossed up and fell down. So for a given height, there could be two different times when the ball was at that height. The input height can result in more than one output.
Number of cars	Number of tires	Yes	For any input of a specific number of cars, there is one specific output representing the number of tires.
Number of tires	Number of cars	Yes	For any input of a specific number of tires, there is one specific output representing the number of cars.

? Exercise

Which of the following situations describes a function?

A. Your age and your weight at noon on your birthday each year.
B. The number of people on a professional baseball team and the name of the team.
C. The diameter of a cookie and the number of chocolate chips in it.

Answer

A. Correct. Age only increases while weight can change. On each birthday, you have just one weight at noon, so for every input, there is only one output.
B. Incorrect. Professional baseball teams all have the same number of players, so the number of players is not a function of the team's name. The correct answer is your age and your weight at noon on your birthday each year.
C. Incorrect. Although bigger cookies can hold more chips, the exact number in any size of cookie will vary with the recipe and how evenly the batter is mixed and distributed. A single input of cookie size will produce different outputs of chips. The correct answer is your age and your weight at noon on your birthday each year.

Relations can be written as ordered pairs of numbers or as numbers in a table of values. By examining the inputs (x-coordinates) and outputs (y-coordinates), you can determine whether or not the relation is a function. Remember, in a function each input has only one output. A couple of examples follow.

✓ Example

Is the relation given by the set of ordered pairs below a function?

$\{(-3,-6),(-2,-1),(1,0),(1,5),(2,0)\}$

Solution

x	y
-3	-6
-2	-1
1	0
1	5
2	0

Organizing the ordered pairs in a table can help.
By definition, the inputs in a function have only one output.
The input 1 has two outputs: 0 and 5.

The relation is not a function.

✓ Example

Is the relation given by the set of ordered pairs below a function?

$\{(-3,4),(-2,4),(-1,4),(2,4),(3,4)\}$

Solution

x	y
-3	4
-2	4
-1	4
2	4
3	4

You could reorganize the information by creating a table.
Each input has only one output, and the fact that it is the same output (4) does not matter.

This relation is a function.

Remember that in a function, the input value must have one and only one value for the output.

Domain and Range

There is a name for the set of input values and another name for the set of output values for a function. The set of input values is called the **domain of the function**. And the set of output values is called the **range of the function**.

If you have a set of ordered pairs, you can find the domain by listing all of the input values, which are the x-coordinates. And to find the range, list all of the output values, which are the y-coordinates.

So for the following set of ordered pairs,

$\{(-2,0),(0,6),(2,12),(4,18)\}$

You have the following:

Domain: $\{-2,0,2,4\}$

Range: $\{0,6,12,18\}$

? Exercise

Jamie plans to sell homemade pies for $10 each at a local farm stand. The amount of money he makes is a function of how many pies he sells: $0 if he sells 0 pies, $10 if he sells 1 pie, $20 if he sells 2 pies, and so on. He does not want the pies to spoil before he is able to sell them, so he will not make (or sell) more than 9 pies. What is the domain and range for that function?

A. Domain: $\{0,10,20,30,40,50,60,70,80,90\}$ Range: $\{0,1,2,3,4,5,6,7,8,9\}$
B. Domain: $\{0,1,2,3,4,5,6,7,8,9\}$ Range: $\{0,10,20,30,40,50,60,70,80,90\}$
C. Domain: $\{0,1,2\}$ Range: $\{0,10,20\}$
D. Domain: all numbers greater than or equal to 0

Answer

A. Incorrect. The number of pies is the input, and the amount of money is the output of the function. That means that the domain is all possible number of pies, and the range is all possible money made from those pies. The correct answer is Domain: $\{0,1,2,3,4,5,6,7,8,9\}$ Range: $\{0,10,20,30,40,50,60,70,80,90\}$
B. Correct. The number of pies Jamie can sell is the input, and that can be any whole number from 0 to the maximum he would make, 9. The money he gets from those pies is always a multiple of 10: 0 for 0 pies, 10 for 1 pie, 20 for 2 pies, and so on.
C. Incorrect. Both the domain and range continue beyond those values. Jamie can sell as many as 9 pies, and as a result he can earn more than $20. You must include all possible values of the domain and range. The correct answer is Domain: $\{0,1,2,3,4,5,6,7,8,9\}$ Range: $\{0,10,20,30,40,50,60,70,80,90\}$.
D. Incorrect. Jamie doesn’t sell fractions of pies, so the only possible inputs are whole numbers from 0 to 9, and the only possible outputs are 0 and multiples of 10 up to 90. The correct answer is Domain: $\{0,1,2,3,4,5,6,7,8,9\}$ Range: $\{0,10,20,30,40,50,60,70,80,90\}$.

✓ Example

State the domain and range of the following function. $\{(-3,5),(-2,5),(-1,5),(0,5),(1,5),(2,5)\}$

Solution

$\{-3,-2,-1,0,1,2\}$	The domain is all the x-coordinates.
$\{5\}$	The range is all the y-coordinates. Each ordered pair has the same y-coordinate. It only needs to be listed once.

Domain: $\{-3,-2,-1,0,1,2\}$

Range: $\{5\}$

Example

Find the domain and range for the function.

x	y
-5	-6
-2	-1
-1	0
0	3
5	15

Solution

$\{-5,-2,-1,0,5\}$	The domain is the set of inputs or x-coordinates.
$\{-6,-1,0,3,15\}$	The range is the set of outputs of y-coordinates.

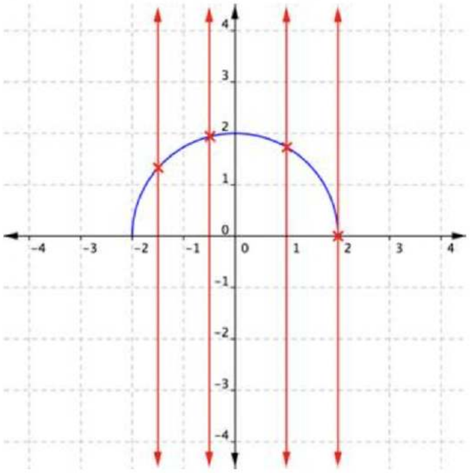
Domain: $\{-5,-2,-1,0,5\}$

Range: $\{-6,-1,0,3,15\}$

Using the Vertical Line Test

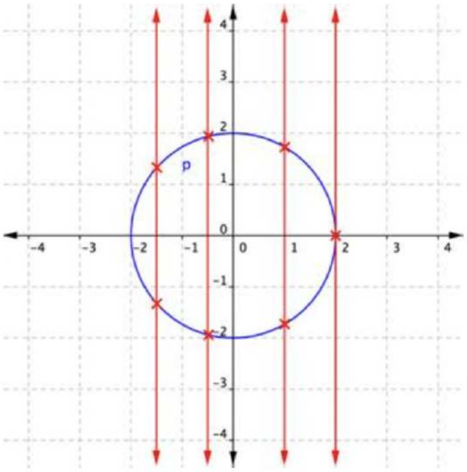
When both the independent quantity (input) and the dependent quantity (output) are real numbers, a function can be represented by a graph in the coordinate plane. The independent value is plotted on the x-axis and the dependent value is plotted on the -axis. The fact that each input value has exactly one output value means graphs of functions have certain characteristics. For each input on the graph, there will be exactly one output.

For example, the graph of the function below drawn in blue looks like a semi-circle. You know that y is a function of x because for each x-coordinate there is exactly one y-coordinate.



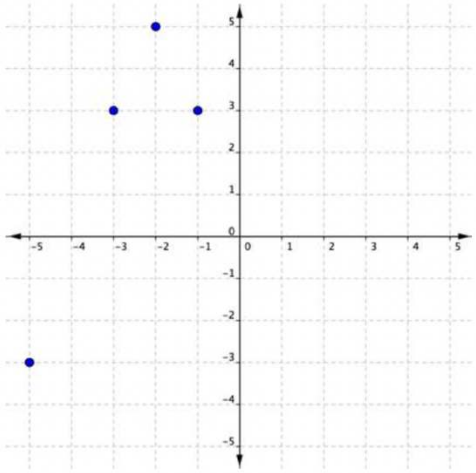
If you draw a vertical line across the plot of the function, it only intersects the function once for each value of x. That is true no matter where the vertical line is drawn. Placing or sliding such a line across a graph is a good way to determine if it shows a function.

Compare the previous graph with this one, which looks like a blue circle. This relationship cannot be a function, because some of the x-coordinates have two corresponding y-coordinates.



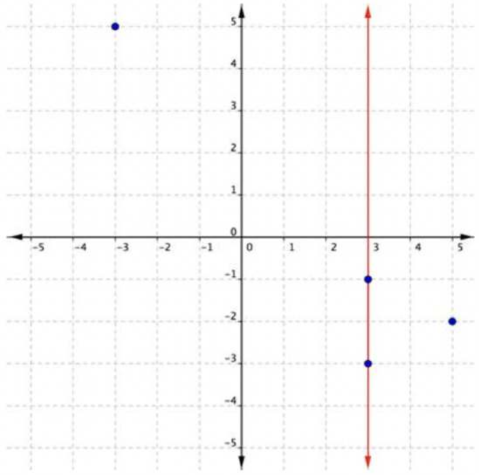
When a vertical line is placed across the plot of this relation, it intersects the graph more than once for some values of x. If a graph shows two or more intersections with a vertical line, then an input (x-coordinate) can have more than one output (y-coordinate), and y is not a function of x. Examining the graph of a relation to determine if a vertical line would intersect with more than one point is a quick way to determine if the relation shown by the graph is a function. This method is often called the “vertical line test.”

The vertical line method can also be applied to a set of ordered pairs plotted on a coordinate plane to determine if the relation is a function. Consider the ordered pairs $\{(-1,3),(-2,5),(-3,3),(-5,-3)\}$, plotted on the graph below.



Here, you can see that in the set of pairs just listed, every independent value has one and only one dependent value. You can also check that a vertical line running through any point would not intersect with another point. A horizontal line would intersect two of the points, but that is just fine. (Remember, it’s a *vertical* line test not a *horizontal* line test that determines if a relation is a function!)

In another set of ordered pairs, $\{(3,-1),(5,-2),(3,-3),(-3,5)\}$, one of the inputs, 3, can produce two different outputs, -1 and -3. You know what that means: this set of ordered pairs is not a function. A plot confirms this.



Notice that a vertical line passes through two plotted points. One x-coordinate has multiple y-coordinates. This relation is not a function.

? Exercise

Which of the following is a set of ordered pairs representing a function?

A. $\{2,4,4,8,8,16,16,32\}$
B. $\{(0,0),(1,1),(1,-1),(2,2),(2,-2)\}$
C. $\{(5,-10),(5,-3),(5,0),(5,2),(5,17)\}$
D. $\{(-2,2),(-1,1),(0,0),(1,1),(2,2)\}$

Answer

A. Incorrect. These numbers are not grouped into ordered pairs. Without proper notation, it is impossible to know which values are the inputs and which are the outputs. The correct answer is $\{(-2,2),(-1,1),(0,0),(1,1),(2,2)\}$.
B. Incorrect. Some x-coordinates are repeated and have different y-coordinates. This is not a function. The correct answer is $\{(-2,2),(-1,1),(0,0),(1,1),(2,2)\}$.
C. Incorrect. This set contains five ordered pairs, each with an x-coordinate of 5 and different y-coordinates as outputs. This is not a function, since a function requires each input to have only one output. The correct answer is $\{(-2,2),(-1,1),(0,0),(1,1),(2,2)\}$.
D. Correct. No x-coordinate is repeated and each has exactly one y-coordinate, so $\{(-2,2),(-1,1),(0,0),(1,1),(2,2)\}$ is a function.

Summary

In real life and in Algebra, different variables are often linked. When a change in value of one variable causes a change in the value of another variable, their interaction is called a relation. A relation has an input value which corresponds to an output value. When each input value has one and only one output value, that relation is a function. Functions can be written as ordered pairs, tables, or graphs. The set of input values is called the domain, and the set of output values is called the range.

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SECTION OVERVIEW

17.2: Using Functions

Topic hierarchy

17.2.1: Evaluating Functions

17.2.2: Graphing Types of Functions

17.2.3: Finding Domain and Range

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17.2.1: Evaluating Functions

Learning Objectives

- Given a function, described by an equation, find function values (outputs) for specified inputs.

Introduction

Throughout this course, you have been working with algebraic equations. Many of these equations are functions. For example, $y = 4x + 1$ is an equation that represents a function. When you input values for x, you can determine a single output for y. In this case, if you substitute $x = 10$ into the equation you will find that y must be 41; there is no other value of y that would make the equation true.

Rather than using the variable y, the equations of functions can be written using **function notation**. Function notation is very useful when you are working with more than one function at a time, and substituting more than one variable in for x.

Function Notation

Some people think of functions as "mathematical machines." Imagine you have a machine that changes a number according to a specific rule, such as "multiply by 3 and add 2" or "divide by 5, add 25, and multiply by -1." If you put a number into the machine, a new number will pop out the other end, having been changed according to the rule. The number that goes in is called the input, and the number that is produced is called the output.

You can also call the machine "f" for function. If you put x into the machine, $f(x)$ comes out. Mathematically speaking, x is the input, or the "independent variable," and $f(x)$ is the output, or the "dependent variable," since it depends on the value of x.

$f(x) = 4x + 1$ is written in function notation and is read "f of x equals $4x$ plus 1." It represents the following situation: A function named f acts upon an input, x, and produces $f(x)$ which is equal to $4x + 1$. This is the same as the equation $y = 4x + 1$.

Function notation gives you more flexibility because you don't have to use y for every equation. Instead, you can use $f(x)$ or $g(x)$ or $c(x)$. This can be a helpful way to distinguish equations of functions when you are dealing with more than one at a time.

You could write the formula for perimeter, $P = 4s$, as the function $p(x) = 4x$, and the formula for area, $A = x^2$, as $a(x) = x^2$. This would make it easy to graph both functions on the same graph without confusion about the variables.

Exercise

Which two equations represent the same function?

A. $y = 2x - 7$ and $f(x) = 7 - 2x$
B. $3x = y - 2$ and $f(x) = 3x - 2$
C. $f(x) = 3x^2 + 5$ and $y = 3x^2 + 5$
D. None of the above

Answer

A. Incorrect. These equations look similar but are not the same. The first has a slope of 2 and a y-intercept of -7. The second function has a slope of -2 and a y-intercept of 7. It slopes in the opposite direction. They do not produce the same graph, so they are not the same function. The correct answer is $f(x) = 3x^2 + 5$ and $y = 3x^2 + 5$.
B. Incorrect. These equations represent two different functions. If you rewrite the first equation in terms of y, you'll find the equation of the function is $y = 3x + 2$. The correct answer is $f(x) = 3x^2 + 5$ and $y = 3x^2 + 5$.
C. Correct. The expressions after $f(x) =$ and $y =$ are the same, so these are two different ways to write the same function: $f(x) = 3x^2 + 5$ and $y = 3x^2 + 5$.
D. Incorrect. Look at the expressions that follow $f(x) =$ and $y =$. If the expressions are the same, then the equations represent the same exact function. The correct answer is $f(x) = 3x^2 + 5$ and $y = 3x^2 + 5$.

Evaluating Functions

Equations written using function notation can also be evaluated. With function notation, you might see a problem like this.

Given $f(x)=4x+1$, find $f(2)$.

You read this problem like this: “given f of x equals $4x$ plus one, find f of 2.” While the notation and wording is different, the process of evaluating a function is the same as evaluating an equation: in both cases, you substitute 2 for x, multiply it by 4 and add 1, simplifying to get 9. In both a function and an equation, an input of 2 results in an output of 9.

$$f(x)=4x+1$$
$$f(2)=4(2)+1=8+1=9$$

You can simply apply what you already know about evaluating expressions to evaluate a function. It’s important to note that the parentheses that are part of function notation do not mean multiply. The notation $f(x)$ does not mean f multiplied by x. Instead the notation means “ f of x” or “the function of x” To evaluate the function, take the value given for x, and substitute that value in for x in the expression. Let’s look at a couple of examples.

✓ Example

Given $f(x)=3x-4$, find $f(5)$.

Solution

$f(5)=3(5)-4$	Substitute 5 in for x in the function.
$f(5)=15-4$ $f(5)=11$	Simplify the expression on the right side of the equation.

Given $f(x)=3x-4, f(5)=11$.

Functions can be evaluated for negative values of x, too. Keep in mind the rules for integer operations.

✓ Example

Given $p(x)=2x^2+5$, find $p(-3)$.

Solution

$p(-3)=2(-3)^2+5$	Substitute -3 in for x in the function.
$p(-3)=2(9)+5$ $p(-3)=18+5$ $p(-3)=23$	Simplify the expression on the right side of the equation.

Given $p(x)=2x^2+5, p(-3)=23$.

You may also be asked to evaluate a function for more than one value as shown in the example that follows.

✓ Example

Given $f(x)=3x^2+2x+1$, find $f(0)$, $f(2)$, and $f(-1)$.

Solution

$f(0)=3(0)^2+2(0)+1$ $f(0)=0+0+1$ $f(0)=1$	Treat each of these like three separate problems. In each case, you substitute the value in for x and simplify. Start with $x=0$.
$f(2)=3(2)^2+2(2)+1$ $f(2)=3(4)+4+1$ $f(2)=12+4+1$ $f(2)=17$	Evaluate for $x=2$.

$f(-1)=3(-1)^2+2(-1)+1$ $f(-1)=3(1)+(-2)+1$ $f(-1)=3-2+1$ $f(-1)=1+1$ $f(-1)=2$	Evaluate for $x=-1$.

Given $f(x)=3x^2+2x+1$, $f(0)=1$, $f(2)=17$, and $f(-1)=2$.

? Exercise

Given $h(x)=4x+7$, find $h(-10)$.

A. $-40h+7$
B. -33
C. $4x+17$
D. 47

Answer

A. Incorrect. $h(-10)$ means “ h of negative ten” not “ h times negative ten.” To evaluate the function, substitute -10 for x. The correct answer is -33.
B. Correct. $h(-10)=4(-10)+7=-40+7=-33$.
C. Incorrect. To find $h(-10)$, substitute -10 in for x in the right side of the equation and simplify. The correct answer is -33.
D. Incorrect. Evaluate the function for $h(-10)$, not $h(10)$. The correct answer is -33.

Evaluating Functions with Variable Inputs

So far, you have evaluated functions for inputs that have been constants. Functions can also be evaluated for inputs that are variables or expressions. The process is the same, but the simplified answer will contain a variable. The following examples show how to evaluate a function for a variable input.

✓ Example

Given $f(x)=3x^2+2x+1$, find $f(b)$.

Solution

$f(b)=3b^2+2b+1$	This problem is asking you to evaluate the function for b. This means substitute b in the equation for x. (That’s it! You're done.)

Given $f(x)=3x^2+2x+1$, $f(b)=3b^2+2b+1$.

In the following example, you evaluate a function for an expression. So here you will substitute the entire expression in for x and simplify.

✓ Example

Given $f(x)=4x+1$, find $f(h+1)$.

Solution

$f(h+1)=4(h+1)+1$	This time, you substitute $(h+1)$ into the equation for x.
$f(h+1)=4h+4+1$ $=4h+5$	Use the distributive property on the right side, and then combine like terms to simplify.

Given $f(x)=4x+1, f(h+1)=4h+5$.

Summary

Function notation takes the form such as $f(x)=18x-10$ and is read “ f of x equals 18 times x minus 10.” Function notation can use letters other than f, such as $c(x)$, $g(x)$, or $h(x)$. As you go further in your study of functions, this notation will provide you more flexibility, allowing you to examine and compare different functions more easily. Just as an algebraic equation written in x and y can be evaluated for different values of the input x, an equation written in function notation can also be evaluated for different values of x. To evaluate a function, substitute in values for x and simplify to find the related output.

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17.2.2: Graphing Types of Functions

Learning Objectives

- Graph linear functions.
- Graph quadratic functions.
- Graph radical functions.

Introduction

When both the input (independent variable) and the output (dependent variable) are real numbers, a function can be represented by a coordinate graph. The input is plotted on the x-axis and the output is plotted on the y-axis.

Linear Functions

A helpful first step in graphing a function is to make a table of values. This is particularly useful when you don't know the general shape the function will have. You probably already know that a linear function will be a straight line, but let's make a table first to see how it can be helpful.

When making a table, it's a good idea to include negative values, positive values, and zero to ensure that you do have a linear function.

Example

Make a table of values for $f(x) = 3x + 2$.

Solution

$\begin{array}{c\|c} x & f(x) \\ \hline & \end{array}$	Make a two-column table. Label the columns x and $f(x)$.
$\begin{array}{c\|c} x & f(x) \\ \hline -2 & \\ -1 & \\ 0 & \\ 1 & \\ 3 & \end{array}$	Choose several values for x and put them as separate rows in the x column. **Tip:** It's always good to include 0, positive values, and negative values, if you can.
$\begin{array}{c\|c} x & f(x) \\ \hline -2 & -4 \\ -1 & -1 \\ 0 & 2 \\ 1 & 5 \\ 3 & 11 \end{array}$	Evaluate the function for each value of x, and write the result in the $f(x)$ column next to the x value you used. When $x = 0, f(0) = 3(0) + 2 = 2$, $f(1) = 3(1) + 2 = 5$, $f(-1) = 3(-1) + 2 = -3 + 2 = -1$ and so on.
$\begin{array}{c\|c} x & f(x) \\ \hline -2 & -4 \\ -1 & -1 \\ 0 & 2 \\ 1 & 5 \\ 3 & 11 \end{array}$	(Note that your table of values may be different from someone else's. You may each choose different numbers for x.)

Now that you have a table of values, you can use them to help you draw both the shape and location of the function. **Important:** The graph of the function will show all possible values of x and the corresponding values of y. This is why the graph is a line and not just the dots that make up the points in our table.

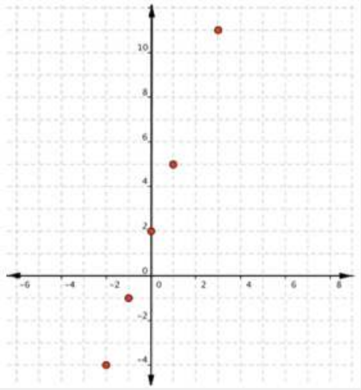
Example

Graph $f(x)=3x+2$.

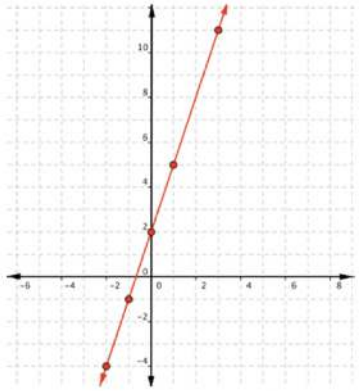
Solution

x	$f(x)$
-2	-4
-1	-1
0	2
1	5
3	11

Start with a table of values, like the one in the previous example. If you think of $f(x)$ as y, each row forms an ordered pair that you can plot on a coordinate grid.



Plot the points.



Since the points lie on a line, use a straight edge to draw the line. Try to go through each point without moving the straight edge.

Let's try another one.

Example

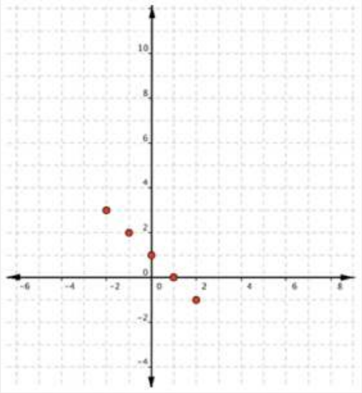
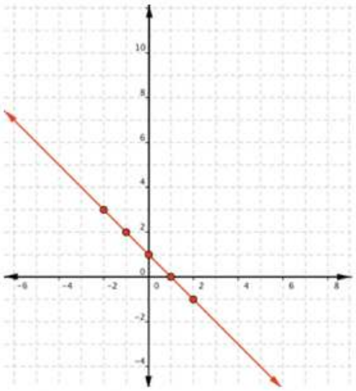
Graph $f(x)=-x+1$.

Solution

$$f(-2)=-(-2)+1=2+1=3$$
$$f(-1)=-(-1)+1=1+1=2$$
$$f(0)=-(0)+1=0+1=1$$
$$f(1)=-(1)+1=-1+1=0$$
$$f(2)=-(2)+1=-2+1=-1$$

x	$f(x)$
-2	3
-1	2
0	1
1	0
2	-1

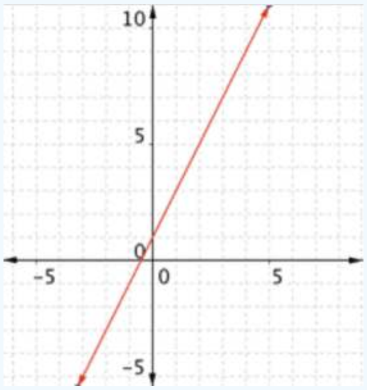
Start with a table of values. You can choose different values for x, but once again, it's helpful to include 0, some positive values, and some negative values.f you think of $f(x)$ as y, each row forms an ordered pair that you can plot on a coordinate grid.

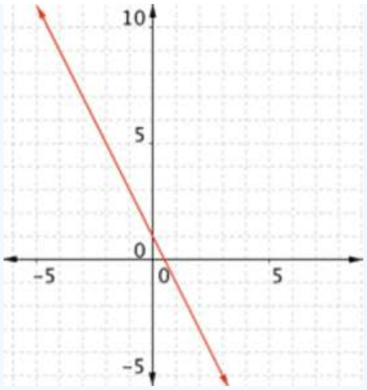
	Plot the points.
	Since the points lie on a line, use a straight edge to draw the line. Try to go through each point without moving the straight edge.

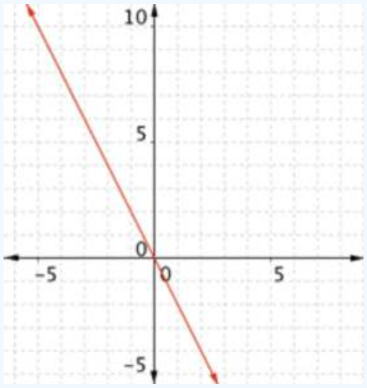
These graphs are representations of a linear function. Remember that a function is a correspondence between two variables, such as x and y.

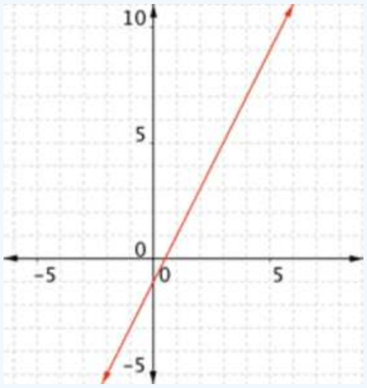
? Exercise

Graph $f(x)=2x-1$. Which of the following graphs is correct?

A. 

B. 

C. 

D. 

Answer

A. Incorrect.

Two good points to check are when $x = 0$ and when $y = 0$. In this case, that would be $(0, -1)$ and $\left(\frac{1}{2}, 0\right)$. The correct answer is Graph D.

B. Incorrect.

Two good points to check are when $x = 0$ and when $y = 0$. In this case, that would be $(0, -1)$ and $\left(\frac{1}{2}, 0\right)$. The correct answer is Graph D.

C. Incorrect.

Two good points to check are when $x = 0$ and when $y = 0$. In this case, that would be $(0, -1)$ and $\left(\frac{1}{2}, 0\right)$. The correct answer is Graph D.

D. Correct.

Two good points to check are when $x = 0$ and when $y = 0$. In this case, that would be $(0, -1)$ and $\left(\frac{1}{2}, 0\right)$. So, Graph D is the correct answer.

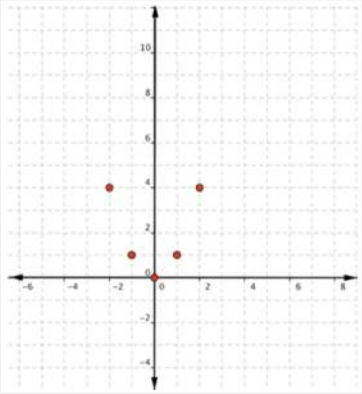
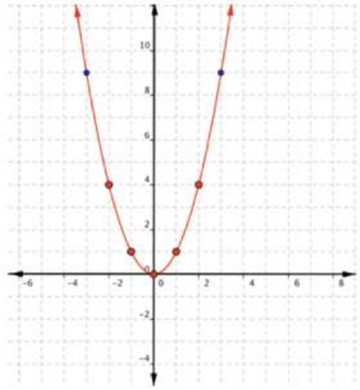
Quadratic Functions

Nonlinear functions can also be graphed. It’s helpful to have an idea what the shape should be, so you can be sure that you’ve chosen enough points to plot as a guide. Let’s start with the most basic quadratic function, $f(x)=x^2$.

Example

Graph $f(x)=x^2$.

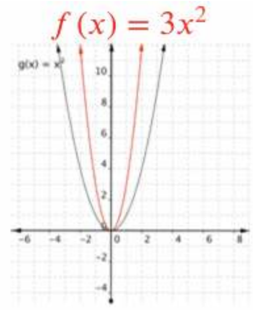
Solution

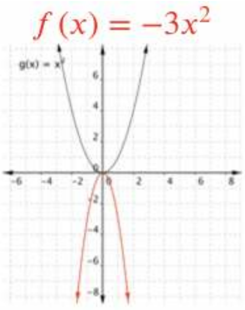
x : $f(x)$ -2 : 4 -1 : 1 0 : 0 1 : 1 2 : 4	Start with a table of values. Then think of the table as ordered pairs.
	Plot the points.
	Since the points are *not* on a line, you can’t use a straight edge. Connect the points as best you can, using a *smooth curve* (not a series of straight lines). You may want to find and plot additional points (such as the ones in blue here).

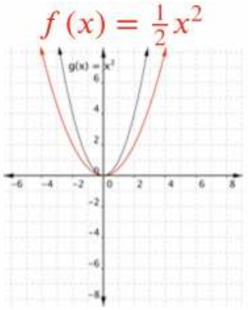
Notice that the shape is like the letter U. This is called a **parabola**. One-half of the parabola is a mirror image of the other half. The line that goes down the middle is called the **line of reflection**. In this case that line is the y-axis.

The equations for quadratic functions have the form $f(x)=ax^2+bx+c$ where $a\neq 0$. In the basic graph above, $a=1$, $b=0$, and $c=0$. Here are some graphs with different values for a, b, and c. Notice how each value changes the shape and location of the parabola.

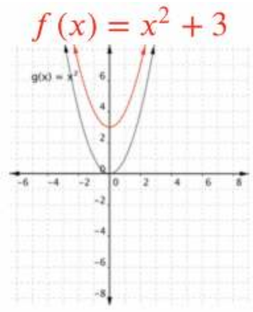
Changing a changes the width of the parabola and whether it opens up $(a>0)$ or down $(a<0)$.

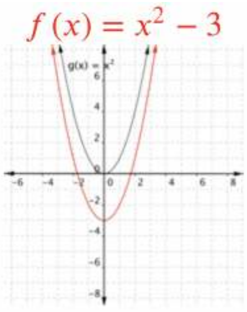


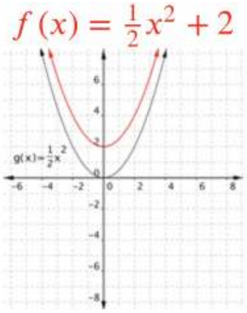




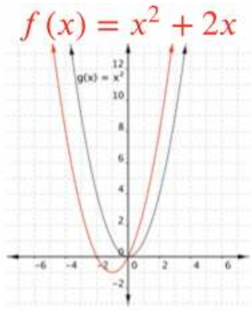
If there is no b term, changing c moves the parabola up or down so that the y intercept is $(0, c)$.

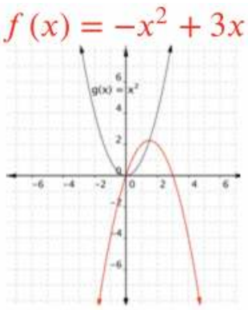


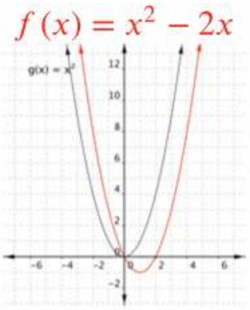




Changing b moves the line of reflection. The way it moves depends also on a.







Properties of a Parabola

For $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers.

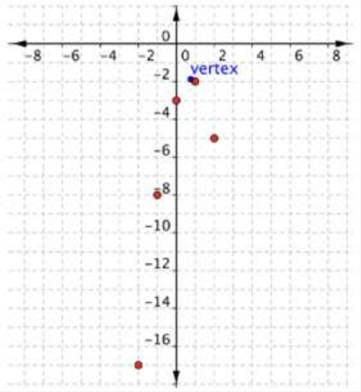
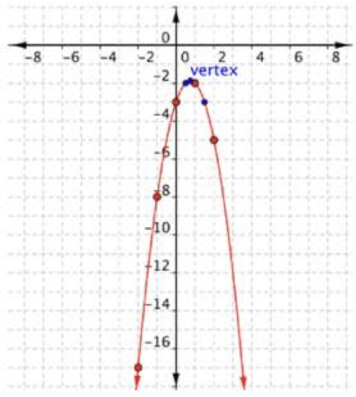
- The parabola opens upward if $a > 0$ and downward if $a < 0$.
- a changes the width of the parabola. The parabola gets narrower if $|a| > 1$ and wider if $|a| < 1$.
- The **vertex** depends on the values of a, b, and c. The vertex is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

Example

Graph $f(x) = -2x^2 + 3x - 3$**.**

Solution

$a = -2$, so the graph will open down and be thinner than $f(x) = x^2$. $c = -3$, so it will move to intercept the y-axis at $(0, -3)$.	Before making a table of values, look at the values of a and c to get a general idea of what the graph should look like.

<table>
<tr><td>$$\begin{array}{c|c} x & f(x) \\ \hline -2 & -17 \\ -1 & -8 \\ 0 & -3 \\ 1 & -2 \\ 2 & -5 \end{array}$$</td><td>Create a table of values. Notice that in this table, the x values increase. The y values increase and then start to decrease again. That indicates a parabola.</td></tr>
<tr><td>$\text{Vertex formula } =\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$
x-coordinate of vertex: $\frac{-b}{2a}=\frac{-(3)}{2(-2)}=\frac{-3}{-4}=\frac{3}{4}$
y-coordinate of vertex:
$$\begin{aligned} f\left(\frac{-b}{2a}\right) &= f\left(\frac{3}{4}\right) \\ f\left(\frac{3}{4}\right) &= -2\left(\frac{3}{4}\right)^2+3\left(\frac{3}{4}\right)-3 \\ &= -2\left(\frac{9}{16}\right)+\frac{9}{4}-3 \\ &= \frac{-18}{16}+\frac{9}{4}-3 \\ &= \frac{-9}{8}+\frac{18}{8}-\frac{24}{8} \\ &= -\frac{15}{8} \end{aligned}$$
Vertex: $\left(\frac{3}{4},-\frac{15}{8}\right)$</td><td>To find the vertex of the parabola, use the formula $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$. Finding the vertex may make graphing the parabola easier.</td></tr>
<tr><td> </td><td>Use the table pairs to plot points. Note the vertex, $\left(\frac{3}{4},-\frac{15}{8}\right)$, in blue.</td></tr>
<tr><td> </td><td>Connect the points as best you can, using a *smooth curve*. Remember that the parabola is two mirror images, so if your points don’t have pairs with the same value, you may want to include additional points (such as the ones in blue here). Plot points on either side of the vertex.
$x=\frac{1}{2}$ and $x=\frac{3}{2}$ are good values to include.</td></tr>
</table>

? Exercise

Would the graph for $f(x)=-3x^2+2x-1$ open up or down, and why?

A. Up, because a (the coefficient of x^2) is negative.
B. Down, because a (the coefficient of x^2) is negative.
C. Up, because b (the coefficient of x) is positive.
D. Down, because b (the coefficient of x) is positive.

Answer

A. Up, because a (the coefficient of x^2) is negative.

Incorrect. The coefficient of x^2 is the correct place to look, but a negative coefficient means the graph opens down. The correct answer is down, because a (the coefficient of x^2) is negative.

B. Down, because a (the coefficient of x^2) is negative.

Correct. A negative coefficient of x^2 means the graph opens down.

C. Up, because b (the coefficient of x) is positive.

Incorrect. Look at the coefficient for x^2, not the coefficient for x. The correct answer is down, because a (the coefficient of x^2) is negative.

D. Down, because b (the coefficient of x) is positive.

Incorrect. Look at the coefficient for x^2, not the coefficient for x. The correct answer is down, because a (the coefficient of x^2) is negative.

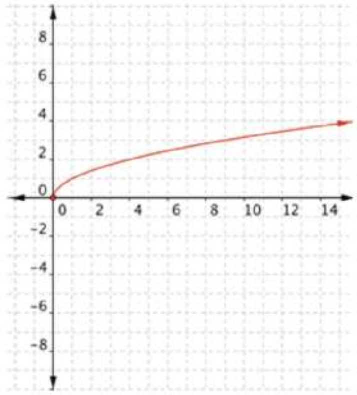
Radical Functions

You can also graph radical functions (such as square root functions) by choosing values for x and finding points that will be on the graph. Again, it's helpful to have some idea about what the graph will look like.

Think about the basic square root function, $f(x)=\sqrt{x}$. Let's take a look at a table of values for x and y and then graph the function. (Notice that all the values for x in the table are perfect squares. Since you are taking the square root of x, using perfect squares makes more sense than just finding the square roots of 0 ,1, 2 ,3 ,4, etc.)

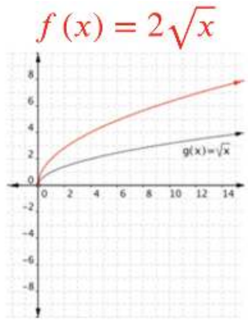
x	$f(x)$
0	0
1	1
4	2
9	3
16	4

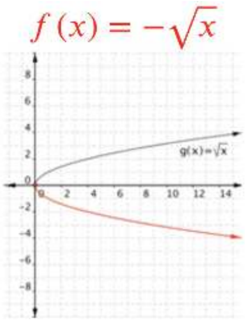
Recall that x can never be negative here because the square root of a negative number would be imaginary, and imaginary numbers cannot be graphed. There are also no values for x that will result in y being a negative number. Take a look at the graph.

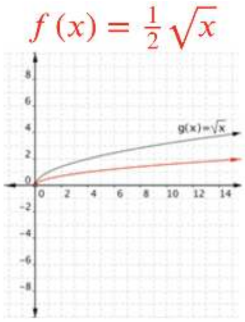


As with parabolas, multiplying and adding numbers makes some changes, but the basic shape is still the same. Here are some examples.

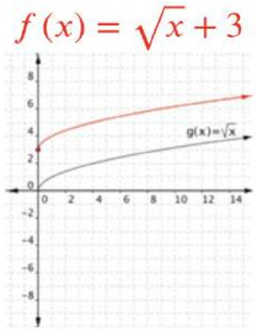
Multiplying $\sqrt{x}$ by a positive value changes the width of the half-parabola. Multiplying $\sqrt{x}$ by a negative number gives you the other half of a horizontal parabola.

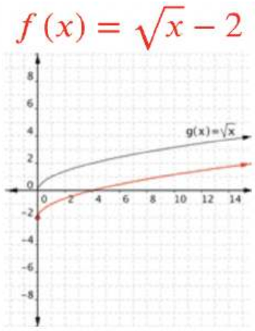




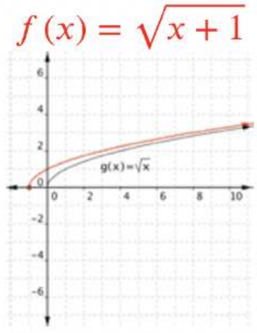


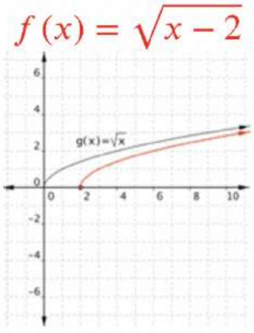
Adding a value *outside* the radical moves the graph up or down. Think about it as adding the value to the basic y value of $\sqrt{x}$, so a positive value added moves the graph up.





Adding a value *inside* the radical moves the graph left or right. Think about it as adding a value to x before you take the square root, so the y value gets moved to a different x value. For example, for $f(x) = \sqrt{x}$, the square root is 3 if $x = 9$. For $f(x) = \sqrt{x+1}$, the square root is 3 when $x+1$ is 9, so x is 8. Changing x to $x+1$ shifts the graph to the left by 1 unit (from 9 to 8). Changing x to $x-2$ shifts the graph to the right by 2 units.





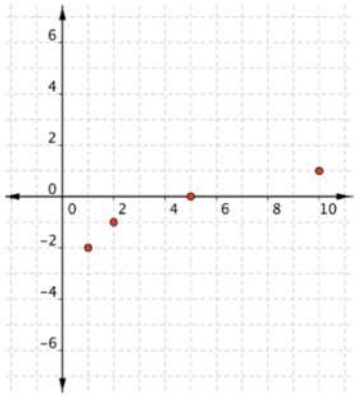
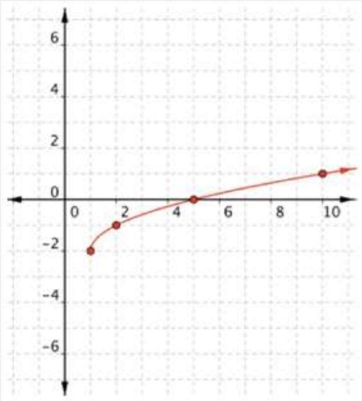
Notice that as x gets greater, adding or subtracting a number inside the square root has less of an effect on the value of y!

✓ Example

Graph $f(x) = -2 + \sqrt{x-1}$**.**

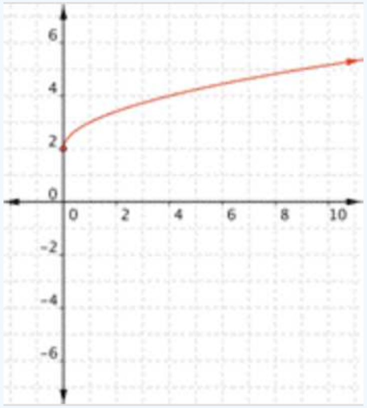
Solution

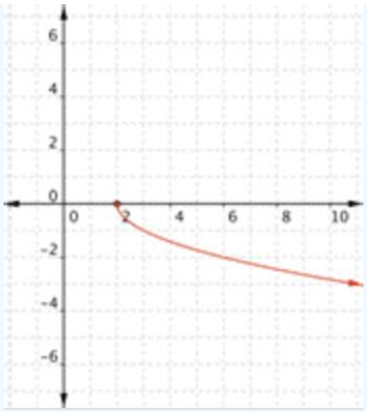
Inside the square root, you're subtracting 1, so the graph will move to the right 1 from the basic $f(x) = \sqrt{x}$ graph. You're also adding -2 outside the square root, so the graph will move down two from the basic $f(x) = \sqrt{x}$ graph.	Before making a table of values, look at the function equation to get a general idea what the graph should look like.

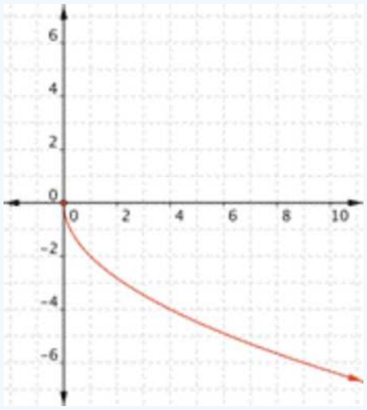
<table>
<tr>
<td>
<table>
<tr><th>x</th><th>$f(x)$</th></tr>
<tr><td>1</td><td>-2</td></tr>
<tr><td>2</td><td>-1</td></tr>
<tr><td>5</td><td>0</td></tr>
<tr><td>10</td><td>1</td></tr>
</table>
</td>
<td>Create a table of values. Choose values that will make your calculations easy. You want $x-1$ to be a perfect square, such as 0, 1, 4, 9, and so on, so you can take the square root.
Since values of x less than 1 makes the value inside the square root negative, there will be no points on the coordinate graph to the left of $x=1$.
There is no need to choose x values less than 1 for your table!</td>
</tr>
<tr>
<td>
</td>
<td>Use the table pairs to plot points.</td>
</tr>
<tr>
<td>
</td>
<td>Connect the points as best you can, using a *smooth curve*.</td>
</tr>
</table>

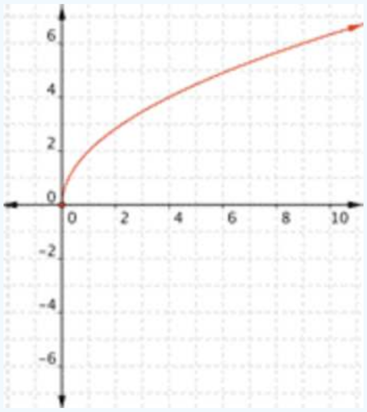
? Exercise

Which of the following is a graph of $f(x)=-2\sqrt{x}$?

A. 

B. 

C. 

D. 

Answer

A. Incorrect.

Recall that the graph of $f(x) = -\sqrt{x}$ is entirely under the x-axis, in Quadrant IV. The graph of $f(x) = -2\sqrt{x}$ will be similar but wider. The correct answer is Graph C.

B. Incorrect.

In the function $f(x) = -2\sqrt{x}$, $f(0) = 0$. This point is not represented on this graph. Find a graph that looks similar to Graph B, but which goes through the point $(0, 0)$. The correct answer is Graph C.

C. Correct.

The graph of $f(x) = -2\sqrt{x}$ will be similar to the graph of $f(x) = -\sqrt{x}$, although it will be a bit wider.

D. Incorrect.

This is the graph of $f(x) = 2\sqrt{x}$. The graph of $f(x) = -2\sqrt{x}$ will be below the x-axis. The correct answer is Graph C.

Summary

Creating a graph of a function is one way to understand the relationship between the inputs and outputs of that function. Creating a graph can be done by choosing values for x, finding the corresponding y values, and plotting them. However, it helps to understand the basic shape of the function. Knowing the effect of changes to the basic function equation is also helpful.

One common shape you will see is a parabola. Parabolas have the equation $f(x)=ax^2+bx+c$, where a, b, and c are real numbers and $a\neq 0$. The value of a determines the width and the direction of the parabola, while the vertex depends on the values of a, b, and c. The vertex is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

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17.2.3: Finding Domain and Range

Learning Objectives

- Find the domain of a square root function.
- Find the domain and range of a function from the algebraic form.

Introduction

Functions are a correspondence between two sets, called the **domain** and the **range**. When defining a function, you usually state what kind of numbers the domain (x) and range $(f(x))$ values can be. But even if you say they are real numbers, that doesn't mean that *all* real numbers can be used for x. It also doesn't mean that all real numbers can be function values, $f(x)$. There may be restrictions on the domain and range. The restrictions partly depend on the *type* of function.

In this topic, all functions will be restricted to real number values. That is, only real numbers can be used in the domain, and only real numbers can be in the range.

Restricting the Domain

There are two main reasons why domains are restricted.

- You can't divide by 0.
- You can't take the square root (or other even root) of a negative number, as the result will not be a real number.

In what kind of functions would these two issues occur?

Division by 0 could happen whenever the function has a variable in the *denominator* of a rational expression. That is, it's something to look for in rational functions. Look at these examples, and note that "division by 0" doesn't necessarily mean that x is 0!

Function	**Notes**
$f(x)=\frac{1}{x}$	If $x=0$, you would be dividing by 0, so $x\neq 0$.
$f(x)=\frac{2+x}{x-3}$	If $x=3$, you would be dividing by 0, so $x\neq 3$.
$f(x)=\frac{2(x-1)}{x-1}$	Although you can simplify this function to $f(x)=2$, when $x=1$ the original function would include division by 0. So $x\neq 1$.
$f(x)=\frac{x+1}{x^2-1}$	Both $x=1$ and $x=-1$ would make the denominator 0. Again, this function can be simplified to $f(x)=\frac{1}{x-1}$, but when $x=1$ or $x=-1$ the *original* function would include division by 0, so $x\neq 1$ and $x\neq -1$.
$f(x)=\frac{2(x-1)}{x^2+1}$	This is an example with no domain restrictions, even though there is a variable in the denominator. Since $x^2\geq 0$, x^2+1 can never be 0. The least it can be is 1, so there is no danger of division by 0.

Square roots of negative numbers could happen whenever the function has a variable under a radical with an even root. Look at these examples, and note that "square root of a negative variable" doesn't necessarily mean that the value under the radical sign is negative! For example, if $x=-4$, then $-x=-(-4)=4$, a positive number.

Function	**Restrictions to the Domain**
$f(x)=\sqrt{x}$	If $x<0$, you would be taking the square root of a negative number, so $x\geq 0$.
$f(x)=\sqrt{x+10}$	If $x<-10$, you would be taking the square root of a negative number, so $x\geq -10$.

$f(x)=\sqrt{-x}$	When is $-x$ negative? Only when x is positive. (For example, if $x=-3$, then $-x=3$. If $x=1$, then $-x=-1$.) This means $x \leq 0$.
$f(x)=\sqrt{x^2-1}$	x^2-1 must be positive, $x^2-1>0$. So $x^2>1$. This happens only when x is greater than 1 or less than -1: $x \leq -1$ or $x \geq 1$.
$f(x)=\sqrt{x^2+10}$	There are no domain restrictions, even though there is a variable under the radical. Since $x^2 \geq 0$, x^2+10 can never be negative. The least it can be is 10, so there is no danger of taking the square root of a negative number.

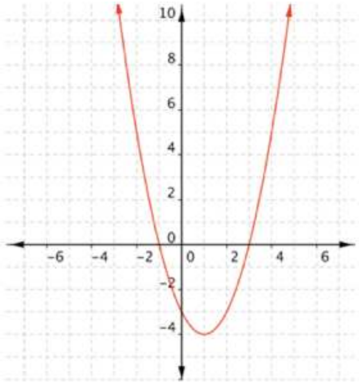
Domains can be restricted if:

- the function is a rational function and the denominator is 0 for some value or values of x
- the function is a radical function with an even index (such as a square root), and the radicand can be negative for some value or values of x.

Range

Remember, here the range is restricted to all real numbers. The range is also determined by the function and the domain. Consider these graphs, and think about what values of y are possible, and what values (if any) are not. In each case, the functions are real-valued: that is, x and $f(x)$ can only be real numbers.

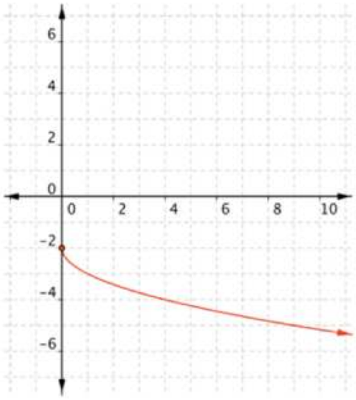
Quadratic function, $f(x)=x^2-2x-3$



Remember the basic quadratic function: $f(x)=x^2$ must always be positive, so $f(x) \geq 0$ in that case. In general, quadratic functions always have a point with a maximum or greatest value (if it opens down) or a minimum or least value (it if opens up, like the one above). That means the range of a quadratic function will always be restricted to being above the minimum value or below the maximum value. For the function above, the range is $f(x) \geq -4$.

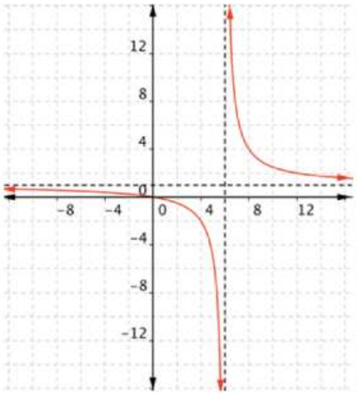
Other polynomial functions with even degrees will have similar range restrictions. Polynomial functions with *odd* degrees, like $f(x)=x^3$, will not have restrictions.

Radical function, $f(x)=-\sqrt{x}-2$



Square root functions look like half of a parabola, turned on its side. The fact that the square root portion must always be positive restricts the range of the basic function, $f(x)=\sqrt{x}$, to only positive values. Changes to that function, such as the negative in front of the radical or the subtraction of 2, can change the range. The range of the function above is $f(x) \leq -2$.

Rational function, $f(x)=\frac{x}{x-6}$



Rational functions may seem tricky. There is nothing in the function that obviously restricts the range. However, rational functions have **asymptotes**: lines that the graph will get *close* to, but never cross or even touch. In the graph above, the domain restriction provides one asymptote, $x=6$. The other is the line $y=1$, which provides a restriction to the range. In this case, there are no values of x for which $f(x)=1$. So, the range for this function is all real numbers except 1.

Determining Domain and Range

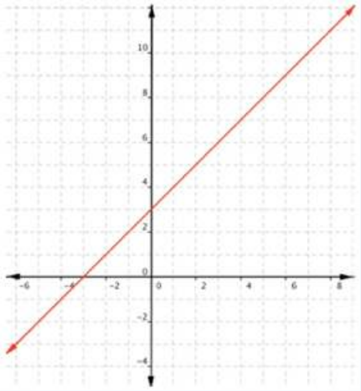
Finding the domain and range of different functions is often a matter of asking yourself, what values can this function *not* have?

✓ Example

What are the domain and range of the real-valued function $f(x)=x+3$?

Solution

This is a *linear* function. Remember that linear functions are lines that continue forever in each direction.



Any real number can be substituted for x and get a meaningful output. For *any* real number, you can always find an x value that gives you that number for the output. Unless a linear function is a constant, such as $f(x) = 2$, there is no restriction on the range.

The domain and range are all real numbers.

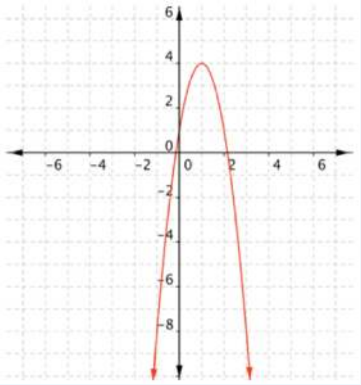
✓ Example

What are the domain and range of the real-valued function $f(x) = -3x^2 + 6x + 1$?

Solution

This is a *quadratic* function. There are no rational or radical expressions, so there is nothing that will restrict the domain. Any real number can be used for x to get a meaningful output.

Because the coefficient of x^2 is negative, it will open downward. With quadratic functions, remember that there is either a maximum (greatest) value, or a minimum (least) value. In this case, there is a maximum value.



The **vertex**, or turning point, is at $(1, 4)$. From the graph, you can see that $f(x) \leq 4$.

The domain is all real numbers, and the range is all real numbers $f(x)$ such that $f(x) \leq 4$.

You can check that the vertex is indeed at $(1, 4)$. Since a quadratic function has two mirror image halves, the line of reflection has to be in the middle of two points with the same y value. The vertex must lie on the line of reflection, because it's the only point that does not have a mirror image!

In the previous example, notice that when $x = 2$ and when $x = 0$, the function value is 1. (You can verify this by evaluating $f(2)$ and $f(0)$.) That is, both $(2, 1)$ and $(0, 1)$ are on the graph. The line of reflection here is $x = 1$, so the vertex must be at the point $(1,\ f(1))$. Evaluating $f(1)$ gives $f(1) = 4$, so the vertex is at $(1, 4)$.

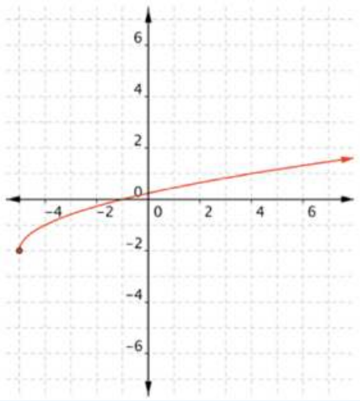
✓ Example

What are the domain and range of the real-valued function $f(x)=-2+\sqrt{x+5}$?

Solution

This is a *radical* function. The domain of a radical function is any x value for which the radicand (the value under the radical sign) is not negative. That means $x+5 \geq 0$, so $x \geq -5$.

Since the square root must always be positive or 0, $\sqrt{x+5} \geq 0$. That means $-2+\sqrt{x+5} \geq -2$.



The domain is all real numbers x where $x \geq -5$, and the range is all real numbers $f(x)$ such that $f(x) \geq -2$.

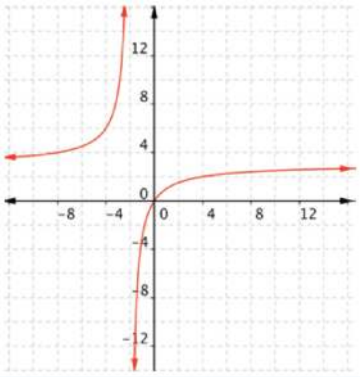
✓ Example

What are the domain and range of the real-valued function $f(x)=\frac{3x}{x+2}$?

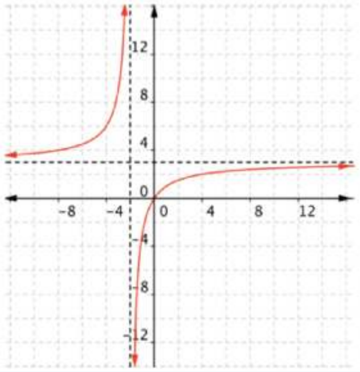
Solution

This is a *rational* function. The domain of a rational function is restricted where the denominator is 0. In this case, $x+2$ is the denominator, and this is 0 only when $x=-2$.

For the range, create a graph using a graphing utility and look for asymptotes:



One asymptote, a vertical asymptote, is at $x=-2$, as you should expect from the domain restriction. The other, a horizontal asymptote, appears to be around $y=3$. (In fact, it is $y=3$.)



The domain is all real numbers except -2, and the range is all real numbers except 3.

You can check the horizontal asymptote, $y=3$. Is it possible for $\frac{3x}{x+2}$ to be equal to 3? Write an equation and try to solve it.

$$\begin{aligned}\frac{3x}{x+2}&=3\\(x+2)\frac{3x}{x+2}&=(x+2)3\\3x&=3x+6\\0&=6\end{aligned}$$

Since the attempt to solve ends with a false statement, 0 cannot be equal to 6, the equation has no solution. There is no value of x for which $\frac{3x}{x+2}=3$, so this proves that the range is restricted.

? Exercise

Find the domain and range of the real-valued function $f(x)=x^2+7$.

A. The domain is all real numbers and the range is all real numbers $f(x)$ such that $f(x)\geq 7$.
B. The domain is all real numbers x such that $x\geq 0$ and the range is all real numbers $f(x)$ such that $f(x)\geq 7$.
C. The domain is all real numbers x such that $x\geq 0$ and the range is all real numbers.
D. The domain and range are all real numbers.

Answer

A. The domain is all real numbers and the range is all real numbers $f(x)$ such that $f(x)\geq 7$.

Correct. Quadratic functions have no domain restrictions. Since $x^2\geq 0$, $x^2+7\geq 7$.

B. The domain is all real numbers x such that $x\geq 0$ and the range is all real numbers $f(x)$ such that $f(x)\geq 7$.

Incorrect. Negative values can be used for x. The correct answer is: The domain is all real numbers and the range is all real numbers $f(x)$ such that $f(x)\geq 7$.

C. The domain is all real numbers x such that $x\geq 0$ and the range is all real numbers.

Incorrect. Negative values can be used for x, but the range is restricted because $x^2\geq 0$. The correct answer is: The domain is all real numbers and the range is all real numbers $f(x)$ such that $f(x)\geq 7$.

D. The domain and range are all real numbers.

Incorrect. While it's true that quadratic functions have no domain restrictions, the range is restricted because $x^2\geq 0$. The correct answer is: The domain is all real numbers and the range is all real numbers $f(x)$ such that $f(x)\geq 7$.

Summary

Although a function may be given as "real valued," it may be that the function has restrictions to its domain and range. There may be some real numbers that can't be part of the domain or part of the range. This is particularly true with rational and radical functions, which can have restrictions to domain, range, or both. Other functions, such as quadratic functions and polynomial functions of even degree, also can have restrictions to their range.

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SECTION OVERVIEW

17.3: Operations with Functions

Topic hierarchy

17.3.1: Arithmetic Operations with Functions

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17.3.1: Arithmetic Operations with Functions

Learning Objectives

- Given two functions, f and g, find their sum, $f+g$.
- Given two functions, f and g, find their difference, $f-g$.
- Given two functions, f and g, find their product, fg.
- Given two functions, f and g, find their quotient, $\frac{f}{g}$.

Introduction

You are used to adding, subtracting, multiplying, and dividing real numbers. You do these operations every day in a variety of situations. You have also learned how to perform these four basic operations on algebraic expressions. So while you may not need to calculate $30x^2+10x$ too often, you do know how to do it.

If you know how to perform the four basic operations on polynomials, then you can also add, subtract, multiply, and divide functions. The notation will look different at first, but knowing a couple of steps can help you arrive at the correct answer.

Understanding Notation

A **function** is a correspondence between two sets: the domain and the range. In addition to evaluating functions, you can do operations with functions.

Let's say you are working with the following two functions.

$$\begin{array}{l} f(x)=9x-5 \\ g(x)=4x+1 \end{array}$$

The sum of these functions can be written $f(x)+g(x)$ or as $(f+g)(x)$. Watch what happens when these two functions are added.

$$\begin{aligned} f(x)&=9x-5 \\ g(x)&=4x+1 \\ (f+g)(x)&=f(x)+g(x) \\ (f+g)(x)&=(9x-5)+(4x+1) \\ (f+g)(x)&=9x+4x-5+1 \\ (f+g)(x)&=13x-4 \end{aligned}$$

That's it! The sum of the two functions is the sum of the two polynomials.

Addition, subtraction, multiplication, and division will all be explained in turn. The table below shows the notation that is used for each type of arithmetic operation.

Addition	$f(x)+g(x)$	$(f+g)(x)$
Subtraction	$f(x)-g(x)$	$(f-g)(x)$
Multiplication	$f(x)\cdot g(x)$	$(f\cdot g)(x)$
Division	$\frac{f(x)}{g(x)}$	$\left(\frac{f}{g}\right)(x)$

Adding and Subtracting

You have already seen one example of adding two functions. Let's look at another one. The domain (x-values) for both functions is all real numbers.

✓ Example

$$\begin{array}{l} f(x)=5x+6 \\ g(x)=3x^2-4x+8 \end{array}$$

Find $(f+g)(x)$.

Solution

$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ &= (5x+6)+\left(3x^2-4x+8\right)\\ &= 3x^2+5x-4x+6+8\\ &= 3x^2+x+14\end{aligned}$	Identify $f(x)$ and $g(x)$. Replace $f(x)$ with $5x+6$, and $g(x)$ with $3x^2-4x+8$. Then add and combine like terms.

$(f+g)(x)=3x^2+x+14$

Subtracting follows the same process. As long as you remember how to subtract one polynomial from another, you can figure out how to subtract one function from another.

✓ Example

$$\begin{aligned}f(x)&=5x+6\\ g(x)&=3x^2-4x+8\end{aligned}$$

Find $(g-f)(x)$.

Solution

$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ &= \left(3x^2-4x+8\right)-(5x+6)\\ &= 3x^2-4x+8-5x-6\\ &= 3x^2-4x-5x+8-6\\ &= 3x^2-9x+2\end{aligned}$	Replace $g(x)$ and $f(x)$ with their respective expressions. Then subtract and combine like terms.

$(g-f)(x)=3x^2-9x+2$

✓ Example

$$\begin{aligned}f(x)&=5x^2+2x-5\\ g(x)&=7x+8\\ h(x)&=4x^2-10\end{aligned}$$

Find $(f-h)(x)$.

Solution

$\begin{aligned}(f-h)(x) &= f(x)-h(x)\\ &= \left(5x^2+2x-5\right)-\left(4x^2-10\right)\\ &= 5x^2+2x-5-4x^2+10\\ &= 5x^2-4x^2+2x-5+10\\ &= x^2+2x+5\end{aligned}$	Notice: $(f-h)(x)=f(x)-h(x)$ You can ignore $g(x)$ since it is not required to solve this problem. Replace the function notations with their appropriate polynomials and subtract.

$(f-h)(x)=x^2+2x+5$

? Exercise

$f(x)=9x^3+2$ and $g(x)=x^3-4x^2-3$. What is $(f-g)(x)$?

A. $10x^3-4x^2-1$
B. $8x^3-4x^2-1$
C. $8x^3+4x^2+5$
D. $-8x^3-4x^2-5$

Answer

A. $10x^3-4x^2-1$

Incorrect. $(f+g)(x)=10^3-4x^2-1$; this question is looking for $(f-g)(x)$. The correct answer is $8x^3+4x^2+5$.

B. $8x^3-4x^2-1$

Incorrect. It looks like you tried to calculate $(f-g)(x)$, but you subtracted incorrectly. Remember: $(f-g)(x) = (9x^3+2)-(x^3-4x^2-3) = 9x^3+2-x^3+4x^2+3$. The correct answer is $8x^3+4x^2+5$.

C. $8x^3+4x^2+5$

Correct. To find $(f-g)(x)$, subtract $g(x)$ from $f(x)$. $(f-g)(x) = (9x^3+2)-(x^3-4x^2-3) = 8x^3+4x^2+5$.

D. $-8x^3-4x^2-5$

Incorrect. It looks like you tried to calculate $(g-f)(x)$. This question is looking for $(f-g)(x)$. The correct answer is $8x^3+4x^2+5$.

Multiplying and Dividing

Multiplying and dividing functions is also just like multiplying and dividing polynomials. Review the following examples.

Example

$f(x) = 2x+1$
$g(x) = 5x-3$

Find the product of f and g.

Solution

$(f\cdot g)(x) = f(x)\cdot g(x)$	To find the product, multiply the functions.
$(f\cdot g)(x) = (2x+1)(5x-3)$ $= 10x^2-6x+5x-3$ $= 10x^2-x-3$	Replace $f(x)$ with $(2x+1)$, and $g(x)$ with $(5x-3)$.

$(f\cdot g)(x) = 10x^2-x-3$

Example

$f(x) = 12x^3+15x^2-6x$
$g(x) = 3x$

Find $\left(\frac{f}{g}\right)(x)$.

Solution

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $= \frac{12x^3+15x^2-6x}{3x}, x\neq 0$ $= \frac{3x\left(4x^2+5x-2\right)}{3x}$ $= 1\cdot\left(4x^2+5x-2\right)$ $= 4x^2+5x-2, x\neq 0$	To find the quotient, divide f by g. Substitute the polynomials in for $f(x)$ and $g(x)$ and divide. We add $x\neq 0$ because $x=0$ would make the denominator $g(x)=0$ and $\frac{f(x)}{g(x)}$ undefined. Remember to rename $\frac{3x}{3x}$ as 1.

$\left(\frac{f}{g}\right)(x) = 4x^2+5x-2$

Operations with three functions work the same way. In the example below, two functions are added and then divided by a third. It is no different than what you have already done with polynomials; just continue to substitute the polynomials in for the correct functions, combine, divide, and simplify.

Example

$f(x) = 8x^3-3x^2$
$g(x) = 4x^3+9x^2$
$h(x) = 3x^2$

Find $\left(\frac{f+g}{h}\right)(x)$.

Solution

$\left(\frac{f+g}{h}\right)(x)=\frac{f(x)+g(x)}{h(x)}$ $=\frac{(8x^3-3x^2)+(4x^3+9x^2)}{3x^2}, x\neq 0$	Replace $f(x)$, $g(x)$, and $h(x)$ with the equivalent polynomials.e add $x\neq 0$ because that would make the denominator $h(x)$ of $\frac{f(x)+g(x)}{h(x)}$ zero and the fraction undefined.
$\left(\frac{f+g}{h}\right)(x)=\frac{3x^2(4x+2)}{3x^2}$ $=1\cdot(4x+2)$ $=4x+2, x\neq 0$	Add $f(x)$ and $g(x)$. Divide by $h(x)$. Pull out a factor of $3x^2$ from the numerator, and then simplify the expression, using $\frac{3x^2}{3x^2}$ $=1$.

$\left(\frac{f+g}{h}\right)(x)=4x+2, x\neq 0$

? Exercise

$f(x)=9x^2$
$g(x)=-4x$
$h(x)=-10x^3$

Find $(f\cdot h)(x)$.

A. $-90x^5$
B. $-36x^3$
C. $40x^4$
D. $360x^6$

Answer

A. $-90x^5$

Correct. $(f\cdot h)(x)=9x^2\cdot(-10x^3)=90x^5$.

B. $-36x^3$

Incorrect. It looks like you found $(f\cdot g)(x)$; this problem is looking for $(f\cdot h)(x)$. The correct answer is $-90x^5$.

C. $40x^4$

Incorrect. It looks like you found $(g\cdot h)(x)$; this problem is looking for $(f\cdot h)(x)$. The correct answer is $-90x^5$.

D. $360x^6$

Incorrect. It looks like you found $(f\cdot g\cdot h)(x)$; this problem is looking for $(f\cdot h)(x)$. The correct answer is $-90x^5$.

Summary

Just like integers and algebraic expressions, functions can be added, subtracted, multiplied, and divided. To perform an arithmetic operation upon two or more functions, replace the indicated function with its respective polynomial, then combine using the regular rules of addition, subtraction, multiplication, and division.

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CHAPTER OVERVIEW

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18.1: Exponential Functions

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18.1.1: Introduction to Exponential Functions

Learning Objectives

- Graph exponential equations and functions.
- Solve applied problems using exponential problems and their graphs.

Introduction

In addition to linear, quadratic, rational, and radical functions, there are **exponential functions**. Exponential functions have the form $f(x)=b^x$, where $b>0$ and $b\neq 1$. Just as in any exponential expression, b is called the **base** and x is called the **exponent**.

An example of an exponential function is the growth of bacteria. Some bacteria double every hour. If you start with 1 bacterium and it doubles every hour, you will have 2^x bacteria after x hours. This can be written as $f(x)=2^x$.

Before you start, $f(0)=2^0=1$

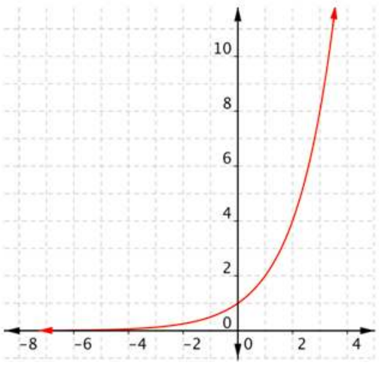
After 1 hour, $f(1)=2^1=2$

In 2 hours, $f(2)=2^2=4$

In 3 hours, $f(3)=2^3=8$

and so on.

With the definition $f(x)=b^x$ and the restrictions that $b>0$ and that (b \neq 1\), the domain of an exponential function is the set of all real numbers. The range is the set of all positive real numbers. The following graph shows $f(x)=2^x$.



Exponential Growth

As you can see above, this exponential function has a graph that gets very close to the x-axis as the graph extends to the left (as x becomes more negative), but never really touches the x-axis. Knowing the general shape of the graphs of exponential functions is helpful for graphing specific exponential equations or functions.

Making a table of values is also helpful, because you can use the table to place the curve of the graph more accurately. One thing to remember is that if a base has a negative exponent, then take the reciprocal of the base to make the exponent positive. For example, $4^{-2}=\left(\frac{4}{1}\right)^{-2}=\left(\frac{1}{4}\right)^2$.

Example

Problem: Make a table of values for $f(x)=3^x$.

Answer

Make a “T” to start the table with two columns. Label the columns x and $f(x)$.

x	$f(x)$

Choose several values for x and put them as separate rows in the x column.

Tip: It’s always good to include 0, positive values, and negative values, if you can.

x	$f(x)$
-2	
-1	
0	
1	
2	

Evaluate the function for each value of x, and write the result in the $f(x)$ column next to the x value you used. For example, when $x=-2, f(x)=3^2=\left(\frac{1}{3}\right)^2=\frac{1}{9}$, so $\frac{1}{9}$ goes in the $f(x)$ column next to -2 in the x column. $f(1)=3^1=3$, so 3 goes in the $f(x)$ column next to 1 in the x column.

x	$f(x)$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9

Note that your table of values may be different from someone else’s, if you chose different numbers for x.

Look at the table of values. Think about what happens as the x values increase: so do the function values ($f(x)$ or y)!

Now that you have a table of values, you can use these values to help you draw both the shape and location of the function. Connect the points as best you can to make a smooth curve (not a series of straight lines). This shows that all of the points on the curve are part of this function.

✓ Example

Problem: Graph $f(x)=3^x$.

Answer

Start with a table of values, like the one in the example above.

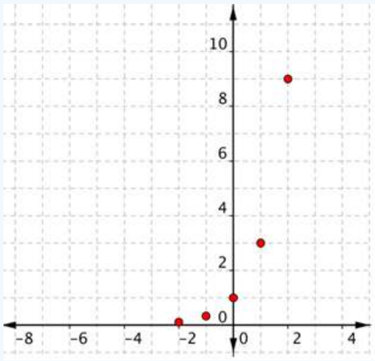
x	$f(x)$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9

If you think of $f(x)$ as y, each row forms an ordered pair that you can plot on a coordinate grid.

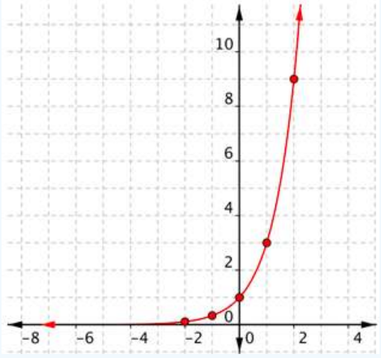
x	$f(x)$	point
-2	$\frac{1}{9}$	(-2, $\frac{1}{9}$)

-1	$\frac{1}{3}$	(-1, $\frac{1}{3}$)
0	1	(0, 1)
1	3	(1, 3)
2	9	(2, 9)

Plot the points.



Connect the points as best you can, using a *smooth curve* (not a series of straight lines). Use the shape of an exponential graph to help you: this graph gets very close to the x-axis on the left, but never really touches the x-axis, and gets steeper and steeper on the right.



This is an example of **exponential growth**. As x increases, $f(x)$ “grows” more quickly. Let’s try another one.

✓ Example

Problem: Graph $f(x) = 4^x$.

Answer

Start with a table of values. You can choose different values, but once again, it’s helpful to include 0, some positive values, and some negative values.

Remember, $4^{-2} = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$.

x	$f(x)$
-2	$\frac{1}{16}$
-1	$\frac{1}{4}$
0	1
1	4

2	16

If you think of $f(x)$ as y each row forms an ordered pair that you can plot on a coordinate grid.

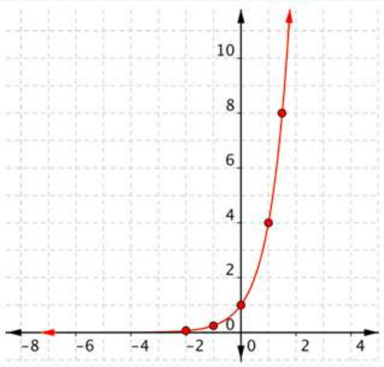
Plot the points.

Notice that the larger base in this problem made the function value skyrocket. Even with x as small as 2, the function value is too large for the axis scale you used before. You can change the scale, but then our other values are very close together. You could also try other points, such as when $x=\frac{3}{2}$. Because you know the square root of 4, you can find that value in this case: $4^{\frac{3}{2}}=(\sqrt{4})^3=2^3=8$

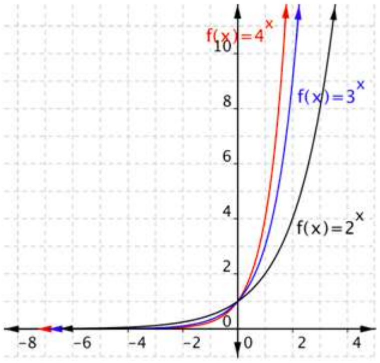
The point ($\frac{3}{2}$, 8) is the blue point on this graph.

For other bases, you might need to use a calculator to help you find the function value.

Connect the points as best you can, using a *smooth curve* (not a series of straight lines). Use the shape of an exponential graph to help you: this graph gets very close to the x-axis on the left, but never really touches the x-axis, and gets steeper and steeper on the right.



Let's compare the three graphs you've seen. The functions $f(x)=2^x$, $f(x)=3^x$, and $f(x)=4^x$ are all graphed below.



Notice that a larger base makes the graph steeper. A larger base also makes the graph closer to the y-axis for $x>0$ and closer to the x-axis for $x<0$. All the graphs go through (0, 1)!

Exponential Decay

Remember that for exponential functions, $b>0$, but $b\neq 1$. In the examples above, $b>1$. What happens when b is between 0 and 1, $0<b<1$?

✓ Example

Problem: Graph $f(x)=\left(\frac{1}{2}\right)^x$.

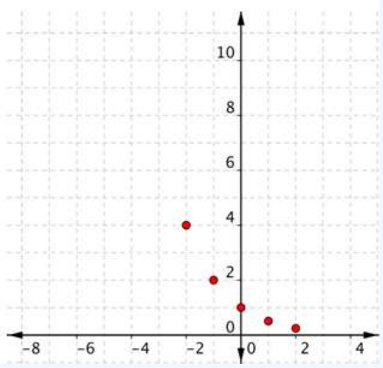
Answer

Start with a table of values.

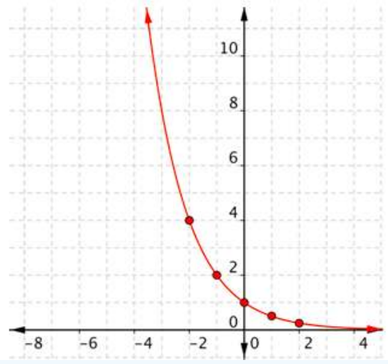
x	$f(x)$
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

Be careful with the negative exponents! Remember to take the reciprocal of the base to make the exponent positive. In this case, $\left(\frac{1}{2}\right)^{-2} = \left(\frac{2}{1}\right)^{2} = 2^2 = 4$, and $\left(\frac{1}{2}\right)^{2} = \frac{1}{4}$.

Use the table as ordered pairs and plot the points.



Since the points are *not* on a line, you can’t use a straightedge. Connect the points as best you can using a *smooth curve* (not a series of straight lines).



Notice that the shape is similar to the shape when $b > 1$, but this time the graph gets closer to the x-axis when $x > 0$, rather than when $x < 0$. This is **exponential decay**. Instead of the function values “growing” as values increase, as they did before, the function values “decay” or decrease as x values increase. They get closer and closer to 0.

✓ Example

Problem: **Graph** $f(x) = \left(\frac{1}{4}\right)^{x}$

Answer

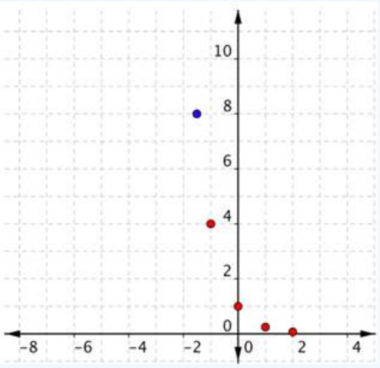
Create a table of values. Again, be careful with the negative exponents. Remember to take the reciprocal of the base to make the exponent positive. $\left(\frac{1}{4}\right)^{-2} = \left(\frac{4}{1}\right)^{2} = 4^2 = 16$.

x	$f(x)$

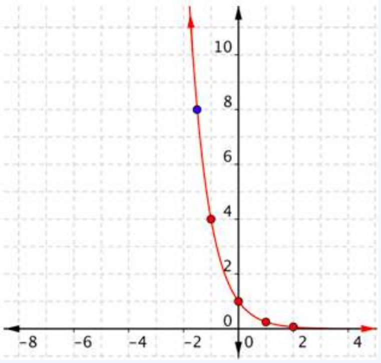
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

Notice that in this table, the x values increase. The y values decrease.

Use the table pairs to plot points. You may want to include new points, especially when one of the points from the table, here (-2, 16), won't fit on your graph. Because you know the square root of 4, try $x = -\frac{3}{2}$. You can find that value in this case: $\left(\frac{1}{4}\right)^{-\frac{3}{2}} = \left(\frac{4}{1}\right)^{\frac{3}{2}} = 4^{\frac{3}{2}} = (\sqrt{4})^3 = 2^3 = 8$. The point (-$frac32$, 8) has been included in blue. You may feel it necessary to include additional points. You also may need to use a calculator, depending on the base.



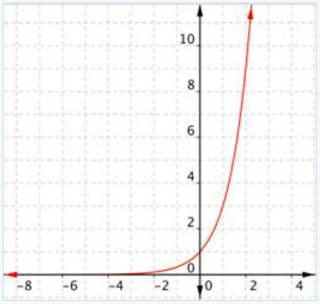
Connect the points as best you can, using a *smooth curve*.

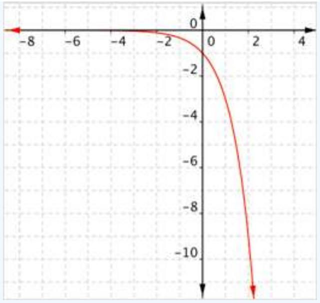


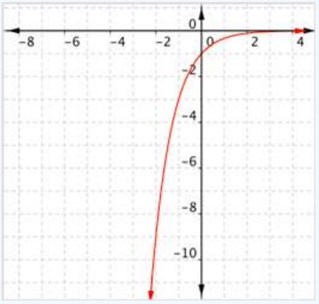
? Exercise

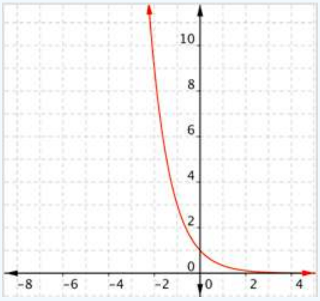
Which of the following is a graph for $f(x) = \left(\frac{1}{3}\right)^x$?

A.



B. 

C. 

D. 

Answer

A. Incorrect. This graph is increasing, because the $f(x)$ or y values increase as the x values increase. (Compare the values for $x = 1$ and $x = 2$.) This graph shows exponential growth, with a base greater than 1. The correct answer is Graph D.
B. Incorrect. This graph is decreasing, but all the function values are negative. The range for an exponential function is always positive values. The correct answer is Graph D.
C. Incorrect. This graph is increasing, but all the function values are negative. The correct graph must be decreasing with positive function values. The correct answer is Graph D.
D. Correct. All the function values are positive, and the graph is decreasing (showing exponential decay).

Applying Exponential Functions

Exponential functions can be used in many contexts, such as compound interest (money), population growth, and radioactive decay. In most of these, however, the function is not exactly of the form $f(x) = b^x$. Often, this is adjusted by adding or multiplying constants.

For example, the compound interest formula is $A = P\left(1 + \frac{r}{m}\right)^{mt}$, where P is the **principal** (the initial investment that is gathering interest) and A is the amount of money you would have, with interest, at the end of t years, using an annual interest rate of r (expressed as a decimal) and m compounding periods per year. In this case, the base is the value represented by the expression $1 + \frac{r}{m}$ and the exponent is mt, a product of two values.

Example

Problem: If you invest $1,000 in an account paying 4% interest, compounded quarterly, how much money will you have after 3 years?

Answer

First identify which of A, P, r, m, and t is being asked for, then determine values for the remaining variables.

The principal is \$1,000. The rate is 4% = 0.04. The time in years is 3. Compounded quarterly means 4 times a year.

The money you will have after 3 years will be A.

$$\begin{aligned} P &= \$1000 \\ r &= 0.04 \\ m &= 4 \\ t &= 3 \end{aligned}$$

To find the amount, use the formula.

$$\begin{aligned} A &= P\left(1+\frac{r}{m}\right)^{mt} \\ &= 1,000 \cdot \left(1+\frac{0.04}{4}\right)^{4 \cdot 3} \\ &= 1,000 \cdot (1+0.01)^{12} \\ &= 1,000 \cdot (1.01)^{12} \\ &= 1,000 \cdot 1.12682503 \\ &= 1,126.82503 \end{aligned}$$

Round the number to the nearest cent (hundredth). Notice that this means the amount of interest earned after three years is \$126.83. (\$1,126.83, minus the principal, \$1,000).

You will have \$1,126.83 after 3 years.

Radioactive decay is an example of exponential decay. Radioactive elements have a **half-life**. This is the amount of time it takes for half of a mass of the element to decay into another substance. For example, uranium-238 is a slowly decaying radioactive element with a half-life of about 4.47 *billion* years. That means it will take that long for 100 grams of uranium-238 to turn into 50 grams of uranium-238 (the other 50 grams will have turned into another element). That's a long time! On the other extreme, radon-220 has a half-life of about 56 seconds. What does this mean? 100 grams of radon-220 will turn into 50 grams of radon-220 and 50 grams of something else in less than a minute!

Since the amount is halved each half-life, an exponential function can be used to describe the amount remaining over time. The formula $R = A\left(\frac{1}{2}\right)^{\frac{t}{h}}$ gives the remaining amount R from an initial amount A, where h is the half-life of the element and t is the amount of time passed (using the same time unit as the half-life).

✓ Example

Problem: Cesium-137 is a radioactive element used in medical applications. It has a half-life of about 30 years. Suppose a laboratory has 10 grams of cesium-137. If they don't use it, how much will still be cesium-137 in 60 years?

Answer

Identify the values known in the formula.

R: This is the remaining value, what you are trying to find.

A: The initial amount was 10 grams.

h: The half-life is 30 years.

t: The amount of time passed is 60 years. (Note that this is in the same unit, years, as the half-life.)

Use the formula.

$$\begin{aligned} R &= A\left(\frac{1}{2}\right)^{\frac{t}{h}} \\ &= 10\left(\frac{1}{2}\right)^{\frac{60}{30}} \\ &= 10\left(\frac{1}{2}\right)^{2} \\ &= 10\left(\frac{1}{4}\right) \\ &= \frac{10}{4} = \frac{5}{2} \end{aligned}$$

There will be 2.5 grams of cesium-137 in 60 years.

? Exercise

Billy's mother put $100 into a bank account for him when he was born. The account gained interest at a rate of 3% per year, compounded monthly. Assuming no more money was deposited and none was withdrawn, how much money will be in the account when Billy turns 18?

A. $170.24
B. $171.49
C. $8,561.76
D. $20,718.34

Answer

A. Incorrect. This is the amount when the annual rate is compounded annually ($m = 1$). In this case, we are compounding monthly, $m = 12$. The correct answer is $171.49.
B. Correct. Using a rate r of 0.03 and 12 compounding periods per year ($m = 12$), the formula gives $171.49.
C. Incorrect. You may have used $r = 3$ rather than $r = 0.03$ for the annual rate, then misread the result from your calculator. The correct answer is $171.49.
D. Incorrect. You may have used $r = 0.3$ rather than $r = 0.03$ for the annual rate. The correct answer is $171.49.

Summary

Exponential functions of the form $f(x) = b^x$ appear in different contexts, including finance and radioactive decay. The base b must be a positive number and cannot be 1. The graphs of these functions are curves that increase (from left to right) if $b > 1$, showing exponential growth, and decrease if $0 < b < 1$, showing exponential decay.

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18.2: Logarithmic Functions

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18.2.1: Introduction to Logarithmic Functions

Learning Objectives

- Convert logarithmic equations to exponential equations.
- Convert exponential equations to logarithmic equations.
- Graph logarithmic functions.
- Solve logarithmic equations.

Introduction

A useful family of functions that is related to **exponential functions** is the **logarithmic functions**. You have been calculating the result of b^x, and this gave us the exponential functions. A **logarithm** is a calculation of the **exponent** in the equation $y = b^x$. Put another way, finding a logarithm is the same as finding the exponent to which the given base must be raised to get the desired value. The exponent becomes the *output* rather than the *input*.

Calculating Exponents

Consider these tables of values using a base of 2.

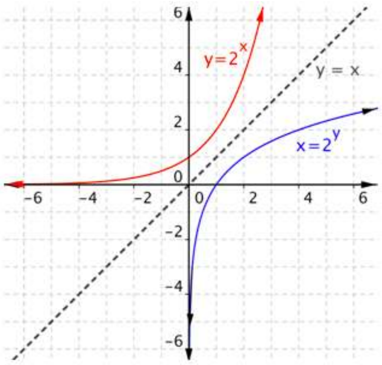
Table 1

Input x, an exponent: x	Output y: $y = 2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

Input x, a number that is a power of 2: $x = 2y$	Output y, the exponent of 2: y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Note the two tables are the same except the columns are reversed: the point (1,2) taken from the first table will be the point (2,1) in the second table.

The graphs of these two relationships should have the same general shape. As shown in the graph, the two curves are symmetrical about the line $y = x$. Another way to put it, if you rotate the red curve about the line $y = x$, it will coincide with the blue curve. (This makes sense, because y in the first table becomes in the second table, and vice versa.)



The equation $x=2^y$ is often written as a logarithmic function (called log function for short). The logarithmic function for $x=2^y$ is written as $y=\log_2 x$ or $f(x)=\log_2 x$. The number 2 is still called the **base**. In general, $y=\log_b x$ is read, “y equals log to the base b of x,” or more simply, “y equals log base b of x.” As with exponential functions, $b>0$ and $b\neq 1$.

You can see from the graph that the range (y values) of the exponential function (in red) is positive real numbers. Since the input and output have been switched, the *domain* (x values) of the logarithmic function (in blue) is positive real numbers.

Similarly, the domain of the exponential function (in red) is all real numbers. The *range* of the logarithmic function (in blue) is all real numbers.

Definition of Logarithm

The logarithm of x in base b is written $\log_b x$ and is defined as:

$\log_b x=y$ if and only if $b^y=x$, where $x>0$ and $b>0$, $b\neq 1$.

Connecting Exponential and Logarithmic Equations

It’s important to remember that the result of a logarithm is the exponent! That is, $\log_b x$ asks, “What exponent on the base b will give the result x?”

$$\underbrace{y=\log_b x}_{\text{logarithmic form}} \quad \text{means} \quad \underbrace{b^y=x}_{\text{exponential form}}$$

Sometimes, you will need to convert $\log_b x\ =y$ to $b^y=x$. Other times, you will convert $b^y=x$ to $\log_b x\ =y$. The examples in the table show the logarithmic form and the corresponding exponential form of several equations.

Logarithmic Form	Exponential Form
$\log_2 16=4$	$2^4=16$
$\log_7 1=0$	$7^0=1$
$\log_5 5=1$	$5^1=5$
$\log_4(\frac{1}{4}$=-1	$4^{-1}=\frac{1}{4}$
$\log_{10} 0.01=\log_{10}(\frac{1}{100})=-2$	$10^{-2}=0.01$

Knowing this connection is essential to understanding, and working with, logarithms.

✓ Example

Problem: Rewrite $\log_3 9=2$ as an exponential equation.

Answer

You need to change $\log_3 9=2$ to the form $b^y=x$.

$\log_b x = y \Leftrightarrow b^y = x$

Here the base is 3 and the exponent is 2.

$$\begin{array}{llll} \log_3 & 9 & = & 2 \\ b = 3 & x = 9 & & y = 2 \end{array}$$

Substitute for b, y, and x in the exponential equation, $b^y = x$.

$b^y = x \Rightarrow 3^2 = 9$

Check the result: Is 3^2 actually equal to 9? Yes!

$3^2 = 9$

Although the input of a logarithm must always be positive, the result (an exponent!) can be negative.

✓ Example

Problem: Convert $\log_4\left(\frac{1}{16}\right) = -2$ into an exponential equation.

Answer

You need to rewrite $\log_4\left(\frac{1}{16}\right) = -2$ in the form $b^y = x$.

$\log_b x = y \Leftrightarrow b^y = x$

Here the base is 4 and the exponent is -2.

$$\begin{array}{llll} \log_4 & \frac{1}{16} & = & -2 \\ b = 4 & x = \frac{1}{16} & & y = -2 \end{array}$$

Substitute for b, y, and x in the exponential equation, $b^y = x$.

$b^y = x \Rightarrow 4^{-2} = \frac{1}{16}$

Check the result: $4^{-2} = \left(\frac{1}{4}\right)^2 = \frac{1}{4^2} = \frac{1}{16}$

$4^{-2} = \frac{1}{16}$

? Exercise

Convert $\log_3\left(\frac{1}{81}\right) = -4$ into an exponential equation.

A. $3^4 = \frac{1}{81}$
B. $81^{\frac{1}{4}} = 3$
C. $3^{-4} = \frac{1}{81}$
D. $3^4 = 81$

Answer

A. Incorrect. The exponent is negative, not positive. The correct answer is $3^{-4} = \frac{1}{81}$.
B. Incorrect. Although this is a true statement, the base in this answer is 81, while the base in the original logarithmic equation is 3. The correct answer is $3^{-4} = \frac{1}{81}$.
C. Correct. The base is 3 and the exponent is -4, so $3^{-4} = \frac{1}{81}$.
D. Incorrect. Although this is a true statement, the exponent in the original logarithmic equation is -4, not 4. The correct answer is $3^{-4} = \frac{1}{81}$.

As you have just seen, it is possible to convert log equations into exponential equations. It is also possible to convert exponential equation into log equations.

Example

Problem: Rewrite $5^3 = 125$ as a logarithmic equation.

Answer

You need to rewrite $5^3 = 125$ in the form $\log_b x = y$.

$b^y = x \Leftrightarrow \log_b x = y$

Here the base is 5 and the exponent is 3.

$5^3 = 125$

Substitute for b, y, and x in the logarithmic equation, $\log_b x = y$.

$\log_b x = y \Rightarrow \log_5 125 = 3$

$\log_b 125 = 3$

Example

Problem: Convert $10^{-3} = \frac{1}{1{,}000}$ into a logarithmic equation.

Answer

You need to rewrite $10^{-3} = \frac{1}{1{,}000}$ in the form $\log_b x = y$.

$b^y = x \Leftrightarrow \log_b x = y$

Here the base is 10 and the exponent is -3.

$10^{-3} = \frac{1}{1{,}000}$
Where $b = 10^{-3}$ and $y = -3$ and
$x = \frac{1}{1{,}000}$

Substitute for b, y, and x in the logarithmic equation, $\log_b x = y$.

$\log_b x = y \Rightarrow \log_{10}\left(\frac{1}{1{,}000}\right) = -3$

$\log_{10} \frac{1}{1{,}000} = -3$

Remember that roots are exponents, too.

Example

Problem: Convert $\sqrt{49} = 7$ into a logarithmic equation.

Answer

Rewrite the square root using a fractional exponent. You need to rewrite $49^{\frac{1}{2}} = 7$ in the form $\log_b x = y$.

$\sqrt{49} = 49^{\frac{1}{2}}$

$b^y = x \Leftrightarrow \log_b x = y$
$49^{\frac{1}{2}} = 7$

Here, the base is 49 and the exponent is $\frac{1}{2}$.

Where $b = 49$ and $y = \left(\frac{1}{2}\right)$ and $x = 7$

Substitute for b, y, and x in the logarithmic equation, $\log_b x = y$.

$\log_{49} 7 = \frac{1}{2}$

Exercise

Convert $11^2 = 121$ into a logarithmic equation.

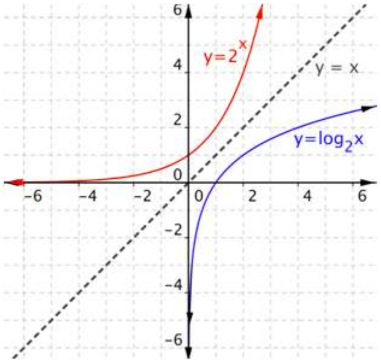
A. $\log_{11} 2 = 121$
B. $\log_{121} 11 = 2$
C. $\log_{11} 121 = 2$
D. $\log_{121} 2 = 11$

Answer

A. Incorrect. Finding a logarithm is the same as finding the exponent to which the given base must be raised to get the desired value, so 2 should be by itself on one side on the equation. The correct answer is $\log_{11} 121 = 2$.
B. Incorrect. The base of the exponential equation, 11, is also the base (the small subscript number at the end of "log") in the logarithmic equation. The correct answer is $\log_{11} 121 = 2$.
C. Correct. The base is 11, and the result of the logarithm, 2, is the exponent.
D. Incorrect. The base of the exponential equation, 11, is also the base (the small subscript number at the end of "log") in the logarithmic equation. The correct answer is $\log_{11} 121 = 2$.

Graphing Logarithmic Functions

Look at the graph below. It should look familiar: you saw it earlier in this topic! The blue graph is the logarithmic function, and the red graph is the corresponding exponential function.



When graphing logarithmic functions, it's important to remember the following:

- The graph can only appear to the right of the y-axis. This is because the domain is restricted to positive values of x.
- The graph gets close to the y-axis for small values of x (close to $x = 0$).

Remember that logarithmic functions almost act like exponential functions. You just have to switch the x and y values. For example, the second bulleted item above is like the exponential function getting close to the x-axis (close to $y = 0$).

Example

Problem: Graph $f(x) = \log_3 x$.

Answer

To make the calculations easier, first make this a logarithmic equation in y. Then convert the logarithm into an exponential equation.

Graph $y = \log_3 x$. This is the same as $3^y = x$.

Start with a table of values. With logarithmic functions, it's usually easier to choose the y values instead of the x values. Be sure to include some negative values for y.

$x = 3^y$	y
	-2

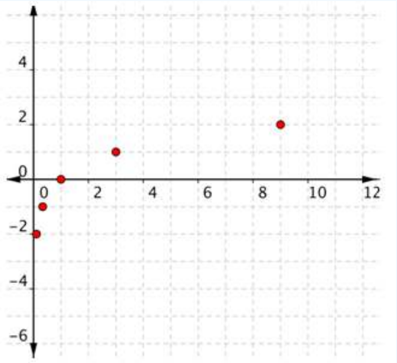
	-1
	0
	1
	2

Be careful with the negative exponents!

$x=3^y$	y
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2

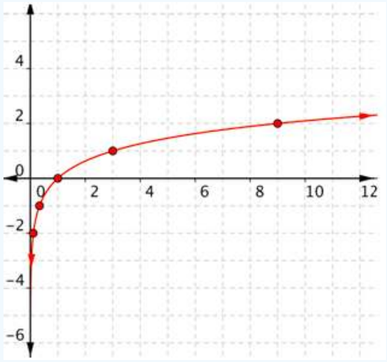
Use the table as ordered pairs. Remember that the graph of the function will show all $x-y$ correspondences, so any pair that could be in the table must be on the graph.

Plot the points.



Since the points are *not* on a line, you can't use a straightedge. Connect the points as best you can, using a *smooth curve* (not a series of straight lines).

Remember that logarithmic functions get close to the y-axis (but won't touch or cross it).



Example

Problem: Graph $f(x)=\log_4 x$.

Answer

To make the calculations easier, first make this a logarithmic equation in y. Then convert the logarithm into an exponential equation.

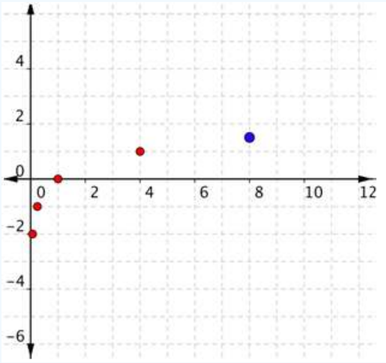
Graph $y=\log_4 x$. This is the same as $4^y=x$.

Start with a table of values, choosing the y values and calculating x. Be careful with the negative exponents!

$x=4^y$	y
$\frac{1}{16}$	-2
$\frac{1}{4}$	-1
1	0
4	1
16	2

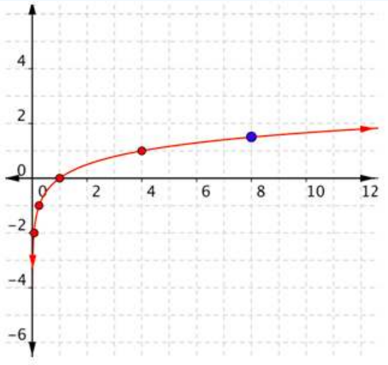
Use the table as ordered pairs and plot the points.

In this case, (16, 2) won't show on the graph. Because you know the square root of 4, try $x=\frac{3}{2}$. In this case, it would be $4^{\frac{3}{2}}=(\sqrt{4})^3=2^3=8$. That point is shown in blue.

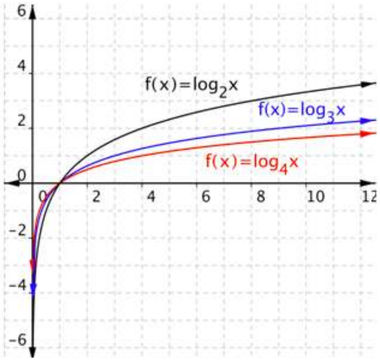


Since the points are *not* on a line, you can't use a straightedge. Connect the points as best you can, using a *smooth curve* (not a series of straight lines).

Remember that logarithmic functions get close to the y-axis (but won't touch or cross it).



Let's compare the logarithmic graphs you've seen: $f(x)=\log_2 x$, $f(x)=\log_3 x$, and $f(x)=\log_4 x$.



Notice that a larger base makes the graph *less* steep. (This is the opposite from exponential functions, where a larger base meant a steeper graph.) A larger base also makes the graph closer to the x-axis for $y > 0$ (or $x > 1$) and closer to the y-axis for $y < 0$ (or x < 1). All the graphs go through (1, 0), while exponential functions always go through (0, 1).

With exponential functions, the graphs were *increasing* (exponential *growth*) when the base b was greater than 1. The graphs were *decreasing* (exponential *decay*) when the base b was less than 1. The logarithmic graphs above are all increasing, and they all have a base greater than 1. Let's see what happens with a base less than 1.

✓ Example

Problem: Graph $f(x) = \log_{\frac{1}{4}} x$.

Answer

Write an exponential equation in y to help you.

$y = \log_{\frac{1}{4}} x$
$\left(\frac{1}{4}\right)^y = x$

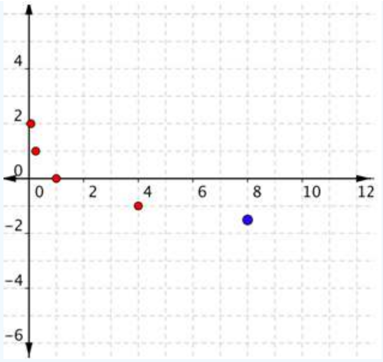
Create a table of values. Again, remember to be careful with the negative exponents. Remember to take the reciprocal of the base to make the exponent positive. $\left(\frac{1}{4}\right)^{-2} = \left(\frac{4}{1}\right)^2 = 4^2 = 16$

x	y
16	-2
4	-1
1	0
$\frac{1}{4}$	1
$\frac{1}{16}$	2

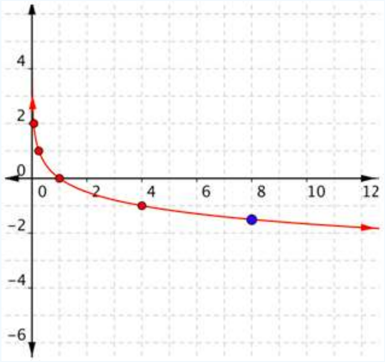
Notice that in this table, the x values decrease as the y values increase.

Use the table pairs to plot points. You may want to include new points, especially when one of the points from the table won't fit on your graph. (16, -2) is off the graph. Because you know the square root of 4, try $x = -\frac{3}{2}$: $\left(\frac{1}{4}\right)^{-\frac{3}{2}} = \left(\frac{4}{1}\right)^{\frac{3}{2}} = 4^{\frac{3}{2}} = (\sqrt{4})^3$ $= 2^3 = 8$

The point (8, -$\frac{3}{2}$) has been included in blue. You may not feel it necessary to include additional points. You also may need to use a calculator, depending on the base.



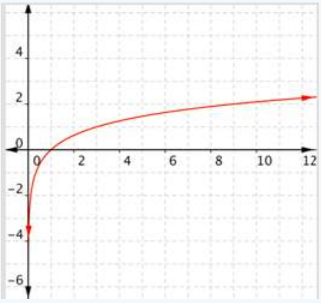
Connect the points as best you can, using a *smooth curve*.

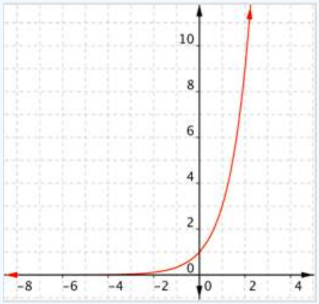


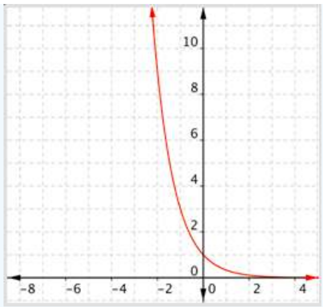
If the base is less than 1, the logarithmic function is decreasing. The graph gets close to the y-axis when x is small, but with *positive* y values instead of negative ones.

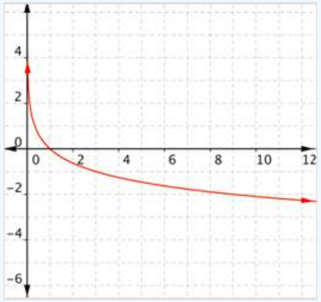
? Exercise

Which of the following is a graph for $f(x) = \log_{\frac{1}{3}} x$?

A. 

B. 

C. 

D. 

Answer

A. Incorrect. This graph is increasing because the $f(x)$ values are increasing as x increases. (Compare the values for $x = 1$ and $x = 2$.) This is a graph of $f(x) = \log_3 x$. The correct answer is Graph D.
B. Incorrect. This function has a domain of all real numbers and a range of positive real numbers. Logarithmic functions have domains of positive real numbers and ranges of all real numbers. The correct answer is Graph D.
C. Incorrect. This graph is decreasing, but it has a domain of all real numbers and a range of positive real numbers. Logarithmic functions have domains of positive real numbers and ranges of all real numbers. The correct answer is Graph D.
D. Correct. The base is less than 1, so the graph shows a decreasing function. The domain is positive real numbers, and the range is all real numbers.

Solving Logarithmic Equations

As you've seen, there are three essential quantities in a logarithmic equation $y = \log_b x$: the base b, the exponent y, and the input x. Any of these could be missing in an equation you need to solve. Often the easiest way to solve these is to convert the logarithmic equation to an exponential equation.

✓ Example

Problem: Solve $4 = \log_5 x$.

Answer

Convert the logarithmic equation to an exponential equation.

$4 = \log_5 x \text{ means } 5^4 = x$

In this case, you only need to evaluate 5^4 to solve for x.

$$\begin{aligned} 5^4 &= 5 \cdot 5 \cdot 5 \cdot 5 \\ 5^4 &= 25 \cdot 25 \\ 5^4 &= 625 \end{aligned}$$

$x = 625$

When solving for b or y, it's helpful to know a lot of exponential relationships, such as $5^3 = 125$ and $2^5 = 32$, off the top of your head. However, even if you don't, you can do some thinking to approximate, or even find exactly, the solutions.

✓ Example

Problem: Solve $3 = \log_b 64$.

Answer

Convert the logarithmic equation to an exponential equation.

$3 = \log_b 64 \text{ means } b^3 = 64$

You may not know what number to the third power is 64. You can try a few and zero in on it quickly, though. 1^3 and 10^3 are easy to calculate, so let’s start there.

$$\begin{aligned} 1^3 &= 1 \\ 10^3 &= 1,000 \end{aligned}$$

64 is much closer to 1 than to 1,000, so the correct base will be closer to 1 than to 10.

Let’s try some other low values. Since 64 is even, you know you don’t need to try any *odd* numbers. An odd number to any power will always be odd, and an even number to any power will always be even.

So let’s try 2 and 4.

$$\begin{aligned} 2^3 &= 8 \\ 4^3 &= 4 \cdot 16 = 64 \end{aligned}$$

Found it! Be sure you are providing the correct number for your answer. It would be easy to mistakenly answer “64” here!

$b = 4$

Let’s try one more:

✓ Example

Problem: Solve $y = \log_5 125$.

Answer

Convert the logarithmic equation to an exponential equation.

$y = \log_5 125 \text{ means } 5^y = 125$

What power of 5 is 125? You might know, but if you don’t, make a table of values for y and 5^y. Look for 125, or close to it, in the 5^y column.

y	5^y
1	5
2	25
3	125

Again, be sure you provide the correct number. It would be easy to make a mistake and write 125 here!

$y = 3$

? Exercise

Solve $4 = \log_2 x$ for x.

A. $\frac{1}{2}$
B. 2
C. 8
D. 16

Answer

A. $\frac{1}{2}$. Incorrect. The equivalent exponential equation is $2^4 = x$. You may have incorrectly written $4^x = 2$. The correct answer is 16.
B. 2. Incorrect. The equivalent exponential equation is $2^4 = x$. You may have incorrectly written $2^x = 4$. The correct answer is 16.
C. 8. Incorrect. The equivalent exponential equation is $2^4 = x$. The correct answer is 16.
D. 16. Correct. The equivalent exponential equation is $2^4 = x$, which is 16.

Summary

Logarithmic functions are closely related to exponential equations. In fact, $\log_b x = y$ says the same thing as $b^y = x$. The graphs look similar, but with characteristics such as the domain and range switched between the x values and the y values. Converting a logarithmic equation to the equivalent exponential equation is helpful with both graphing and solving equations.

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18.2.2: Properties of Logarithmic Functions

Learning Objectives

- Express the logarithm of a product as a sum of logarithms.
- Express the logarithm of a quotient as a difference.
- Express the logarithm of a power as a product.
- Simplify logarithmic expressions.

Introduction

Throughout your study of algebra, you have come across many properties such as the commutative, associative, and distributive properties. These properties help you take a complicated expression or equation and simplify it.

The same is true with **logarithms**. There are a number of properties that will help you simplify complex logarithmic expressions. Since logarithms are so closely related to exponential expressions, it is not surprising that the properties of logarithms are very similar to the properties of exponents. As a quick refresher, here are the **exponent** properties.

Properties of Exponents

Product of powers: $b^m b^n = b^{m+n}$

Quotient of powers: $\frac{b^m}{b^n} = b^{m-n}$

Power of a power: $(b^m)^n = b^{mn}$

One important but basic property of logarithms is $\log_b b^x = x$. This makes sense when you convert the statement to the equivalent exponential equation. The result? $b^x = b^x$.

Let's find the value of y in $\log_3 3^2 = y$. Remember $\log_b x = y \Leftrightarrow b^y = x$, so $\log_3 3^2 = y$ means $3^y = 3^2$ and y must be 2, which means $\log_3 3^2 = 2$. You will get the same answer that $\log_3 3^2$ equals 2 by using the property that $\log_b b^x = x$.

Logarithm of a Product

Remember that the properties of exponents and logarithms are very similar. With exponents, to multiply two numbers with the same **base**, you add the exponents. With logarithms, the logarithm of a product is the sum of the logarithms.

Logarithm of a Product

The logarithm of a product is the sum of the logarithms: $\log_b(MN) = \log_b M + \log_b N$

Let's try the following example.

Example

Problem: Use the product property to rewrite $\log_2(4 \cdot 8)$.

Answer

Use the product property to write as a sum.

$\log_2(4 \cdot 8) = \log_2 4 + \log_2 8$

Simplify each addend, if possible. In this case, you can simplify both addends. First rewrite $\log_2 4$ as $\log_2 2^2$ and $\log_2 8$ as $\log_2 2^3$, and then use the property $\log_b b^x = x$.

Or, rewrite $\log_2 4 = y$ as $2^y = 4$ to find $y = 2$, and $\log_2 8 = y$ as $2^y = 8$ to find $y = 3$.

Use whatever method makes sense to you.

$$\begin{aligned} \log_2 4 + \log_2 8 &= \\ \log_2 2^2 + \log_2 2^3 &= \\ 2 + 3 &= 5 \end{aligned}$$

$\log_2(4\cdot 8)=5$

Another way to simplify $\log_2(4\cdot 8)$ would be to multiply 4 and 8 as a first step.

$\log_2(4\cdot 8)=\log_2 32=5 \text{ because } 2^5=32$

You get the same answer$\log_2(4\cdot 8)=5$ as in the example!

Notice the similarity to the exponent property: $b^m b^n=b^{m+n}$, while $\log_b(MN)=\log_b M+\log_b N$. In both cases, a product becomes a sum.

✓ Example

Problem: Use the product property to rewrite $\log_3(9x)$.

Answer

Use the product property to write as a sum.

$\log_3(9x)=\log_3 9+\log_3 x$

Simplify each addend, if possible. In this case, you can simplify $\log_3 9$ but not $\log_3 x$. Rewrite \(\log _{3} 9\ as $\log_3 3^2$ and then use the property $\log_b b^x=x$.

Or, simplify $\log_3 9$ by converting $\log_3 9=y$ to $3^y=9$ and finding that $y=2$. Use whatever method makes sense to you.

$\log_3 9+\log_3 x=\log_3 3^2+\log_3 x=2+\log_3 x$

$\log_3(9x)=2+\log_3 x$

If the product has many factors, you just add the individual logarithms:

$\log_b(ABCD)=\log_b A+\log_b B+\log_b C+\log_b D$

? Exercise

Rewrite $\log_2 8a$, then simplify.

A. $3\log_2 a$
B. $\log_2 3a$
C. $\log_2(3+a)$
D. $3+\log_2 a$

Answer

A. Incorrect. The individual logarithms must be added, not multiplied. The correct answer is $3+\log_2 a$.
B. Incorrect. You found that $\log_2 8=3$, but you must first apply the logarithm of a product property. The correct answer is $3+\log_2 a$.
C. Incorrect. The logarithm of a product property says you separate the 8 and a into separate logarithms. The correct answer is $3+\log_2 a$.
D. Correct. The logarithm of a product property says $\log_2 8a=\log_2 8+\log_2 a$, and $\log_2 8=3$.

Logarithm of a Quotient

You can use the similarity between the properties of exponents and logarithms to find the property for the logarithm of a quotient. With exponents, to multiply two numbers with the same base, you add the exponents. To divide two numbers with the same base, you subtract the exponents. What do you think the property for the logarithm of a quotient will look like?

As you may have suspected, the logarithm of a quotient is the difference of the logarithms.

Logarithm of a Quotient

$\log_b\left(\frac{M}{N}\right)=\log_b M-\log_b N$

With both properties: $\frac{b^m}{b^n}=b^{m-n}$ and $\log_b\left(\frac{M}{N}\right)=\log_b M-\log_b N$, a quotient becomes a difference.

Example

Problem: Use the quotient property to rewrite $\log_x\left(\frac{x}{2}\right)$.

Answer

Use the quotient property to rewrite as a difference.

$\log_2\left(\frac{x}{2}\right)=\log_2 x-\log_2 2$

The first expression can't be simplified further. However, the second expression can be simplified. What exponent on the base (2) gives a result of 2? Since $2^1=2$, you know $\log_2 2=1$.

$\log_2\left(\frac{x}{2}\right)=\log_2 x-1$

Exercise

Which of these is equivalent to $\log_3\left(\frac{81}{a}\right)$?

A. $4-\log_3 a$
B. $\frac{4}{\log_3 a}$
C. $\log_3(4-a)$
D. $\log_3\left(\frac{4}{a}\right)$

Answer

A. Correct. The logarithm of a quotient property states $\log_3\left(\frac{81}{a}\right)=\log_3 81-\log_3 a$, and $\log_3 81=4$.
B. Incorrect. The individual logarithms must be subtracted, not divided. The correct answer is $4-\log_3 a$.
C. Incorrect. The logarithm of a quotient property says you separate the 81 and a into separate logarithms. The correct answer is $4-\log_3 a$.
D. Incorrect. You found that $\log_3 81=4$, but you must first apply the logarithm of the quotient property. The correct answer is $4-\log_3 a$.

Logarithm of a Power

The remaining exponent property was power of a power: $(b^m)^n=b^{mn}$. The similarity with the logarithm of a power is a little harder to see.

Logarithm of a Power

$\log_b M^n=n\log_b M$

With both properties, $(b^m)^n=b^{mn}$ and $\log_b M^n=n\log_b M$, the power "n" becomes a factor.

Example

Problem: Use the power property to simplify $\log_3 9^4$.

Answer

You *could* find 9^4 but that wouldn't make it easier to simplify the logarithm. Use the power property to rewrite $\log_3 9^4$ as $4\log_3 9$.

$\log_3 9^4=4\log_3 9$

You may be able to recognize by now that since $3^2=9$, $\log_3 9=2$.

$4\log_3 9 = 4 \cdot 2$

Multiply the factors.

$\log_3 9^4 = 8$

Notice in this case that you also could have simplified it by rewriting it as 3 to a power: $\log_3 9^4 = \log_3 \left(3^2\right)^4$. Using exponent properties, this is $\log_3 3^8$ and by the property $\log_b b^x = x$, this must be 8!

Example

Problem: Use the properties of logarithms to rewrite $\log_4 64^x$.

Answer

Use the power property to rewrite $\log_4 64^x$ as $x\log_4 64$.

$64 = 4 \cdot 4 \cdot 4 = 4^3$

Rewrite $\log_4 64$ as $\log_4 4^3$, then use the property $\log_b b^x = x$ to simplify $\log_4 4^3$.

Or, you may be able to recognize by now that since $4^3 = 64$, $\log_4 64 = 3$.

$$\begin{aligned}\log_4 64^x &= x\log_4 64\\ &= x\log_4 4^3\\ &= x \cdot 3\end{aligned}$$

Multiply the factors.

$\log_4 64^x = 3x$

Exercise

Which of these is equivalent to $\log_2 x^8$?

A. $\log_2 3x$
B. $8\log_2 x$
C. $\log_2 8x$
D. $3\log_2 x$

Answer

A. Incorrect. The exponent becomes a factor *outside* the logarithm. The correct answer is $8\log_2 x$.
B. Correct. By the power property, $\log_2 x^8 = 8\log_2 x$. You can't simplify this further.
C. Incorrect. The exponent becomes a factor *outside* the logarithm. The correct answer is $8\log_2 x$.
D. Incorrect. You probably noticed that $\log_2 8 = 3$, so you used 3 instead of 8 when you pulled the exponent out to be a factor. However, the exponent must be pulled outside the logarithm to be a factor without any other changes. The correct answer is $8\log_2 x$.

Simplifying Logarithmic Expressions

The properties can be combined to simplify more complicated expressions involving logarithms.

Example

Problem: Use the properties of logarithms to expand $\log_{10}\left(\frac{ab}{cd}\right)$ into four simpler terms.

Answer

Use the quotient property to rewrite $\log_{10}\left(\frac{ab}{cd}\right)$ as a difference of logarithms.

$\log_{10}\left(\frac{ab}{cd}\right) = \log_{10}(ab) - \log_{10}(cd)$

Now you have two logarithms, each with a product. Apply the product rule to each.

Be careful with the subtraction! Since all of $\log_{10} cd$ is subtracted, you have to subtract both parts of the term: $(\log_{10} c + \log_{10} d)$

$\log_{10}(ab) - \log_{10}(cd) = \log_{10} a + \log_{10} b - (\log_{10} c + \log_{10} d)$

$\log_{10}\left(\frac{ab}{cd}\right) = \log_{10} a + \log_{10} b - \log_{10} c - \log_{10} d$

✓ Example

Problem: Simplify $\log_6 (ab)^4$, writing it as two separate terms.

Answer

Use the power property to rewrite $\log_6 (ab)^4$ as $4\log_6(ab)$.

You are taking the log of a product, so apply the product property.

Be careful: the value 4 is multiplied by the *whole* logarithm, so use parentheses when you rewrite $\log_6(ab)$ as $(\log_6 a + \log_6 b)$.

$$\begin{aligned}\log_6 (ab)^4 &= 4\log_6(ab)\\ &= 4(\log_b a + \log_6 b)\end{aligned}$$

Use the distributive property.

$\log_6 (ab)^4 = 4\log_6 a + 4\log_6 b$

? Exercise

Simplify $\log_3 x^2 y$.

A. $2(\log_3 x + \log_3 y)$
B. $\log_3 x^2 + \log_3 y$
C. $2\log_3 xy$
D. $2\log_3 x + \log_3 y$

Answer

A. Incorrect. You may have started incorrectly by applying the power property, or you may have started correctly with the product property but then incorrectly applied the power property. The correct answer is $2\log_3 x + \log_3 y$.
B. Incorrect. While you correctly applied the product property first, $\log_3 x^2$ can be simplified further. The correct answer is $2\log_3 x + \log_3 y$.
C. Incorrect. You probably started incorrectly by applying the power property. Start with the product property. The correct answer is $2\log_3 x + \log_3 y$.
D. Correct. $\log_3 x^2 y = \log_3 x^2 + \log_3 y = 2\log_3 x + \log_3 y$

Summary

Like exponents, logarithms have properties that allow you to simplify logarithms when their inputs are a product, a quotient, or a value taken to a power. The properties of exponents and the properties of logarithms have similar forms.

	Exponents	Logarithms
Product Property	$b^m b^n = b^{m+n}$	$\log_b(MN) = \log_b M + \log_b N$
Quotient Property	$\frac{b^m}{b^n} = b^{m-n}$	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$
Power Property	$(b^m)^n = b^{mn}$	$\log_b M^n = n\log_b M$

Notice how the product property leads to addition, the quotient property leads to subtraction, and the power property leads to multiplication for both exponents and logarithms.

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SECTION OVERVIEW

18.3: Natural Logarithms

Topic hierarchy

18.3.1: Introduction to Natural and Common Logarithms

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18.3.1: Introduction to Natural and Common Logarithms

Learning Objectives

- Use a calculator to find logarithms or powers of base e.
- Graph exponential and logarithmic functions of base e.
- Find logarithms to bases other than e or 10 by using the change of base formula.

Introduction

In both exponential functions and logarithms, any number can be the base. However, there are two bases that are used so frequently that mathematicians have special names for their logarithms, and scientific and graphing calculators include keys specifically for them! These are the common and natural logarithms.

Common Logarithms and e

A **common logarithm** is any logarithm with base 10. Recall that our number system is base 10; there are ten digits from 0 through 9, and place value is determined by groups of ten. You can remember a “common logarithm,” then, as any logarithm whose base is our “common” base, 10.

Natural logarithms are different than common logarithms. While the base of a common logarithm is 10, the base of a natural logarithm is the special number e. Although this looks like a variable, it represents a fixed irrational number approximately equal to 2.718281828459. (Like pi, it continues without a repeating pattern in its digits.) e is sometimes called Euler's number or Napier’s constant, and the letter e was chosen to honor the mathematician Leonhard Euler (pronounced *oiler*).

e is a complicated but interesting number. Let’s take a closer look at it through the lens of a formula you have seen before: compound interest.

The formula for compound interest is $A=P\left(1+\frac{r}{m}\right)^{mt}$, where A is the amount of money after t years, P is the principal or initial investment, r is the annual interest rate (expressed as a decimal, not a percent), m is the number of compounding periods in a year, and t is the number of years.

Imagine what happens when the compounding happens frequently. If interest is compounded annually, then $m=1$. If compounded monthly, then $m=12$. Compounding daily would be represented by $m=365$; hourly would be represented by $m=8,760$. You can see that as the frequency of the compounding periods increases, the value of m increases quickly. Imagine the value of m if interest were compounded each minute or each *second*!

You can even go more frequently than each second, and eventually get compounding continuously. Look at the values in this table, which looks a lot like the expression multiplied by P in the above formula. As x gets greater, the expression more closely resembles continuous compounding.

x	$\left(1+\frac{1}{x}\right)^{x}$
1	2
10	2.59374 ...
100	2.70481 ...
1000	2.71692 ...
10,000	2.71814 ...
100,000	2.71826 ...
1,000,000	2.71828 ...

Notice that although x is increasing a lot (multiplying by 10 each time!), the value of $\left(1+\frac{1}{x}\right)^{x}$ is not increasing wildly. In fact, it is getting closer and closer to 2.718281828459... or the value now called e.

The function $f(x)=e^{x}$ has many applications in economics, business, and biology. e is an important number for this reason.

Working with Bases of e and 10

Scientific and graphing calculators all have keys that help you work with e. Look on your calculator and find one labeled “e” or “exp.” (Some graphing calculators may require you to use a menu to find e. If you can’t see the key, consult your manual or ask your instructor.)

How to evaluate exponential expressions using e (such as e^3) depends on your calculator. On some calculators, you have to press the $[e^x]$ key first then enter the exponent and press enter. On others you enter the exponent first then press the $[e^x]$ key. It is important that you know how your calculator works. With your calculator, try finding e^3. The result should be 20.0855369... (the number of digits displayed will also depend on your calculator).

✓ Example

Find $e^{1.5}$ using a calculator. Round your answer to the nearest hundredth.

Solution

	Enter the keystrokes needed for your calculator. If you are having trouble getting the correct answer, consult your manual or instructor.
4.4816890...	Calculator result. Then round the answer to the nearest hundredth.
4.48	To see this worked out on a calculator, see the Worked Examples for this topic.

You can find powers of 10 (the *common base*) in the same way. Some calculators have a $[10^\wedge]$ or $[10^x]$ key that you can use to find powers of 10. Another way to find powers of 10 is to use the $[x^y]$ or the $[y^x]$ key that will work with any base (although if you use this method, you will have to key in two numbers: the base, 10, and whatever exponent you are raising it to).

✓ Example

Find $10^{1.5}$, using a calculator. Round your answer to the nearest hundredth.

Solution

	Enter the keystrokes needed for your calculator. If you are having trouble getting the correct answer, consult your manual or instructor.
31.6227766...	Calculator result. Then round the answer to the nearest hundredth.
31.62	To see this worked out on a calculator, see the Worked Examples for this topic.

Natural logarithms (using e as the base) and common logarithms (using 10 as the base) are also available on scientific and graphing calculators. When a logarithm is written without a base then you should assume the base is 10. For example:

$$\log 100 = \log_{10} 100 = 2$$

Natural logarithms also have their own symbol: $\ln$.

$$\ln 100 = \log_e 100 = 4.60517\ldots$$

The logarithm keys are often easier to find, but they may work differently from one calculator to the next. Most handheld scientific calculators require you to provide the input *first*, then press the $[\log]$ (common) or $[\ln]$ (natural) key. Other calculators work in reverse: press the $[\log]$ or $[\ln]$ key, and then provide the input and press [Enter] or [=].

On your calculator, find the common logarithm ($[\log]$ or $[\log_{10}]$) key and the natural logarithm ($[\ln]$) key and verify that $\ln 100 = \log_e 100 = 4.60517\ldots$

Example

Find $\ln 3$, using a calculator. Round your answer to the nearest hundredth.

Solution

	Remember $\ln$ means "natural logarithm," or $\log_e$. Enter the keystrokes needed for your calculator. If you are having trouble getting the correct answer, consult your manual or instructor.
1.098612...	Calculator result. Then round the answer to the nearest hundredth.
1.10	To see this worked out on a calculator, see the Worked Examples for this topic.

Example

Find $\log 34$, using a calculator. Round your answer to the nearest hundredth.

Solution

	Remember, when no base is specified, this is the common logarithm (base 10). Enter the keystrokes needed for your calculator. If you are having trouble getting the correct answer, consult your manual or instructor.
1.5314789...	Calculator result. Then round the answer to the nearest hundredth.
1.53	To see this worked out on a calculator, see the Worked Examples for this topic.

Exercise

Use a calculator to find $\ln 7$.

A. 0.845098...
B. c
C. 1096.633...
D. 10,000,000

Answer

A. 0.845098...

Incorrect. You found the value of $\log 7$, that is, $\log_{10} 7$. The correct answer is 1.945910...

B. 0.845098...

Correct. You correctly identified the keys on your calculator and found the natural $\log$ of 7.

C. 1096.633...

Incorrect. You found the value of e^7. The correct answer is 1.945910...

D. 10,000,000

Incorrect. You found the value of 10^7. The correct answer is 1.945910...

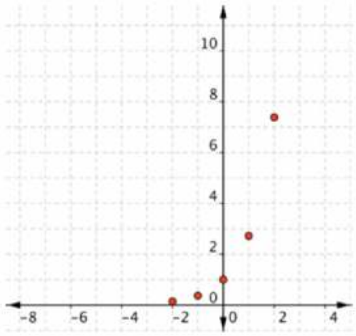
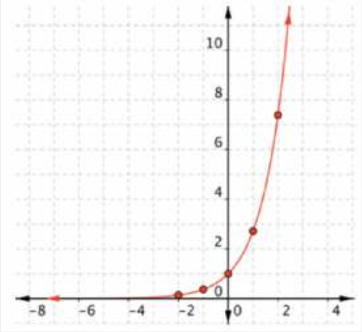
Graphing Exponential and Logarithmic Functions of Base e

Graphing functions with the base e is no different than graphing other exponential and logarithmic functions: create a table of values, plot the points, and connect them with a smooth curve. You will want to use a calculator when creating the table.

✓ Example

Graph $f(x)=e^x$.

Solution

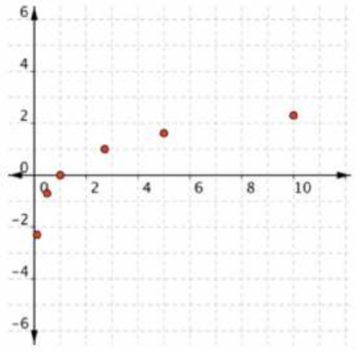
$\begin{array}{c\|c} x & f(x) \\ \hline -2 & 0.1353\ldots \\ -1 & 0.3678\ldots \\ 0 & 1 \\ 1 & 2.7182\ldots \\ 2 & 7.3890\ldots \end{array}$	Start with a table of values. Don't forget to choose positive and negative values for x. Use a calculator to find the $f(x)$ values.
$\begin{array}{\|c} (x,y) \text{ pairs} \\ \hline (-2, 0.1353\ldots) \\ (-1, 0.3678\ldots) \\ (0,1) \\ (1, 2.7182\ldots) \\ (2, 7.3890\ldots) \end{array}$	If you think of $f(x)$, as y, each row forms an ordered pair that you can plot on a coordinate grid.
	Plot the points.
	Connect the points as best you can, using a *smooth curve* (not a series of straight lines). Use the shape of an exponential graph to help you: the graph gets close to the x-axis on the left, and gets steeper and steeper on the right.

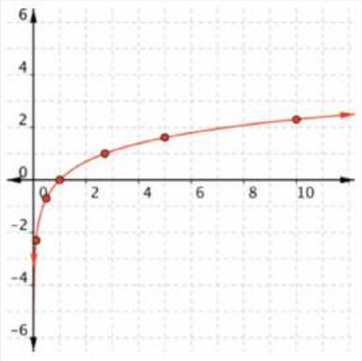
The same process works for logarithmic functions. Choose x values and use a calculator to find the y values.

✓ Example

Graph $f(x)=\ln x$.

Solution

<table>
<tr><td>
<table>
<tr><th>x</th><th>$f(x)$</th></tr>
<tr><td>0.1</td><td>$-2.30\ldots$</td></tr>
<tr><td>0.5</td><td>$-0.69\ldots$</td></tr>
<tr><td>1</td><td>0</td></tr>
<tr><td>e</td><td>1</td></tr>
<tr><td>5</td><td>$1.60\ldots$</td></tr>
<tr><td>10</td><td>$2.30\ldots$</td></tr>
</table>
</td><td>Start with a table of values. If you choose x values, remember that x must be greater than 0. Choose values greater than and less than the base (the base and 1 are also good choices for x values).</td></tr>
<tr><td>
<table>
<tr><th>(x, y) pairs</th></tr>
<tr><td>$(0.1, -2.30\ldots)$</td></tr>
<tr><td>$(0.5, -0.69\ldots)$</td></tr>
<tr><td>$(1, 0)$</td></tr>
<tr><td>$(e, 1)$</td></tr>
<tr><td>$(5, 1.60\ldots)$</td></tr>
<tr><td>$(10, 2.30\ldots)$</td></tr>
</table>
</td><td>If you think of $f(x)$ as y, each row forms an ordered pair that you can plot on a coordinate grid.</td></tr>
<tr><td>


</td><td>Plot the points.</td></tr>
<tr><td>


</td><td>Connect the points as best you can, using a *smooth curve* (not a series of straight lines). Use the shape of a logarithmic graph to help you: the graph gets close to the y-axis for x near 0.</td></tr>
</table>

Sometimes the inputs to the logarithm, or the exponent on the base, will be more complicated than just a single variable. In those cases, be sure to use the correct input on the calculator.

Note: If your calculator uses the “input last” method for logarithms, either calculate the input separately and write it down, or use parentheses to be sure the correct input is used. For example, when calculating $\log(3x)$ when $x = 4$, the correct answer is 1.079... . If you don’t use the parentheses, the calculator will find $\log 3$, and multiply that by 4 to get 1.908... .

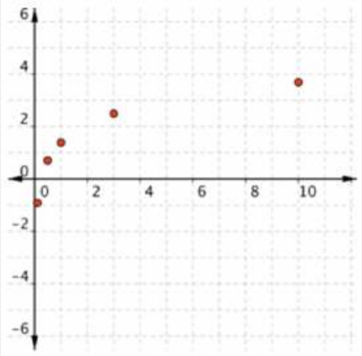
✓ Example

Graph $f(x) = \ln 4x$.

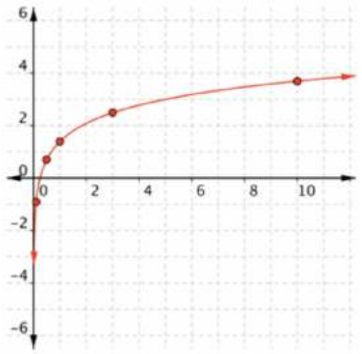
Solution

x	$4x$	$f(x)$
0.1	0.4	$-0.91\ldots$
0.5	2	$0.69\ldots$
1	4	$1.38\ldots$
3	12	$2.48\ldots$
10	40	$3.68\ldots$

Create a table of values. Although everything could be done using the calculator, let’s include a column for the input of the logarithm. This helps you avoid calculator errors.



Use the table pairs to plot points. You may want to choose additional values for the table to give a better idea for the entire visible graph.

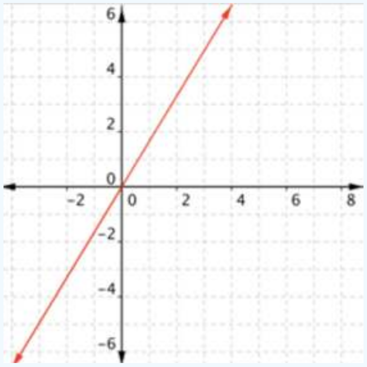


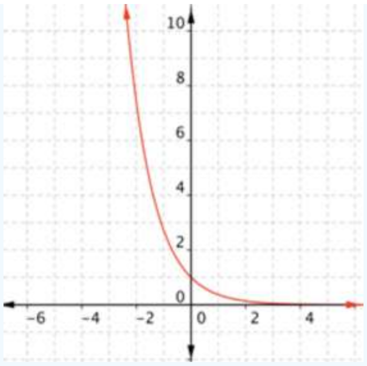
Connect the points as best you can, using a *smooth curve*.

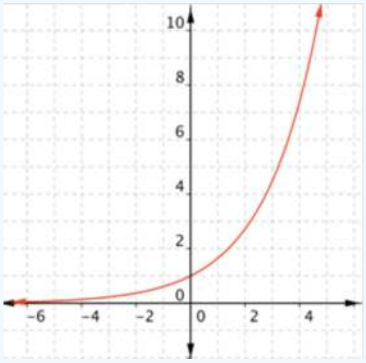
? Exercise

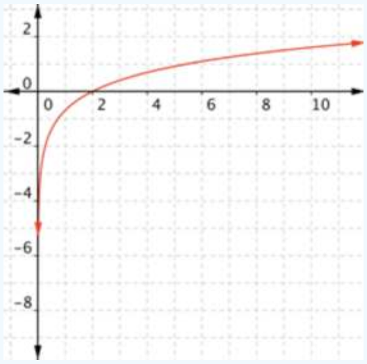
Which of the following is a graph for $f(x) = e^{0.5x}$?

A.

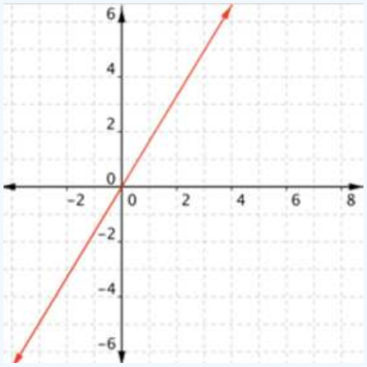


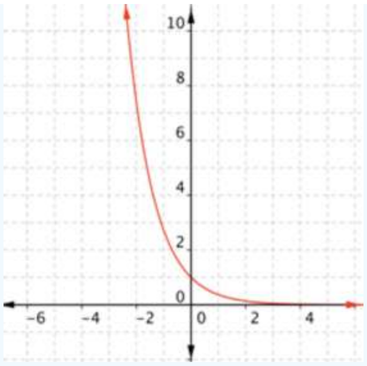
B. 
10
8
6
4
2
0
−6
−4
−2
0
2
4

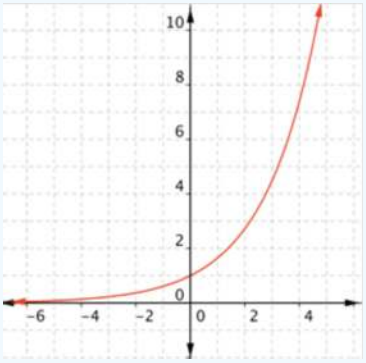
C. 
10
8
6
4
2
0
−6
−4
−2
0
2
4

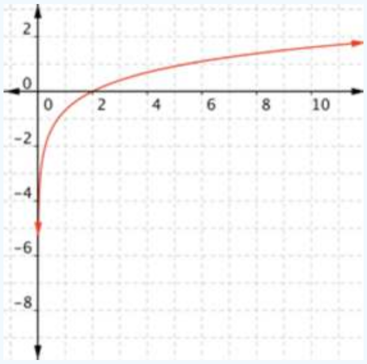
D. 
2
0
0
2
4
6
8
10
-2
-4
-6
-8

Answer

A. 
6
4
2
0
−2
0
2
4
6
8
-2
-4
-6

B. 

C. 

D. 

Finding Logarithms of Other Bases

Now you know how to find base 10 and base e logarithms of any number. What if you wanted to calculate $\log_7 36$? Converting to an exponential equation, you have $7^x = 36$. You know 7^1 is 7, and 7^2 is 49, so you can reason that x must be between 1 and 2, probably very close to 2. But how close? You don't have a key for base 7, so you use a Change of Base formula to change the base of a log function to another base.

Change of Base formula

$\log_b x = \frac{\log_a x}{\log_a b}$

Notice that a appears as the base in both logarithms on the right side of the formula. For example, $\log_7 36 = \frac{\log 36}{\log 7}$, using a new base of 10. You could also say $\log_7 36 = \frac{\ln 36}{\ln 7}$, or even $\log_7 36 = \frac{\log_{362} 36}{\log_{362} 7}$. Of course, that last one isn't any easier to calculate than the original expression, but using the $[\log]$ or $[\ln]$ keys on a calculator, you can use $\frac{\log 36}{\log 7}$ or $\frac{\ln 36}{\ln 7}$ to find $\log_7 36$.

Example

Find $\log_7 36$.

Solution

$\log_7 36 = \frac{\log 36}{\log 7}$	Use the Change of Base formula. You can use common logarithms or natural logarithms. For this example, let's use common logarithms.
$\frac{\log 36}{\log 7} = \frac{1.55630...}{0.84509...} = 1.84156\ldots$	Use the calculator to evaluate the quotient.

$1.84156\ldots$

If you had used natural logarithms, you would have gotten the same answer: $\log_7 36 = \frac{\ln 36}{\ln 7} = \frac{3.5835...}{1.9459...} = 1.84156\ldots$

✓ Example

Find $\log_3 25.9$.

Solution

$\log_3 25.9 = \frac{\ln 25.9}{\ln 3}$	Use the Change of Base formula. This time, let's use natural logarithms.
$\frac{\ln 25.9}{\ln 3} = \frac{3.2542...}{1.0986...} = 2.9621\ldots$	Evaluate the quotient.

2.9621

? Exercise

Find $\log_5 200$.

A. 40
B. 0.303...
C. 3.292...
D. 2.301...

Answer

A. 40

Incorrect. You found the value of $200 \div 5$. The correct answer is 3.292...

B. 0.303...

Incorrect. When using the Change of Base formula, the log of the original base is the denominator: $\log_5 200 = \frac{\log 200}{\log 5} = \frac{\ln 200}{\ln 5}$. The correct answer is 3.292...

C. 3.292...

Correct. $\log_5 200 = \frac{\log 200}{\log 5} = \frac{\ln 200}{\ln 5} = 3.292\ldots$

D. 2.301...

Incorrect. You found the value of $\log 200$. The correct answer is 3.292...

Summary

Common logarithms (base 10, written $\log x$ without a base) and natural logarithms (base e, written $\ln x$) are used often. Scientific and graphing calculators have keys or menu items that allow you to easily find $\log x$ and $\ln x$, as well as 10^x and e^x. Using these keys and the change of base formula, you can find logarithms in any base.

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SECTION OVERVIEW

18.4: Logarithmic and Exponential Equations

Topic hierarchy

18.4.1: Solving Exponential and Logarithmic Equations

18.4.2: Mathematical Modeling with Exponential and Logarithmic Functions

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18.4.1: Solving Exponential and Logarithmic Equations

Learning Objectives

- Solve exponential equations.
- Solve logarithmic equations.

Introduction

As you know, algebra often requires you to solve equations to find unknown values. This is also true for exponential and logarithmic equations. There are some strategies that you can use, along with some properties you've learned, that you can use to solve those equations.

Solving Exponential Equations

You may be able to look at an equation like $4^x = 16$ and solve it by asking yourself, "4 to what power is 16? 4^2 is 16, so $x = 2$." Equations like $4^x = 17$ are a bit more difficult. You know x must be a little more than 2, because 17 is just a little more than 16. One way to find x with more precision, though, is by using **logarithms**.

When you have solved other algebraic equations, you often relied on the idea that you can change both sides of the equation in the same way and still get a true equation. This is true with logarithms, too: If $x = y$, then $\log_b x = \log_b y$, no matter what b is.

Let's look at this with an equation whose solution you already know: $4^x = 16$. You can use either the **common log** or the **natural log**. In the following example, you will use the common log.

Example

Solve $4^x = 16$.

Solution

$4^x = 16$ $\log 4^x = \log 16$	Take the common logarithm of both sides. (Remember, when no base is written, that means the base is 10.)
$\log 4^x = \log 16$ $x \log 4 = \log 16$	What can you do with that new equation? Use the power property of logarithms to simplify the logarithm on the left side of the equation.
$x \log 4 = \log 16$ $x = \frac{\log 16}{\log 4}$	Remember that $\log 4$ is a number. You can divide both sides of the equation by $\log 4$ to get x by itself.
$x = \frac{1.204\ldots}{0.602\ldots} = 2$	Use a calculator to evaluate the logarithms and the quotient.

Just as you knew, $x = 2$. Now let's try it with our more difficult example, $4^x = 17$. The procedure is exactly the same.

Example

Solve $4^x = 17$.

Solution

$4^x = 17$ $\log 4^x = \log 17$	Take the common logarithm of both sides.
$\log 4^x = \log 17$ $x \log 4 = \log 17$	Use the power property of logarithms to simplify the logarithm on the left.
$x \log 4 = \log 17$ $x = \frac{\log 17}{\log 4}$	Divide both sides by $\log 4$ to get x by itself.
$x = \frac{1.230\ldots}{0.602\ldots} = 2.043\ldots$	Use a calculator to evaluate the logarithms and the quotient.

You could have used either the common log or the natural log with the example above. You use one of these two bases, as you can then use your calculator to find the values.

✓ Example

Solve $e^{2x}=54$.

Solution

$e^{2x}=54$ $\ln e^{2x}=\ln 54$	Since the base is e, use the natural logarithm. (If the base were 10, using common logarithms would be better.)
$\ln e^{2x}=\ln 54$ $2x=\ln 54$	Remember that logarithms and exponential functions are inverses. When you have $\log_b b^m$, the logarithm undoes the exponent, and the result is just m. So $\ln e^{2x}=\log_e e^{2x}=2x$.
$x=\frac{\ln 54}{2}$	Divide both sides by 2 to get x by itself.
$x=1.99449\ldots$	Use a calculator to evaluate the logarithm and quotient on the right and you're done!

Another kind of exponential equation has exponential expressions on both sides. When the bases are the same, or the exponents are the same, you can just compare the parts that are different. Look at these examples.

✓ Example

Solve $3^{2x+5}=3^{3x-2}$.

Solution

$3^{2x+5}=3^{3x-2}$	Here are two exponential expressions with the same base. If the two expressions are equal, then their exponents must be equal. (Think about that: if you have 3^a and 3^b, and $a\neq b$, then 3^a can't have the same value as 3^b.)
$2x+5=3x-2$	So, write a new equation that sets the exponents equal to each other.
$5=x-2$ $7=x$	Solve the linear equation as you normally would.
$3^{2(7)+5}=3^{3(7)-2}$ $3^{19}=3^{19}$	Test the solution in the original equation. No need to find 3^{19}. When both sides say the same thing, you know it's correct!

$x=7$

✓ Example

Solve $(x+4)^8=7^8$.

Solution

$(x+4)^8=7^8$	Again, you have two exponential expressions that are equal to each other. In this case, both sides have the same exponent, and this means the *bases* must be equal.
$x+4=7$	Write a new equation that sets the bases equal to each other.
$x=3$	Solve the linear equation as you normally would.

$(3+4)^8 = 7^8$ $7^8 = 7^8$	Test the solution in the original equation. No need to find 7^8. When both sides say the same thing, you know it’s correct!

$x = 3$

Exercise

Solve $10^{3x-2} = 13$.

A. $x = 5$
B. $x = 1.03798\ldots$
C. $x = 1.52164\ldots$
D. $x = 3.11394\ldots$

Answer

A. $x = 5$

Incorrect. You probably forgot to take the logarithm of 13 as well as of 10^{3x-2}. The correct answer is $x = 1.03798\ldots$.

B. $x = 1.03798\ldots$

Correct. 10^{3x-2} can be rewritten as $\log 10^{3x-2} = \log 13$, and $\log 10^{3x-2} = 3x-2$, so $3x-2 = \log 13$. This yields $x = \frac{\log 13+2}{3}$.

C. $x = 1.52164\ldots$

Incorrect. You probably used a natural logarithm on the right side $(\ln 13)$, but common logarithm on the left side $(\log 10^{3x-2} = 3x-2)$. The correct answer is $x = 1.03798\ldots$.

D. $x = 3.11394\ldots$

Incorrect. You probably took the logarithms correctly and added 2 to both sides, but you forgot to divide by 3. The correct answer is $x = 1.03798\ldots$.

Solving Logarithmic Equations

There are several strategies you can use to solve logarithmic equations. The first is one you have used before: Rewrite the logarithmic equation as an exponential equation!

Example

Solve $\ln x = 4.657$. Give x to the thousandths place.

Solution

$\ln x = 4.657$ $\log_e x = 4.657$ $e^{4.657} = x$	Remember that natural logarithms have a base of e. Rewrite this logarithm as an exponential equation.
$105.3196\ldots = x$ $x \approx 105.320$	Use a calculator to evaluate $e^{4.657}$, and round to the nearest thousandth.

This works regardless of the base.

Example

Solve $\log_7 x = 3.843$. Give x to the thousandths place.

Solution

$\log_7 x = 3.843$ $7^{3.843} = x$	Rewrite this logarithm as an exponential equation.
$1768.9345\ldots = x$ $x \approx 1768.935$	Use a calculator to evaluate $7^{3.843}$ and round to the nearest thousandth.

Logarithmic equations may also involve inputs where the variable has a coefficient other than 1, or where the variable itself is squared. In these cases, you need to complete a few more steps in solving for the variable.

✓ Example

Solve $\log_5 3x^2 = 1.96$. Give x to the hundredths place.

Solution

$5^{1.96} = 3x^2$	Rewrite this logarithmic equation as an exponential equation.
$23.44127\ldots = 3x^2$	Evaluate $5^{1.96}$.
$7.81375\ldots = x^2$ $x = \pm 2.7953\ldots$ $x \approx \pm 2.80$	Solve as you normally would. In this case, divide both sides by 3, then use the square root property to find the possible values for x. Don’t forget that when using the square root property, both positive and negative roots must be considered. Round to the nearest hundredth.
$\log_5 3x^2 = 1.96$ $\log_5 3(-2.80)^2 = 1.96$ $\log_5 3(7.84) = 1.96$ $\log_5 23.52 = 1.96$	Check your answer by substituting the value of x into the original equation. Because $(-2.80)^2$ and $(+2.80)^2$ are both positive, we don't need to check $+2.80$ separately.
$\frac{\log 23.52}{\log 5} = 1.96$ $\frac{1.3714\ldots}{0.6989\ldots} = 1.96$ $1.96 = 1.96$	Apply the change of base formula to switch from base 5 to base 10. The check shows that with rounding accounted for, a true statement results, so you know that the answer is correct.

$x \approx \pm 2.80$

The equations may also include more than one logarithm. You can use the properties of logarithms to combine these logarithms into one logarithm. **Note:** You’ll find it helpful to record which properties you use at each step, both to help you be sure you’re using them properly and as a way to help you find errors.

✓ Example

Solve $2\log 3 + \frac{1}{2}\log 16 - \log 3 = \log x$.

Solution

$2\log 3 + \frac{1}{2}\log 16 - \log 3 = \log x$	First notice that all of the logarithms have the same base. (These are common logarithms, so the bases are all 10). When using the properties, it is *absolutely necessary* that the bases are the same.
$\log 3^2 + \log 16^{\frac{1}{2}} - \log 3 = \log x$	Use the power property to rewrite $2\log 3$ as $\log 3^2$ and $\frac{1}{2}\log 16$ as $\log 16^{\frac{1}{2}}$.
$\log 9 + \log 4 - \log 3 = \log x$	Evaluate the exponents.
$\log(9 \cdot 4) - \log 3 = \log x$ $\log 36 - \log 3 = \log x$	Use the product property $\log_b MN = \log_b M + \log_b N$ to combine $\log 9 + \log 4$.

$\log\left(\frac{36}{3}\right)=\log x$ $\log 12=\log x$	Use the quotient property $\log_b\left(\frac{M}{N}\right)=\log_b M-\log_b N$ to combine $\log 36-\log 3$.
$x=12$	Since the logarithm of 12 and the logarithm of x are equal, x must equal 12.

? Exercise

Solve $\log x+\log 3=\log 24$.

A. $0.460\ldots$
B. $2.892\ldots$
C. 8
D. 21

Answer

A. $0.460\ldots$

Incorrect. Use the product property to combine $\log x+\log 3$ into one logarithm. The correct answer is 8.

B. $2.892\ldots$

Incorrect. Use the product property to combine $\log x+\log 3$ into one logarithm. The correct answer is 8.

C. 8

Correct. $\log x+\log 3=\log 3x$. So $\log 3x=\log 24$, $3x=24$ and $x=8$.

D. 21

Incorrect. Use the product property to combine $\log x+\log 3$ into one logarithm. The correct answer is 8.

Summary

There are several strategies that can be used to solve equations involving exponents and logarithms. Taking logarithms of both sides is helpful with exponential equations. Rewriting a logarithmic equation as an exponential equation is a useful strategy. Using properties of logarithms is helpful to combine many logarithms into a single one.

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18.4.2: Mathematical Modeling with Exponential and Logarithmic Functions

Learning Objectives

- Solve applied problems involving logarithmic functions.
- Solve applied problems involving exponential functions.

Introduction

You have seen many different kind of functions. You know that each one can be used to model some kind of situation in the real world. Exponential and logarithmic functions are no exception!

Using Logarithmic Functions

Much of the power of logarithms is their usefulness in solving exponential equations. Some examples of this include sound (decibel measures), earthquakes (Richter scale), the brightness of stars, and chemistry (pH balance, a measure of acidity and alkalinity).

Let’s look at the Richter scale, a logarithmic function that is used to measure the magnitude of earthquakes. The magnitude of an earthquake is related to how much energy is released by the quake. Instruments called seismographs detect movement in the earth; the smallest movement that can be detected shows on a seismograph as a wave with **amplitude** A_0.

A - the measure of the amplitude of the earthquake wave

A_0 - the amplitude of the smallest detectable wave (or standard wave)

From this you can find R, the Richter scale measure of the magnitude of the earthquake using the formula:

$$\boldsymbol{R}=\log\left(\frac{A}{A_0}\right)$$

The intensity of an earthquake will typically measure between 2 and 10 on the Richter scale. Any earthquakes registering below a 5 are fairly minor; they may shake the ground a bit, but are seldom strong enough to cause much damage. Earthquakes with a Richter rating of between 5 and 7.9 are much more severe, and any quake above an 8 is likely to cause massive damage. (The highest rating ever recorded for an earthquake is 9.5 during the 1960 Valdivia earthquake in Chile.)

Example

Problem: An earthquake is measured with a wave amplitude 392 times as great as A_0. What is the magnitude of this earthquake using the Richter scale, to the nearest tenth?

Answer

Use the Richter scale equation.

$R=\log\left(\frac{A}{A_0}\right)$

Since A is 392 times as large as A_0, $A=392A_0$. Substitute this expression in for A.

$\boldsymbol{R}=\log\left(\frac{392A_0}{A_0}\right)$

Simplify the expression $\left(\frac{392A_0}{A_0}\right)=392$. Use a calculator to evaluate the logarithm.

$$\begin{aligned}R&=\log 392\\R&=2.5932\ldots\\R&\cong 2.6\end{aligned}$$

The magnitude of this earthquake is 2.6 on the Richter scale.

A difference of 1 point on the Richter scale equates to a 10-fold difference in the amplitude of the earthquake (which is related to the wave strength). This means that an earthquake that measures 3.6 on the Richter scale has 10 times the amplitude of one that measures 2.6.

Let’s look back at the example just shown. In that example, the wave amplitude of the earthquake was 392 times normal. What if it were 10 times that, or 3,920 times normal? To find the measurement of that size earthquake on the Richter scale, you find $\log 3920$.

A calculator gives a value of 3.5932... or 3.6, when rounded to the nearest tenth. One extra point on the Richter scale can mean a lot more shaking!

Sound is measured in a logarithmic scale using a unit called a decibel. The formula looks similar to the Richter scale:

$$d = 10 \log \left(\frac{P}{P_0}\right)$$

where P is the power or intensity of the sound and P_0 is the weakest sound that the human ear can hear.

✓ Example

Problem: One hot water pump has a noise rating of 50 decibels. One dishwasher, however, has a noise rating of 62 decibels. The dishwasher noise is how many times more intense than the hot water pump noise?

Answer

You can't easily compare the two noises using the formula, but you can compare them to P_0. Start by finding the intensity of noise for the hot water pump. Use h for the intensity of the hot water pump's noise.

$50 = 10 \log \left(\frac{h}{P_0}\right)$

Divide the equations by 10 to get the log by itself.

$5 = \log \left(\frac{h}{P_0}\right)$

Rewrite the equation as an exponential equation.

$10^5 = \frac{h}{P_0}$

Multiply by P_0 to get h by itself.

$h = 10^5 P_0$

Repeat the same process to find the intensity of the noise for the dishwasher.

$$\begin{aligned} 62 &= 10 \log \left(\frac{d}{P_0}\right) \\ 6.2 &= \log \left(\frac{d}{P_0}\right) \\ 10^{6.2} &= \frac{d}{P_0} \\ d &= 10^{6.2} P_0 \end{aligned}$$

To compare d to h, you can divide. (Think: if the dishwasher's noise is twice as intense as the pump's, then d should be $2h$; that is, $\frac{d}{h}$ should be 2.) Use the laws of exponents to simplify the quotient.

$$\begin{aligned} \frac{d}{h} &= \frac{10^{6.2} P_0}{10^5 P_0} \\ &= 10^{1.2} \end{aligned}$$

The dishwasher's noise is $10^{1.2}$ (or about 15.85) times as intense as the hot water pump.

With decibels, every increase of 10 means the sound is 10 times more intense. An increase of 20 would be 10 times more intense for the first 10, and another 10 times more intense for the second 10; so a sound that is 75 decibels is 100 times more intense than a sound that is 55 decibels!

Here's one more example of logarithms used in scientific contexts. The measure of acidity of a liquid is called the pH of the liquid. This is based on the amount of hydrogen ions (H+) in the liquid. The formula for pH is:

$$pH = -\log[H+]$$

where [H+] is the concentration of hydrogen ions, given in a unit called mol/L ("moles per liter"; one mole is 6.022×10^{23} molecules or atoms).

Liquids with a low pH (down to 0) are more acidic than those with a high pH. Water, which is neutral (neither acidic nor alkaline, the opposite of acidic) has a pH of 7.0.

Example

Problem: If lime juice has a pH of 1.7, what is the concentration of hydrogen ions (in mol/L) in lime juice, to the nearest hundredth?

Answer

Use the formula for pH.

$pH = -\log[H+]$

Substitute the known pH into the formula, and represent H+ with the variable x.

$1.7 = -\log x$

If $1.7 = -\log x$ then $\log x = -1.7$.

$-1.7 = \log x$

Solve for x.

$$x = 10^{-1.7}$$
$$x = 0.02$$

The concentration of hydrogen ions in lime juice is 0.02.

Exercise

An earthquake monitoring station measured the amplitude of the waves during a recent tremor. It measured the waves as being 100,000 times as large as A_0, the smallest detectable wave. How high did this earthquake measure on the Richter scale?

$$R = \log\left(\frac{A}{A_0}\right)$$

A. 1
B. 3
C. 5
D. 9

Answer

A. Incorrect. An earthquake that measured 1 on the Richter scale would have a wave amplitude of only 10 times greater than normal. To solve this problem, setup the equation as $R = \log\left(\frac{100{,}000 A_0}{A_0}\right)$, and solve for R. The correct answer is 5.
B. Incorrect. An earthquake that measured 3 on the Richter scale would have a wave amplitude of 10^3, or 1,000 times greater than normal. To solve this problem, set up the equation as $\boldsymbol{R} = \log\left(\frac{100{,}000 A_0}{A_0}\right)$, and solve for R. The correct answer is 5.
C. Correct. To find the Richter measure of an earthquake with waves 100,000 times greater than normal, solve $\boldsymbol{R} = \log\left(\frac{100{,}000 A_0}{A_0}\right)$. The answer is 5.
D. Incorrect. An earthquake that measured 9 on the Richter scale would have a wave amplitude of 1,000,000,000 times greater than normal. To solve this problem, set up the equation as $\boldsymbol{R} = \log\left(\frac{100{,}000 A_0}{A_0}\right)$, and solve for R. The correct answer is 5.

Using Exponential Functions

Exponential functions are used for even more contexts, including population and bacterial growth, radioactive decay, compound interest, cooling of objects, and growth of phenomena such as virus infections, Internet usage, and popularity of fads.

For example, recall that the formula for compound interest is $A = P\left(1 + \frac{r}{m}\right)^{mt}$, where P is principal, A is amount, r is the annual rate, m is the number of compounding periods, and t is the number of years.

Example

Problem: How long will it take, to the nearest whole year, for money to double if it is invested at 10% compounded monthly?

Answer

Find values for the variables that you can. Note that you don't know either P or A, but you know the relationship between P and A.

P = initial investment

A = $2P$ (since you want the money to double)

r = 0.1 (10% written as a decimal)

m = 12 (12 compound periods per year)

t = is the value you are looking for.

Use the formula.

$2P = P\left(1+\frac{0.1}{12}\right)^{12t}$

Divide both sides by P to get the exponential expression by itself. Note: This means P can't be 0, since you can't divide by 0. But if P were 0, no money would actually be invested and the problem wouldn't make sense anyway. So it's fine to assume P >0.

$2=\left(1+\frac{0.1}{12}\right)^{12t}$

To solve an equation with the variable in the exponent, take logarithms of both sides. You can use any base, so $\log$ (base 10) or $\ln$ (base e) would be best, then you can use a calculator to evaluate the logarithms.

$\ln 2=\ln\left(1+\frac{0.1}{12}\right)^{12t}$

Use the power property of logarithms to get the variable out of the exponent.

$\ln 2=12t\cdot\ln\left(1+\frac{0.1}{12}\right)$

Divide to get t by itself.

$\frac{\ln 2}{12\ln\left(1+\frac{0.1}{12}\right)}=t$

Use the calculator to evaluate the logarithms. Round to the nearest whole year.

$$\begin{aligned}\frac{\ln 2}{12\ln\left(1+\frac{0.1}{12}\right)}&=t\\ \frac{0.6931\ldots}{12(0.008298\ldots)}&=t\\ 6.96&\approx t\end{aligned}$$

The money will double in about 7 years.

According to the U.S. Census Bureau, the world population in 2011 was about 6.9 billion people, and would grow about 76 million during the year. That is, it would increase about 1.1%.

If the world population grows by 1.1% every year, then each year the population is multiplied by 1.011. (The 1 represents the current population, and the .011 represents the new growth.) After two years, the population would be $6.9(1.011)^2$, and after three years it would be $6.9(1.011)^3$. In general, the world population P (in billions of people) could be estimated for t years after 2011 by this formula:

$$P=6.9(1.011)^t$$

Example

Problem: Using the world population formula $P=6.9(1.011)^t$, where t is the number of years after 2011 and P is the world population in billions of people, estimate:

A) the population in the year 2050 to the nearest hundred million, and

B) by what year the population will be double what it was in 2011.

Answer

Looking at part A first, identify the variables. In this case, 2050 is 39 years after 2011, so $t = 39$.

Part A

t=2050-2011=39. P is what you are looking for.

Use the formula and a calculator to evaluate the exponential expression. (Note that “hundred million” is a tenth of a billion, so to the nearest hundred million is 10.6.) According to the problem, the formula finds P in billions of people, so you have 10.6 billion people.

$$\begin{aligned} P &= 6.9(1.011)^{39} \\ P &= 10.57\ldots \\ P &\approx 10.6 \end{aligned}$$

For part B, you want the population to double the 6.9 from 2011, so $P = 13.8$.

Part B

t is what you are looking for.

$P = 2(6.9) = 13.8$

Use the formula and the value for P.

$13.8 = 6.9(1.011)^t$

Divide by 6.9 to get the exponential expression by itself.

$2 = 1.011^t$

Since the variable t is an exponent, take logarithms of both sides. You can use any base, but base 10 or e will allow you to use the calculator easily.

$\log 2 = \log (1.011)^t$

Use the power property of logarithms to get the variable out of the exponent, and then solve for t. The population will hit the doubling point about halfway through the 63rd year, so you will round up to 64 (since the question asks by what year).

$$\begin{aligned} \log 2 &= t \log 1.011 \\ \frac{\log 2}{\log 1.011} &= t \\ \frac{0.3010\ldots}{0.00475\ldots} &= t \\ 63.359 &= t \end{aligned}$$

It will take about 64 years for the population to double, so you have to add 64 to 2011 to estimate the year the world population will be 13.8 billion people.

$2011 + 64 = 2075$

Be sure to answer the questions asked.

A) The population in 2050 will be about 10.6 billion people.

B) The population will double (13.8 billion people) the population by the year 2075.

? Exercise

A particular bacterial colony doubles its population every 15 hours. A scientist running an experiment is starting with 100 bacteria cells. She expects the number of cells to be given by the formula $c = 100(2)^{\frac{t}{15}}$, where t is the number of hours since the experiment started.

After how many hours would the scientist expect to have 300 bacteria cells? Give your answer to the nearest hour.

A. 2 hours
B. 24 hours

C. $1,048,577$ hours
D. $104,857,699$ hours

Answer

A. Incorrect. You may have forgotten that the exponent is $\frac{t}{15}$ rather than just t. The correct answer is 24 hours.
B. Correct. 24 hours is the solution to $300=100(2)^{\frac{t}{15}}$.
C. Incorrect. You need to find the solution to $300=100(2)^{\frac{t}{15}}$. The correct answer is 24 hours.
D. Incorrect. You need to find the solution to $300=100(2)^{\frac{t}{15}}$. The correct answer is 24 hours.

Summary

Logarithmic and exponential functions can be used to model real-world situations. Logarithmic functions are very helpful when working with phenomena that have a very wide range of values, because they allow you to keep the values you actually work with in a smaller range. Exponential functions are helpful with phenomena that change very quickly, or that grow or decay by a percentage over a particular time period.

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CHAPTER OVERVIEW

19: Trigonometry

19.1: Introduction to Trigonometric Functions
  19.1.1: Identifying the Six Trigonometric Functions
  19.1.2: Right Triangle Trigonometry
  19.1.3: Unit Circle Trigonometry
19.2: Graphing Trigonometric Functions
  19.2.1: Degree and Radian Measure
  19.2.2: Graphing the Sine and Cosine Functions
  19.2.3: Amplitude and Period

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SECTION OVERVIEW

19.1: Introduction to Trigonometric Functions

Topic hierarchy

19.1.1: Identifying the Six Trigonometric Functions

19.1.2: Right Triangle Trigonometry

19.1.3: Unit Circle Trigonometry

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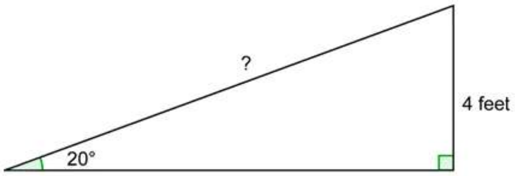
19.1.1: Identifying the Six Trigonometric Functions

Learning Objectives

- Identify the hypotenuse, adjacent side, and opposite side of an acute angle in a right triangle.
- Determine the six trigonometric ratios for a given angle in a right triangle.
- Recognize the reciprocal relationship between sine/cosecant, cosine/secant, and tangent/cotangent.
- Use a calculator to find the value of the six trigonometric functions for any acute angle.
- Use a calculator to find the measure of an angle given the value of a trigonometric function.

Introduction

Suppose you want to build a ramp for access to a loading dock that is 4 feet above ground level. You want to be able to easily push a cart on wheels up the ramp, and the angle of elevation should be no more than 20°. How long does the ramp have to be?

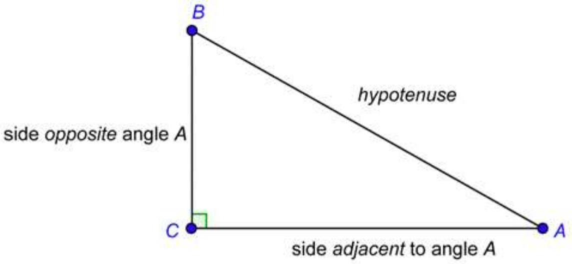


In this diagram, you have a right triangle for which you know the length of one side and the measure of an acute angle. You want to find the length of the **hypotenuse**. You may know that the Pythagorean Theorem enables you to find the length of one side of a right triangle, given the lengths of the other two sides. Now you will learn trigonometry, which is a branch of mathematics that studies the relationship between angles and the sides of triangles. In fact, trigonometry will allow you to find unknown side lengths and angle measures in right triangles in a variety of cases, such as in the problem above.

The Sides of a Right Triangle

In the example above, one of the acute angles has a measure of 20°. You could describe the side (or leg of the right triangle) with length 4 feet as the height of the triangle, or you could say that it is "opposite" the 20° angle. The other leg is said to be "adjacent" to the 20° angle. In trigonometry, this type of relationship between sides and angles is very important.

The general relationship between sides and angles is shown in the diagram below.

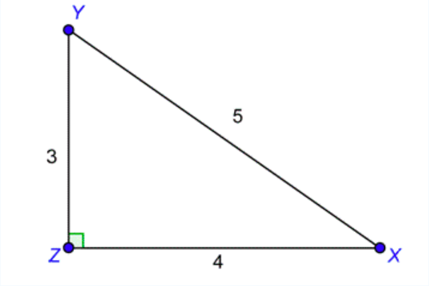


Angle A is formed by the hypotenuse and leg $\overline{CA}$. We say that leg $\overline{CA}$is the side adjacent to angle A. We say that leg $\overline{BC}$ is the side opposite angle A. In other words, the **adjacent side** is the leg that is part of the angle; the **opposite side** is the leg that is not part of the angle.

Example

Problem: What is the length of the side opposite angle X and the length of the side adjacent to angle X?

Answer



The side opposite angle X is $\overline{YZ}$. Its length is 3. The side adjacent to angle X is $\overline{ZX}$. Its length is 4.

opposite side length: 3

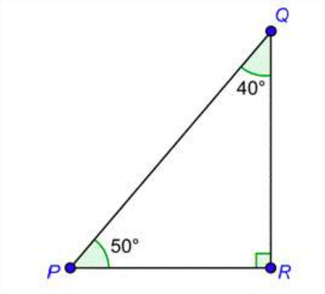
adjacent side length: 4

Keep in mind that the labels “opposite” and “adjacent” depend on which angle you are talking about. The side opposite an angle does not need to be the height of the triangle. Consider the following example:

✓ Example

Problem: What is the name of the side opposite the 40° angle and the name of the side adjacent to the 40° angle?

Answer



The 40° angle is formed by the hypotenuse and $\overline{QR}$, so $\overline{QR}$ is the adjacent side. Since $\overline{PR}$ is not a part of the 40° angle, it is the opposite side.

opposite side: $\overline{PR}$

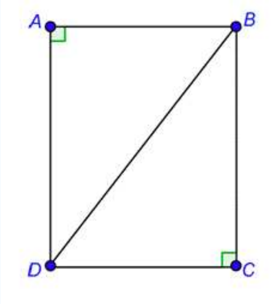
adjacent side: $\overline{QR}$

Each leg in a right triangle is adjacent to one of the acute angles and opposite the other acute angle.

✓ Example

Problem: In $\triangle AB$, side $\overline{DA}$ is adjacent to which angle and opposite which angle?

Answer



Side $\overline{DA}$ and the hypotenuse $\overline{DB}$ together form $\angle ADB$. So $\overline{DA}$ is adjacent to $\angle ADB$. Since $\overline{DA}$ is not part of acute angle $\angle ABD$, $\overline{DA}$ is the side opposite $\angle ABD$.

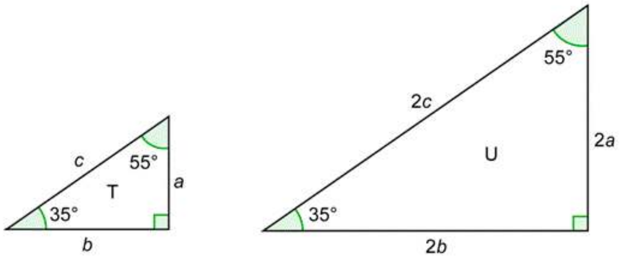
adjacent to $\angle ADB$

opposite $\angle ABD$

The Six Trigonometric Ratios

Suppose your professor asks you and another student to draw a triangle with angle measures 35°, 55°, and 90°. You and your friend will probably draw triangles of different sizes. However, because the triangles will have the same angle measures, they will be similar.

Remember that this means that the triangles will have proportional corresponding sides. For example, one triangle might have sides that are all twice as long as the sides of the other, as seen below.



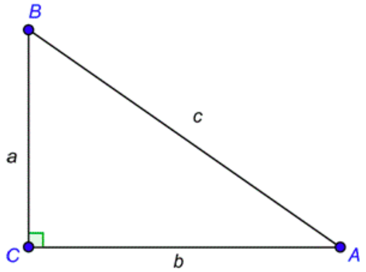
Now suppose that each of you has been asked to find the ratio of the side opposite the 35° angle over the hypotenuse. Even though you are using different triangles and will have different numbers in the numerator and denominator, you will still end up with the same result. The two of you will get:

$$\frac{\text{opposite } 35^{\circ} \text{ in T}}{\text{hypotenuse of T}} = \frac{a}{c} \text{ and } \frac{\text{opposite } 35^{\circ} \text{ in U}}{\text{hypotenuse of U}} = \frac{2a}{2c} = \frac{a}{c}$$

The two ratios are the same because the 2s cancel. If you draw a triangle with the same angles and sides that are three times as long as those of triangle T, the ratio of the side opposite 35° over the hypotenuse will be $\frac{3a}{3c} = \frac{a}{c}$. This ratio will be the same for all similar triangles, and this ratio is called the **sine** of 35°. Sine is abbreviated as $\sin 35^{\circ}$.

The same type of result will happen if you use other ratios of sides. For example, if you take the ratio of the side adjacent to 35° over the hypotenuse, you will get $\frac{b}{c}$ no matter which of the above triangles you use.

In addition to the sine ratio, there are five other ratios that you can compute: cos, tan, cot, sec, and csc. Just as *sin* is an abbreviation for sine, *cos* is short for **cosine**, *tan* is short for **tangent**, *csc* is short for cosecant, *sec* is short for secant, and *cot* is short for cotangent. (When reading these abbreviations aloud, you need to say the complete word.) These six ratios will help you find unknown side lengths and unknown angle measures in right triangles. Let's look at definitions for these six ratios, starting with a typical right triangle like the one below.



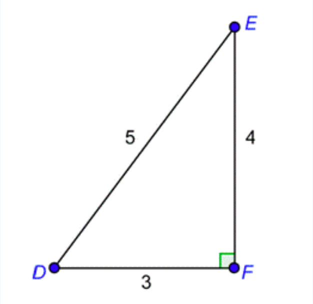
The definitions are as follows:

$$\begin{array}{rr}
\sin A=\dfrac{\text{opposite}}{\text{hypotenuse}}=\dfrac{a}{c} & \csc A=\dfrac{\text{hypotenuse}}{\text{opposite}}=\dfrac{c}{a}\\
\cos A=\dfrac{\text{adjacent}}{\text{hypotenuse}}=\dfrac{b}{c} & \sec A=\dfrac{\text{hypotenuse}}{\text{adjacent}}=\dfrac{c}{b}\\
\tan A=\dfrac{\text{opposite}}{\text{adjacent}}=\dfrac{a}{b} & \cot A=\dfrac{\text{adjacent}}{\text{opposite}}=\dfrac{b}{a}
\end{array}$$

Given these definitions, let’s practice applying them.

✓ Example

Problem: Determine the six trigonometric ratios for angle D in the right triangle below.



Answers

The first thing you need to do is recognize that $\overline{EF}$ is opposite angle D and $\overline{DF}$ is adjacent to angle D. Then write down their lengths.

length of side opposite D = 4

length of side adjacent to D = 3

length of hypotenuse = 5

Substitute these values into the definitions for the six ratios.

$$\begin{array}{r}
\sin D=\frac{\text{opposite}}{\text{hypotenuse}}=\frac{4}{5}\\
\cos D=\frac{\text{adjacent}}{\text{hypotenuse}}=\frac{3}{5}\\
\tan D=\frac{\text{opposite}}{\text{adjacent}}=\frac{4}{3}\\
\csc D=\frac{\text{hypotenuse}}{\text{opposite}}=\frac{5}{4}\\
\sec D=\frac{\text{hypotenuse}}{\text{adjacent}}=\frac{5}{3}\\
\cot D=\frac{\text{adjacent}}{\text{opposite}}=\frac{3}{4}
\end{array}$$

$\sin D=\frac{4}{5}$, $\cos D=\frac{3}{5}$, $\tan D=\frac{4}{3}$

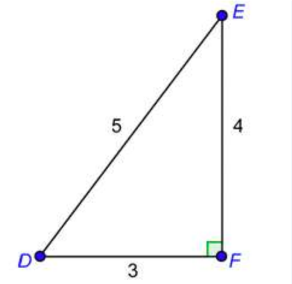
$\csc D=\frac{5}{4}$, $\sec D=\frac{5}{3}$, $\cot D=\frac{3}{4}$

Notice that the values of sine and cosine are between 0 and 1. You found them by dividing the length of a leg by the hypotenuse. The hypotenuse is the longest side, so the numerator is less than the denominator. That means the output of the sine or cosine function is always less than 1.

Keep in mind that the opposite side for one acute angle is the adjacent side of the other acute angle. In the example above, side EF was the opposite side for angle D. But, as you’ll see in the next example, it will be the adjacent side for angle E.

✓ Example

Problem: Determine the six trigonometric ratios for angle in the right triangle below.



Answer

This is the same triangle that you saw in the previous example, so the hypotenuse is the same. The difference is that you are looking at the triangle from the perspective of angle E instead of angle D. So the opposite and adjacent sides switch places. That is, $\overline{EF}$ is adjacent to angle E and $\overline{DF}$ is opposite to angle E.

length of side opposite E = 3

length of side adjacent to E = 4

length of hypotenuse = 5

Substitute the new values into the definitions for the six ratios.

$$\begin{aligned}\sin E &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5}\\ \cos E &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5}\\ \tan E &= \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}\\ \csc E &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3}\\ \sec E &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4}\\ \cot E &= \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}\end{aligned}$$

$\sin E = \frac{3}{5}$, $\cos E = \frac{4}{5}$, $\tan E = \frac{3}{4}$

$\csc E = \frac{5}{3}$, $\sec E = \frac{5}{4}$, $\cot E = \frac{4}{3}$

If you compare the answers to the last two examples, you will see the following:

$$\sin D = \frac{4}{5} = \cos E$$

These two **trigonometric functions** are equal because the opposite side to angle D (which is 4) is the adjacent side to angle E. Because they are the two acute angles in a right triangle, D and E are complementary. That is:

$$D + E = 90^\circ$$

or

$$E = 90^\circ - D$$

Substitute this into the equation above:

$$\sin D = \cos(90^\circ - D)$$

Again, the reason these two functions are equal is that the opposite side to one acute angle is the adjacent side to the other acute angle. This is true in any right triangle. So if A is *any* acute angle, it is always true that:

$$\sin A = \cos(90^\circ - A)$$

Comparing more answers from the last two examples, you can find these relationships:

$$\sec D = \frac{5}{3} = \csc E and \tan D = \frac{4}{3} = \cot E$$

You get these equalities because (1) the adjacent side to angle D is 3, while this is the opposite side to angle E, and (2) the opposite side to angle D is 4, while this is the adjacent side to angle E. These are examples of the general relationship we have stated: the opposite side to one acute angle is the adjacent side to the other acute angle. Using the same reasoning as above, if A is *any* acute angle, it is always true that:

$$\begin{aligned} \sin A &= \cos(90^\circ - A) \\ \sec A &= \csc(90^\circ - A) \\ \tan A &= \cot(90^\circ - A) \end{aligned}$$

An equation, such as any of the three above, that is true for any value of the variable is called an **identity**. Note the full names of these functions: sine and cosine, secant and cosecant, tangent and cotangent. These pairs are referred to as **cofunctions**. The angles A and $90^\circ - A$ are complementary. In other words, the cofunctions of any pair of complementary angles are equal.

You can use these relationships to find values of trigonometric functions from values of other functions without drawing a triangle. Note that you can replace A and $90^\circ - A$ by B and $90^\circ - B$. The different letter will not change the relationship, because these angles are still complementary.

✓ Example

Problem: A right triangle has acute angles A and B. If $\sin A = \frac{7}{25}$ and $\tan B = \frac{24}{7}$, what are $\cos B$ and $\cot A$?

Answer

Since A and B are the acute angles in a right triangle, they are complementary angles.

$$\begin{aligned} A + B &= 90^\circ \\ &\text{or} \\ B &= 90^\circ - A \\ &\text{or} \\ A &= 90^\circ - B \end{aligned}$$

Substitute for B. Use the identity (the cofunctions are equal). Substitute the given value.

$$\begin{aligned} \cos B &= \cos(90^\circ - A) \\ &= \sin A \\ &= \frac{7}{25} \end{aligned}$$

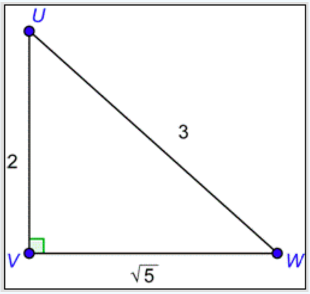
Substitute for A. The cofunctions of any pair of complementary angles are equal. Substitute the given value.

$$\begin{aligned} \cot A &= \cot(90^\circ - B) \\ &= \tan B \\ &= \frac{24}{7} \end{aligned}$$

\(\cos B = \frac{7}{25} , \cot A = \frac{24}{7}

? Exercise

What are the values of $\cos U$ and $\csc U$?



A. $\cos U = \frac{\sqrt{5}}{3}$ and $\csc U = \frac{3}{2}$
B. $\cos U = \frac{3}{2}$ and $\csc U = \frac{\sqrt{5}}{3}$
C. $\cos U = \frac{2}{3}$ and $\csc U = \frac{3}{\sqrt{5}}$
D. $\cos U = \frac{3}{\sqrt{5}}$ and $\csc U = \frac{2}{3}$

Answer

A. $\cos U = \frac{\sqrt{5}}{3}$ and $\csc U = \frac{3}{2}$. Incorrect. You probably used the acute angle W, and found $\cos W$ and $\csc W$. Remember that you get different ratios for the two acute angles, so pay careful attention to which angle you are using. The correct answer is C.
B. $\cos U = \frac{3}{2}$ and $\csc U = \frac{\sqrt{5}}{3}$. Incorrect. You may have used the acute angle W and also switched cosine and cosecant. Remember that you get different ratios for the two acute angles, so pay careful attention to which angle you are using. The correct answer is C.
C. $\cos U = \frac{2}{3}$ and $\csc U = \frac{3}{\sqrt{5}}$. Correct. Using the definition of cosine, $\cos U = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2}{3}$. Using the definition of cosecant, $\csc U = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{3}{\sqrt{5}}$.
D. $\cos U = \frac{3}{\sqrt{5}}$ and $\csc U = \frac{2}{3}$. Incorrect. It looks like you switched the values of cosine and cosecant. The names are very similar, so be careful to use the right definition. The correct answer is C.

Relationships Among the Trigonometric Functions

The six ratios or functions are usually thought of as two groups of three functions. The first group is:

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{o}{h}, \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{h}, \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{o}{a}$$

One way to remember these three definitions is with a memory device that uses the first letter of each word. The definition of sine is represented by *soh* (sine equals opposite over hypotenuse). Likewise, the definition of cosine is represented by cah (cosine equals adjacent over hypotenuse), and the definition of tangent is represented by toa (tangent equals opposite over adjacent). Putting these together gives you *sohcahtoa*.

The second group is:

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{h}{o}, \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{h}{a}, \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{o}$$

If you compare these three ratios to the three above them, you'll see that these three fractions are the reciprocals of the three fractions above them. That is, cosecant is the reciprocal of sine, secant is the reciprocal of cosine, and cotangent is the reciprocal of tangent. Writing this gives three more identities:

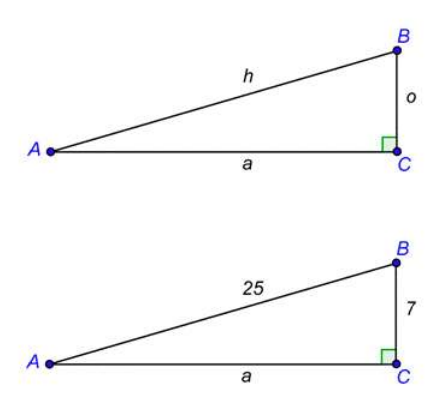
$\csc A = \frac{1}{\sin A}$	$\sec A = \frac{1}{\cos A}$	$\cot A = \frac{1}{\tan A}$

If you remember *sohcahtoa* plus these three identities, you can find the values of any trigonometric functions, as seen in the following example.

✓ Example

Problem: For acute angle A, $\sin A = \frac{7}{25}$ and $\cos A = \frac{24}{25}$. Find the values of the other four trigonometric ratios for angle A.

Answer



The definition of sine tells you that $\sin A = \frac{o}{h} = \frac{7}{25}$. A triangle with o = 7 and h = 25 will have this ratio.

You also know that $\cos A = \frac{a}{h} = \frac{a}{25}$. You are given $\cos A = \frac{24}{25}$, so Extra close brace or missing open brace.

Now you have all three sides of the triangle and you can use the definition of tangent.

$\tan A = \frac{o}{a} = \frac{7}{24}$

Next, use the three reciprocal identities to obtain the other three ratios.

$$\csc A = \frac{1}{\sin A} = \frac{1}{\frac{7}{25}} = \frac{25}{7}$$
$$\sec A = \frac{1}{\cos A} = \frac{1}{24} = \frac{25}{24}$$
$$\cot A = \frac{1}{\tan A} = \frac{1}{\frac{7}{24}} = \frac{24}{7}$$

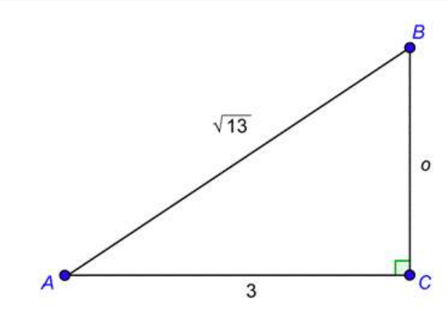
$\tan A = \frac{7}{24}, \csc A = \frac{25}{7}, \sec A = \frac{25}{24}, \cot A = \frac{24}{7}$

The value of any trigonometric function is a ratio, or a fraction. Remember that fractions can be reduced.

✓ Example

Problem: For acute angle A, $\cos A = \frac{3}{\sqrt{13}}$ and $\tan A = \frac{4}{6}$. Find the values of $\sin A$ and $\csc A$.

Answer



You want a right triangle where the ratio of the side adjacent to angle A over the hypotenuse is $\frac{3}{\sqrt{13}}$. A triangle with sides $a = 3$ and $h = \sqrt{13}$ would have this ratio.

You can use the definition of tangent to find the opposite side. Substitute the value you are given for tangent and then solve the equation.

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{o}{3}$$
$$\frac{4}{6} = \frac{o}{3}$$
$$\frac{12}{6} = o$$
$$2 = o$$

Now you have all three sides. Use the definition of sine to find its value.

$\sin A = \frac{o}{h} = \frac{2}{\sqrt{13}}$

Now using the reciprocal identity, the $\csc$ can be found by taking the reciprocal of the $\sin$.

$\sin A = \frac{2}{\sqrt{13}}, \csc A = \frac{\sqrt{13}}{2}$

Remember that the sides of a right triangle satisfy the Pythagorean Theorem. So if a and b are the lengths of the legs, and c is the hypotenuse, you must have $a^2 + b^2 = c^2$. In the last example, the lengths of the legs were 2 and 3, and the hypotenuse was $\sqrt{13}$, and it is true that $2^2 + 3^2 = \sqrt{13}^2$.

? Exercise

Which of the following could be the values of the trigonometric functions of the same angle?

A. $\sin W = \frac{1}{\sqrt{5}}, \tan W = \frac{1}{2}, \csc W = \frac{\sqrt{5}}{2}$

B. $\sin X=\frac{4}{5}, \cos X=\frac{2}{5}, \sec X=\frac{5}{2}$
C. $\sin Y=\frac{12}{13}, \cos Y=\frac{5}{13}, \tan Y=\frac{5}{12}$
D. $\tan Z=1, \cos Z=\frac{1}{\sqrt{2}}, \cot Z=1$

Answer

A. $\sin W=\frac{1}{\sqrt{5}}, \tan W=\frac{1}{2}, \csc W=\frac{\sqrt{5}}{2}$. Incorrect. You can have these values of sine and tangent for the same angle. However, the values of sine and cosecant of the same angle are reciprocals. If $\sin W=\frac{1}{\sqrt{5}}$, then $\csc W=\frac{\sqrt{5}}{1}$, not $\frac{\sqrt{5}}{2}$. The correct answer is D.
B. $\sin X=\frac{4}{5}, \cos X=\frac{2}{5}, \sec X=\frac{5}{2}$. Incorrect. The values of cosine and secant are reciprocals, as they should be. However, you cannot have the given values of sine and cosine for the same angle. If $\sin X=\frac{4}{5}$, you can draw a right triangle with the leg opposite angle X having length 4 and the hypotenuse having length 5. If you have $\cos X=\frac{2}{5}$, then the adjacent leg length is 2. However, the lengths 2, 4, and 5 do not satisfy the Pythagorean Theorem. The correct answer is D.
C. $\sin Y=\frac{12}{13}, \cos Y=\frac{5}{13}, \tan Y=\frac{5}{12}$. Incorrect. You can draw a right triangle with the side opposite angle Y having length 12, the adjacent side having length 5, and the hypotenuse having length 13. This will give you $\sin Y=\frac{12}{13}$ and $\cos Y=\frac{5}{13}$. Using the definition of tangent, $\tan Y=\frac{\text{opposite}}{\text{adjacent}}$, you would then have $\tan Y=\frac{12}{5}$, not $\frac{5}{12}$. The correct answer is D.
D. $\tan Z=1, \cos Z=\frac{1}{\sqrt{2}}, \cot Z=1$. Correct. If $\tan Z=1$, then $\cot Z=1$, because they are reciprocals. You can draw a right triangle with both legs having length 1, and the hypotenuse will have length $\sqrt{2}$ because of the Pythagorean Theorem. Using the definition of cosine, $\cos Z=\frac{\text{adjacent}}{\text{hypotenuse}}$, you will then have $\cos Z=\frac{1}{\sqrt{2}}$.

Using a Calculator to Find the Values of Trigonometric Functions

You know that if you draw similar triangles with angle measures 35°, 55°,and 90°, the ratio of the side opposite 35° to the hypotenuse will be the same for all those triangles. This is $\sin 35^{\circ}$. The easiest way to find what this ratio actually equals is with a scientific or graphing calculator.

Looking at a calculator, you will find a key that says SIN on it. You can use this to find the value of $\sin 35^{\circ}$. Keep this in mind: you need to know that there are different units for measuring angles. For our purposes, make sure that your calculator is set in the "degree mode." (The following instructions are generalized, but you may need to refer to your calculator's instruction manual for how to perform these calculations on your particular calculator.)

If you use a scientific calculator, look in the display and see if it says DEG in small letters above the 0 (as opposed to RAD or GRAD). If it does not, press the DRG key until the display says DEG. Now enter 35, and then press the SIN key. The result is $\sin 35^{\circ}$:

$$\sin 35^{\circ}=0.573576436$$

If you have a graphing calculator, press the MODE key. The third line of the display will say RADIAN DEGREE. Use the arrows to select DEGREE, then press ENTER, 2ND, QUIT. Now the calculator is in degree mode. On a graphing calculator, you enter things the same way as you would write them. So press the keys to give you $\sin(35)$ on the display and then press ENTER. You should now see the value on the next line of the display.

Because sine is a function, given an angle measure X (the input), your calculator will give you the value of $\sin X$ (the output). All the right triangles with acute angle measure X will be similar, so the ratio of the opposite side to the hypotenuse will be the same for all of those triangles. Therefore, the ratio depends only on the value of X; it does not depend on the triangle.

Likewise, the other five trigonometric ratios are functions. You can use your calculator to find the value of those functions. You will notice that next to the SIN key there are COS and TAN keys, which can be used to find the values of cosine and tangent.

✓ Example

Problem: Use your calculator to find the values of $\cos 35^{\circ}$ and $\tan 35^{\circ}$ to the nearest thousandth.

Answer

On a scientific calculator, enter 35, then press COS. Do this in the reverse order for a graphing calculator.

$\cos 35^{\circ}=0.819152044$

Remember to look at the ten thousandths place to help you round to the nearest thousandth.

$\cos 35^\circ \approx 0.819$

Use the same procedure for tangent.

$$\begin{aligned}\tan 35^\circ &= 0.700207538\\ &\approx 0.700\end{aligned}$$

$\cos 35^\circ \approx 0.819 \text{ and } \tan 35^\circ \approx 0.700$

You may have noticed that your calculator has no keys for $\csc$, $\sec$, or $\cot$. You can still use it to find the values of these functions. You can do this by using the calculator in combination with the reciprocal identities. You must first find the value of $\sin$, $\cos$, or $\tan$, and then find the reciprocal, as this next example shows.

✓ Example

Problem: Use your calculator to find the values of $\csc 50^\circ$, $\sec 50^\circ$, and $\cot 50^\circ$ to the nearest thousandth.

Answer

First use your calculator to find the value of $\sin 50^\circ$. Do not round this value until you are writing the final answer.

$\sin 50^\circ = 0.766044443$

Press the key that says $\frac{1}{x}$ or x^{-1}. This will give you the value of cosecant.

$\csc 50^\circ = \frac{1}{\sin 50^\circ} = 1.305407289$

Now round your final answer to the nearest thousandth.

$\csc 50^\circ \approx 1.305$

Find the value of $\cos 50^\circ$. Then find the reciprocal and round off.

$$\begin{aligned}\cos 50^\circ &= 0.64278761\\ \sec 50^\circ &= \frac{1}{\cos 50^\circ}\\ &= 1.555723827\\ &\approx 1.556\end{aligned}$$

Find the value of $\tan 50^\circ$. Then find the reciprocal and round off.

$$\begin{aligned}\tan 50^\circ &= 1.191753593\\ \cot 50^\circ &= \frac{1}{\tan 50^\circ}\\ &= 0.839099631\\ &\approx 0.839\end{aligned}$$

$\csc 50^\circ \approx 1.305, \sec 50^\circ \approx 1.556, \text{ and } \cot 50^\circ \approx 0.839$

? Exercise

What is the value of $\csc 17^\circ$ to the nearest thousandth?

1. 3.420
2. 1.046
3. 0.001
4. -1.040

Answer

1. 3.420. Correct. First use the calculator to find $\sin 17^\circ = 0.292371705$. Find the reciprocal of this: $\csc 17^\circ = 3.42030362$ ≈ 3.420
2. 1.046. Incorrect. You found the value of $\sec 17^\circ$. Remember that cosecant is the reciprocal of sine (not cosine). The correct answer is 3.420.
3. 0.001. Incorrect. You found the value of $\sin\left(\frac{1}{17^\circ}\right)$. You must first find $\sin 17^\circ$, *then* find the reciprocal. The correct answer is 3.420.

4. -1.040. Incorrect. Your calculator was not set to degrees. Before doing any calculations, make sure that you first have it set on degrees. The correct answer is 3.420.

Using a Calculator to Find Angle Measures

So far you have learned the definitions of the six trigonometric functions. Remember that a function has an input and an output. For each of these functions, the input is the angle measure and the output equals a certain ratio of sides. Your calculator can be used to find the values of these functions. For example, if the angle measures 60°, the cosine of the angle is 0.5. This can be represented as $\cos 60^\circ = 0.5$.

Now what if the situation were reversed? What if you knew the value of the ratio and wanted to know the angle that produced it? That is, what if you knew the output of a trigonometric function, and wanted to know the input? For example, you might know that the cosine of some angle is 0.5 and want to find out what the angle is. You can use your calculator to find these values, too.

In general, when you reverse the input and the output of a function, what you get is called an **inverse function**. Your calculator can find the inverses of sine, cosine, and tangent. In the example above, on a scientific calculator you would enter 0.5, press the 2ND key, then press COS. The display would show 60. (Make sure that your calculator is set on degrees!) This tells you that the angle is 60°. On a graphing calculator, you would press 2ND, then COS, then 0.5, and finally ENTER. (Keep in mind that you may need to refer to your calculator's instruction manual for how to perform these calculations on your particular calculator.)

Above the SIN, COS, and TAN keys you will see $\sin^{-1}, \cos^{-1}$ and $\tan^{-1}$. These are the inverse trigonometric functions, and the way to read them out loud is: *arcsine*, *arccosine*, and *arctangent*. The result mentioned above can be written as $\cos^{-1} 0.5 = 60^\circ$ or $\arccos 0.5 = 60^\circ$.

If you were given the value of the sine (or tangent) function and wanted to know what angle produced it, you would follow a procedure similar to that described above. So on a scientific calculator, you would enter the value, press the 2ND key, then press SIN (or TAN).

✓ Example

Problem: Use your calculator to find the angle, to the nearest degree, whose tangent value is 0.75.

Answer

On a scientific calculator, enter 0.75, then press the 2ND key and TAN. Do this in the reverse order for a graphing calculator.

$\tan^{-1} 0.75 = 36.86989765^\circ$

Look at the tenths place to help you round to the nearest degree.

$36.86989765^\circ \approx 37^\circ$

$\tan^{-1} 0.75 \approx 37^\circ$

If you are given the expression $\cos^{-1} 0.24$, for example, you can interpret this as saying, "Find the angle whose cosine equals 0.24."

✓ Example

Problem: Determine $\cos^{-1} 0.24$ to the nearest tenth of a degree.

Answer

On a scientific calculator, enter 0.24, then press the 2ND key and COS. Do this in the reverse order for a graphing calculator.

$\cos^{-1} 0.24 = 76.11345964^\circ$

Look at the hundredths place to help you round to the nearest tenth.

$76.11345964^\circ \approx 76.1^\circ$

Remember that this means $\cos 76.1^\circ \approx 0.24$.

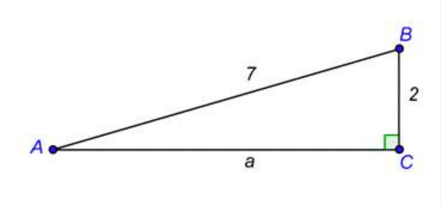
$\cos^{-1} 0.24 \approx 76.1^\circ$

Here is a real-world example using an inverse function.

✓ Example

Problem: A skateboard ramp is 7 feet long with one end on the ground and the other end 2 feet above the ground. What is the angle of elevation to the nearest tenth of a degree?

Answer



The angle of elevation is angle A. Because you know the opposite side and the hypotenuse, you can use the sine function.

Use the definition of sine. The unknown is the input.

$\sin A=\frac{2}{7}$

You can rewrite this equation using arcsine. You need to reverse the input and the output.

$A=\sin^{-1}\left(\frac{2}{7}\right)$

On a scientific calculator, divide 2 by 7, then press the 2ND key and SIN. Do this in the reverse order for a graphing calculator.

$$\begin{aligned}A=&\sin^{-1}0.285714286\\=&16.6015496^{\circ}\\&\approx 16.6^{\circ}\end{aligned}$$

16.6°

Remember that the sine or cosine function cannot have an output greater than 1. With arcsine and arccosine, you are reversing inputs and outputs. Consequently, the input of these functions cannot be a number bigger than 1. If you try to compute $\sin^{-1}2$ with your calculator, for example, you will get an error message.

? Exercise

If $\sin x=\frac{1}{3}$, what is x to the nearest hundredth of a degree?

A. 0.01°
B. 0.34°
C. 19.47°
D. 70.53°

Answer

1. 0.01°. Incorrect. Instead of finding $\sin^{-1}\left(\frac{1}{3}\right)$, you found $\sin\left(\frac{1}{3}\right)$. The correct answer is 19.47°.
2. 0.34°. Incorrect. You did not have your calculator set on degrees. Before doing any calculations, make sure that you first have it set on degrees. The correct answer is 19.47°.
3. 19.47°. Correct. The solution to the equation is given by computing $\sin^{-1}\left(\frac{1}{3}\right)$.
4. 70.53°. Incorrect. Instead of finding $\sin^{-1}\left(\frac{1}{3}\right)$, you found $\cos^{-1}\left(\frac{1}{3}\right)$. The correct answer is 19.47°.

Summary

The six trigonometric functions are defined as ratios of sides in a right triangle. Their values depend only on the angle and not on any particular right triangle. A good way to remember the definitions of sine, cosine, and tangent is with the memory device *sohcahtoa*. The other three functions, cosecant, secant, and cotangent, are reciprocals of the first three.

You can use a calculator to find the values of these functions or ratios. You can also use a calculator to find the values of the inverse trigonometric functions. That is, given the ratio, you can find the angle that produced it.

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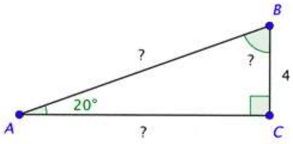
19.1.2: Right Triangle Trigonometry

Learning Objectives

- Use the Pythagorean Theorem to find the missing lengths of the sides of a right triangle.
- Find the missing lengths and angles of a right triangle.
- Find the exact trigonometric function values for angles that measure 30°, 45°, and 60°.
- Solve applied problems using right-triangle trigonometry.

Introduction

Suppose you have to build a ramp and don't know how long it needs to be. You know certain angle measurements and side lengths, but you need to find the missing pieces of information.

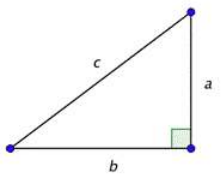


There are six trigonometric functions, or ratios, that you can use to compute what you don't know. You will now learn how to use these six functions to solve right-triangle application problems.

Using the Pythagorean Theorem in Trigonometry Problems

There are several ways to determine the missing information in a right triangle. One of these ways is the **Pythagorean Theorem**, which states that $a^2+b^2=c^2$.

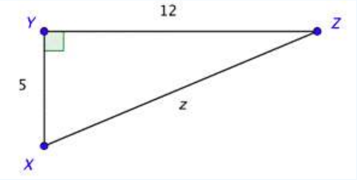
Suppose you have a right triangle in which a and b are the lengths of the legs, and c is the length of the hypotenuse, as shown below.



If you know the length of any two sides, then you can use the Pythagorean Theorem $(a^2+b^2=c^2)$ to find the length of the third side. Once you know all the side lengths, you can compute all of the trigonometric functions.

Example

Problem: Find the values of $tanX$ and $\sec X$.



Answer

You can immediately find the tangent from the definition and the information in the diagram.

$\tan X=\frac{\text{opposite}}{\text{adjacent}}=\frac{12}{5}$

To find the value of the secant, you will need the length of the hypotenuse. Use the Pythagorean Theorem to find the length of the hypotenuse.

$$\begin{aligned} 5^2+12^2 &= z^2 \\ 25+144 &= z^2 \\ 169 &= z^2 \\ 13 &= z \end{aligned}$$

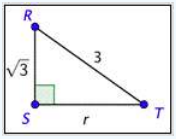
Now calculate $\sec X$ using the definition of secant.

$\sec X = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{z}{5} = \frac{13}{5}$

$\tan X = \frac{12}{5}, \sec X = \frac{13}{5}$

? Exercise

What is the value of $\csc R$?



A. $\frac{\sqrt{3}}{3}$
B. $\frac{3}{\sqrt{3}}$
C. $\frac{3}{\sqrt{6}}$
D. $\frac{3}{\sqrt{12}}$

Answer

A. $\frac{\sqrt{3}}{3}$. Incorrect. You found $\cos R$ instead of $\csc R$. Use the Pythagorean Theorem to find the opposite side length. Then divide the hypotenuse by the opposite length. The correct answer is $\frac{3}{\sqrt{6}}$.

B. $\frac{3}{\sqrt{3}}$. Incorrect. It looks like you used the wrong angle and found $\csc T$. Remember that the two acute angles will give you different trigonometric function values. Use the Pythagorean Theorem to find the opposite side length. Divide the hypotenuse by the opposite length. The correct answer is $\frac{3}{\sqrt{6}}$.

C. $\frac{3}{\sqrt{6}}$. Correct. You need to know the opposite side length, so use the Pythagorean Theorem: $(\sqrt{3})^2 + r^2 = 3^2$. This simplifies to $3 + r^2 = 9$. Solving this equation gives you $r = \sqrt{6}$. Using the definition of cosecant, $\csc R = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{3}{r} = \frac{3}{\sqrt{6}}$.

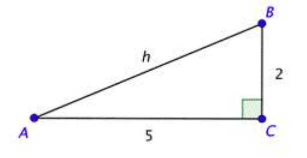
D. $\frac{3}{\sqrt{12}}$. Incorrect. You probably used the correct definition, $\csc R = \frac{\text{hypotenuse}}{\text{opposite}}$, and used the Pythagorean Theorem to find the opposite side length, r, but set up the equation incorrectly. It should be $(\sqrt{3})^2 + r^2 = 3^2$. The correct answer is $\frac{3}{\sqrt{6}}$.

Some problems may provide you with the values of two trigonometric ratios for one angle and ask you to find the value of other ratios. However, you really only need to know the value of one trigonometric ratio to find the value of any other trigonometric ratio for the same angle.

✓ Example

Problem: For acute angle A, $\tan A = \frac{2}{5}$. Find the values of $\cos A$ and $\sec A$.

Answer



First you need to draw a right triangle in which $\tan A = \frac{2}{5}$.

The tangent is the ratio of the opposite side to the adjacent side. The simplest triangle you can use that has that ratio is shown. It has an opposite side of length 2 and an adjacent side of length 5. You could have used a triangle that has an opposite side of length 4 and an adjacent side of length 10. (You just need the ratio to reduce to $\frac{2}{5}$).

You can use the Pythagorean Theorem to find the hypotenuse.

$$\begin{aligned} 5^2+2^2&=h^2\\ 25+4&=h^2\\ 29&=h^2\\ \sqrt{29}&=h \end{aligned}$$

Then use the definition of cosine to find $\cos A$.

$$\cos A=\frac{\text{adjacent}}{\text{hypotenuse}}=\frac{5}{\sqrt{29}}$$

Now use the fact that $\sec A=\frac{1}{\cos A}$ to find $\sec A$.

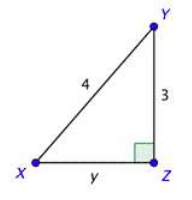
$$\sec A=\frac{1}{\cos A}=\frac{1}{\frac{5}{\sqrt{29}}}=\frac{\sqrt{29}}{5}$$

$$\cos A=\frac{5}{\sqrt{29}}, \sec A=\frac{\sqrt{29}}{5}$$

✓ Example

Problem: If angle X is an acute angle with $\sin X=\frac{3}{4}$, what is the value of $\cot X$?

Answer



In this right triangle, because $\sin X=\frac{3}{4}$, the ratio of the opposite side to the hypotenuse is $\frac{3}{4}$. The simplest triangle we can use that has that ratio would be the triangle that has an opposite side of length 3 and a hypotenuse of length 4.

We can use the Pythagorean Theorem to find the unknown leg length.

$$\begin{aligned} y^2+3^2&=4^2\\ y^2+9&=16\\ y^2&=7\\ y&=\sqrt{7} \end{aligned}$$

You can find the cotangent using the definition.

$$\cot X=\frac{\text{adjacent}}{\text{opposite}}=\frac{y}{3}=\frac{\sqrt{7}}{3}$$

Or you can find the cotangent by first finding tangent and then taking the reciprocal.

$$\begin{aligned} \tan X&=\frac{\text{opposite}}{\text{adjacent}}=\frac{3}{y}=\frac{3}{\sqrt{7}}\\ \cot X&=\frac{1}{\tan X}=\frac{1}{\frac{3}{\sqrt{7}}}=\frac{\sqrt{7}}{3} \end{aligned}$$

$$\cot X=\frac{\sqrt{7}}{3}$$

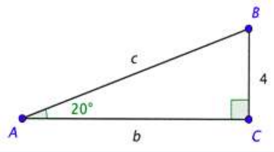
Solving Right Triangles

Determining all of the side lengths and angle measures of a right triangle is known as *solving* a right triangle. Let’s look at how to do this when you’re given one side length and one acute angle measure. Once you learn how to solve a right triangle, you’ll be able to solve many real world applications, such as the ramp problem at the beginning of this lesson, and the only tools you’ll need are the definitions of the trigonometric functions, the Pythagorean Theorem, and a calculator.

Example

Problem: You need to build a ramp with the following dimensions. Solve the right triangle shown below. Use the approximations $\csc 20^\circ \approx 2.92$ and $\tan 20^\circ \approx 0.364$, and give the lengths to the nearest tenth.

Answer



Remember that the acute angles in a right triangle are complementary, which means their sum is 90°. Since $m\angle A = 20^\circ$, it follows that $m\angle B = 90^\circ - 20^\circ = 70^\circ$.

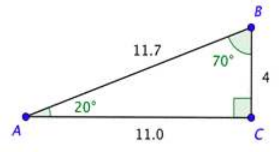
You can use the definition of cosecant to find c. Substitute the measure of the angle on the left side of the equation and use the triangle to set up the ratio on the right. Solving the equation and rounding to the nearest tenth gives you $c \approx 11.7$.

$$\begin{aligned}\csc 20^\circ &= \frac{\text{hypotenuse}}{\text{opposite}}\\ 2.92 &\approx \frac{c}{4}\\ 4(2.92) &\approx c\\ 11.68 &\approx c\end{aligned}$$

In a similar way, you can use the definition of tangent and the measure of the angle to find b. Solving the equation and rounding to the nearest tenth gives you $b \approx 11.0$.

$$\begin{aligned}\tan 20^\circ &= \frac{\text{opposite}}{\text{adjacent}}\\ 0.364 &\approx \frac{4}{b}\\ 0.364b &\approx 4\\ b \approx \frac{4}{0.364} &\approx 10.989\end{aligned}$$

The ramp needs to be 11.7 feet long.

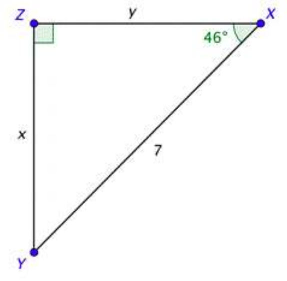


In the problem above, you were given the values of the trigonometric functions. In the next problem, you'll need to use the trigonometric function keys on your calculator to find those values.

Example

Problem: Solve the right triangle shown below. Give the lengths to the nearest tenth.

Answer



The acute angles are complementary, which means their sum is 90°. Since Since $m\angle X = 46^\circ$, it follows that $m\angle Y = 90^\circ - 46^\circ = 44^\circ$.

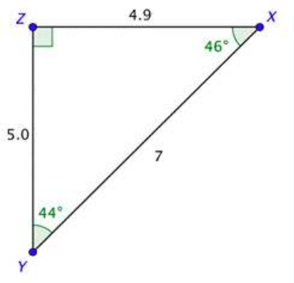
You can use the definition of sine to find x. Use your calculator to find the value of $\sin 46^\circ$ and the triangle to set up the ratio on the right. Solving the equation and rounding to the nearest tenth gives you $x \approx 5.0$.

$$\begin{aligned} \sin 46^\circ &= \frac{\text{opposite}}{\text{hypotenuse}} \\ 0.72 &\approx \frac{x}{7} \\ 7(0.72) &\approx x \\ 5.04 &\approx x \end{aligned}$$

To find y, you can either use another trigonometric function (such as cosine) or you can use the Pythagorean Theorem. Solving the equation and rounding to the nearest tenth gives you $y \approx 4.9$.

$$\begin{aligned} 5.0^2 + y^2 &\approx 7^2 \\ 25 + y^2 &\approx 49 \\ y^2 &\approx 24 \\ y &\approx \sqrt{24} \\ y &\approx 4.899 \end{aligned}$$

We now know all three sides and all three angles. Their values are shown in the drawing.

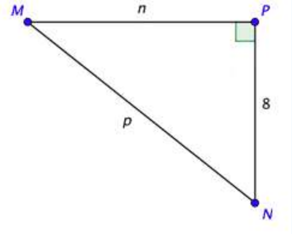


Sometimes you may be given enough information about a right triangle to solve the triangle, but that information may not include the measures of the acute angles. In this situation, you will need to use the inverse trigonometric function keys on your calculator to solve the triangle.

✓ Example

Problem: Solve the right triangle shown below, given that $\cot N = \frac{4}{5}$. Find the exact side lengths and approximate the angles to the nearest degree.

Answer



You are not given an angle measure, but you can use the definition of cotangent to find the value of n.

Use the ratio you are given on the left side and the information from the triangle on the right side. Cross-multiply and solve for n.

$$\begin{aligned} \cot N &= \frac{\text{adjacent}}{\text{opposite}} \\ \frac{4}{5} &= \frac{8}{n} \\ 4n &= 40 \\ n &= 10 \end{aligned}$$

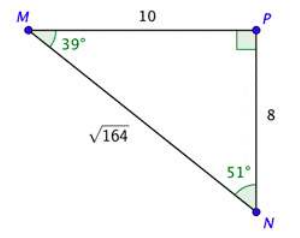
Use the Pythagorean Theorem to find the value of p.

$$\begin{aligned} 10^2+8^2&=p^2\\ 100+64&=p^2\\ 164&=p^2\\ \sqrt{164}&=p \end{aligned}$$

We can use the triangle to find a value of the tangent and the inverse tangent key on your calculator to find the angle that yields that value. Rounding to the nearest degree, $\angle M$ is approximately 39°, $M\approx 39^\circ$. Subtract 39°, from 90° to get $N\approx 51^\circ$.

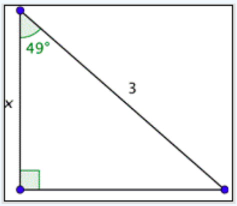
$$\begin{aligned} \tan M&=\tfrac{8}{10}=0.8\\ M&=\tan^{-1}0.8\\ M&\approx 38.7^\circ \end{aligned}$$

We now know all three sides and all three angles. Their values are shown in the drawing.



Exercise

What is the value of x to the nearest hundredth?



A. 4.57
B. 1.97
C. 0.90
D. 0.22

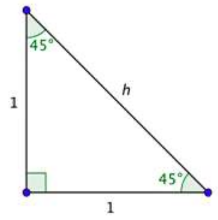
Answer

A. 4.57. Incorrect. You probably set up the ratio incorrectly, equating $\cos 49^\circ$ and $\frac{3}{x}$. A correct way to set up the equation is $\cos 49^\circ=\frac{x}{3}$. The correct answer is 1.97.
B. 1.97. Correct. One way to set up a correct equation is to use the definition of cosine. This will give you $\cos 49^\circ=\frac{x}{3}$. The solution to this equation is: $x=3\cdot\cos 49^\circ\approx 3\cdot(0.656)=1.968$. This rounds to 1.97.
C. 0.90. Incorrect. You probably set up the correct equation, $\cos 49^\circ=\frac{x}{3}$, and solved it correctly. However, your calculator was not set to degrees. The correct answer is 1.97.
D. 0.22. Incorrect. You may have correctly set up your equation as $\cos 49^\circ=\frac{x}{3}$, but then incorrectly solved it as $x=\frac{\cos 49^\circ}{3}$. The correct answer is 1.97.

Special Angles

As a general rule, you need to use a calculator to find the values of the trigonometric functions for any particular angle measure. However, angles that measure 30°, 45°, and 60°, which you will see in many problems and applications, are special. You can find the exact values of these functions without a calculator. Let’s see how.

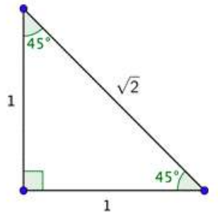
Suppose you had a right triangle with an acute angle that measured 45°. Since the acute angles are complementary, the other one must also measure 45°. Because the two acute angles are equal, the legs must have the same length, for example, 1 unit.



You can determine the hypotenuse using the Pythagorean Theorem.

$$\begin{gathered}1^2+1^2=h^2\\1+1=h^2\\2=h^2\\\sqrt{2}=h\end{gathered}$$

Now you have all the sides and angles in this right triangle.



You can use this triangle (which is sometimes called a 45° - 45° - 90° triangle) to find all of the trigonometric functions for 45°. One way to remember this triangle is to note that the hypotenuse is $\sqrt{2}$ times the length of either leg.

✓ Example

Problem: Find the values of the six trigonometric functions for 45° and rationalize denominators, if necessary.

Answer

Use the definitions of sine, cosine and tangent. Notice that because the opposite and adjacent sides are equal, sine and cosine are equal.

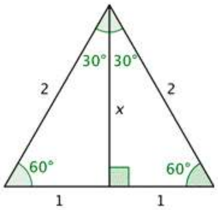
$$\begin{aligned}\sin 45^\circ &= \frac{\text{opposite}}{\text{hypotenuse}}=\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}\\ \cos 45^\circ &= \frac{\text{adjacent}}{\text{hypotenuse}}=\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}\\ \tan 45^\circ &= \frac{1}{1}=1\end{aligned}$$

Use the reciprocal identities. Notice that because the opposite and adjacent sides are equal, cosecant and secant are equal.

$$\begin{aligned}\csc 45^\circ &= \frac{1}{\sin 45^\circ}=\frac{1}{\frac{1}{\sqrt{2}}}=\frac{\sqrt{2}}{1}=\sqrt{2}\\ \sec 45^\circ &= \frac{1}{\cos 45^\circ}=\frac{1}{\frac{1}{\sqrt{2}}}=\frac{\sqrt{2}}{1}=\sqrt{2}\\ \cot 45^\circ &= \frac{1}{\tan 45^\circ}=\frac{1}{1}=1\end{aligned}$$

$$\begin{aligned}&\sin 45^\circ=\cos 45^\circ=\frac{\sqrt{2}}{2}, \tan 45^\circ=1\\ &\csc 45^\circ=\sec 45^\circ=\sqrt{2}, \cot 45^\circ=1\end{aligned}$$

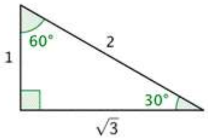
You can construct another triangle that you can use to find all of the trigonometric functions for 30° and 60°. Start with an equilateral triangle with side lengths equal to 2 units. If you split the equilateral triangle down the middle, you produce two triangles with 30°, 60° and 90° angles. These two right triangles are congruent. They both have a hypotenuse of length 2 and a base of length 1.



You can determine the height using the Pythagorean Theorem.

$$x^2+1^2=2^2$$
$$x^2+1=4$$
$$x^2=3$$
$$x=\sqrt{3}$$

Here is the left half of the equilateral triangle turned on its side.



You can use this triangle (which is sometimes called a 30° - 60° - 90° triangle) to find all of the trigonometric functions for 30° and 60°. Note that the hypotenuse is twice as long as the shortest leg, which is opposite the 30° angle, so that $\sin 30^\circ=\frac{1}{2}$. The length of the longest leg, which is opposite the 60° angle is $\sqrt{3}$ times the length of the shorter leg.

✓ Example

Problem: Find the values of $\cos 30^\circ$, $\tan 60^\circ$, $\sec 30^\circ$, and $\cot 60^\circ$. Rationalize denominators, if necessary.

Answer

Use the definitions of sine, cosine and tangent. For each angle, be sure to use the legs that are opposite and adjacent to *that* angle. For example, $\sqrt{3}$ is opposite to 60°, but adjacent to 30°.

$$\cos 30^\circ=\frac{\sqrt{3}}{2}$$
$$\tan 60^\circ=\frac{\sqrt{3}}{1}=\sqrt{3}$$

Remember that secant is the reciprocal of cosine and that cotangent is the reciprocal of tangent. Rationalize the denominators.

$$\sec 30^\circ=\frac{1}{\frac{\sqrt{3}}{2}}=\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$$
$$\cot 60^\circ=\frac{1}{\sqrt{3}}=\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

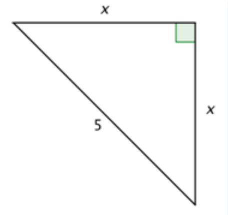
$$\cos 30^\circ=\frac{\sqrt{3}}{2}, \sec 30^\circ=\frac{2\sqrt{3}}{3}$$
$$\tan 60^\circ=\sqrt{3}, \cot 60^\circ=\frac{\sqrt{3}}{3}$$

You can use the information from the 30° - 60° - 90° and 45° - 45° - 90° triangles to solve similar triangles without using a calculator.

✓ Example

Problem: What is the value of x in the triangle below?

Answer



Since the two legs have the same length, the two acute angles must be equal, so they are each 45°.

In a 45° - 45° - 90° triangle, the length of the hypotenuse is times the length of a leg. You can use this relationship to find x. Remember to rationalize the denominator.

$$\begin{aligned}\sqrt{2}\cdot x &= 5\\ x &= \frac{5}{\sqrt{2}}\\ &= \frac{5}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{5\sqrt{2}}{2}\end{aligned}$$

Here is another way you can solve this problem. You can use the definition of sine to find x.

$$\begin{aligned}\sin 45^\circ &= \frac{x}{5}\\ 5\cdot\sin 45^\circ &= x\\ 5\cdot\frac{1}{\sqrt{2}} &= x\\ \frac{5}{\sqrt{2}} &= x\\ x &= \frac{5\sqrt{2}}{2}\end{aligned}$$

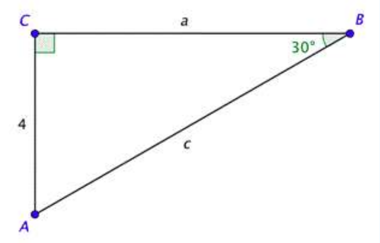
$x = \frac{5\sqrt{2}}{2}$

You also could have solved the last problem using the Pythagorean Theorem, which would have produced the equation $x^2 + x^2 = 5^2$.

✓ Example

Problem: Solve the right triangle shown below.

Answer



The acute angles are complementary, so $m\angle A = 60^\circ$. This is a 30° - 60° - 90° triangle. We can now use the trigonometric functions to find the lengths of the missing sides.

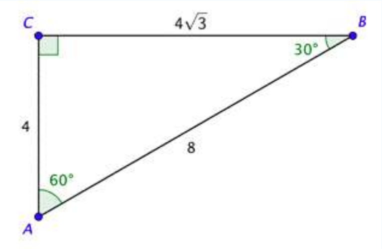
Since we know all the measures of the angles, we now need to find the lengths of the missing sides. To find c (the length of the hypotenuse), we can use the sine function because we know that $\sin 30^\circ = \frac{1}{2}$ and we know the length of the opposite side.

$$\begin{aligned}\sin 30^\circ &= \tfrac{4}{c}\\ \tfrac{1}{2} &= \tfrac{4}{c}\\ c &= 8\end{aligned}$$

To find a (the length of the side opposite angle A), we can use the tangent function because we know that $\tan 60^\circ = \sqrt{3}$ and we know the length of the adjacent side.

$$\begin{aligned} \tan 60^\circ &= \frac{a}{4} \\ \sqrt{3} &= \frac{a}{4} \\ 4\sqrt{3} &= a \end{aligned}$$

We now know all three sides and all three angles. Their values are shown in the drawing.



Exercise

If $\cos x = \frac{1}{2}$, what is the value of $\csc x$?

A. 2
B. $\frac{\sqrt{3}}{3}$
C. $\frac{\sqrt{6}}{3}$
D. $\frac{2\sqrt{3}}{3}$

Answer

A. 2. Incorrect. Refer to the 30° - 60° - 90° triangle. From there you can see that $\cos 60^\circ = \frac{1}{2}$. It follows that $x = 60^\circ$. You may have used the wrong angle and found $\csc 30^\circ$ or used the correct angle but found $\sec 60^\circ$. You need to find $\csc 60^\circ$. The correct answer is $\frac{2\sqrt{3}}{3}$.
B. \(\frac{\Incorrect. Refer to the 30° - 60° - 90° triangle. From there you can see that $\cos 60^\circ = \frac{1}{2}$. It follows that $x = 60^\circ$. You may have mistakenly found $\cot 60^\circ$. You need to find $\csc 60^\circ$. The correct answer is $\frac{2\sqrt{3}}{3}$.
C. $\frac{\sqrt{6}}{3}$. Incorrect. You may have correctly found $\csc 60^\circ$, but made a mistake when rationalizing the denominator. The correct answer is $\frac{2\sqrt{3}}{3}$.
D. $\frac{2\sqrt{3}}{3}$. Correct. Refer to the 30° - 60° - 90° triangle. From there you can see that $\cos 60^\circ = \frac{1}{2}$. It follows that $x = 60^\circ$. So $\csc x = \csc 60^\circ = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

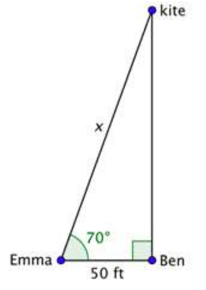
Using Trigonometry in Real-World Problems

There are situations in the real world, such as building a ramp for a loading dock, in which you have a right triangle with certain information about the sides and angles, and you wish to find unknown measures of sides or angles. This is where understanding trigonometry can help you.

Example

Problem: Ben and Emma are out flying a kite. Emma can see that the kite string she is holding is making a 70° angle with the ground. The kite is directly above Ben, who is standing 50 feet away. To the nearest foot, how many feet of string has Emma let out?

Answer



We want to find the length of string let out. It is the hypotenuse of the right triangle shown.

Since the 50-foot distance measures the adjacent side to the 70° angle, you can use the cosine function to find x.

$\cos 70^\circ = \frac{50}{x}$

Solve the equation for x. Use a calculator to find a numerical value. The answer rounds to 146.

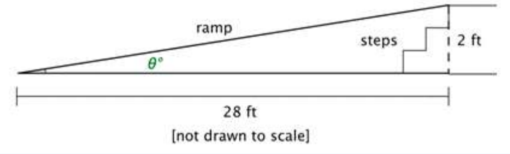
$$\begin{aligned} x \cdot \cos 70^\circ &= 50 \\ x &= \frac{50}{\cos 70^\circ} \\ x &= \frac{50}{0.342\ldots} \\ x &= 146.19\ldots \end{aligned}$$

Emma has let out approximately 146 feet of string.

In the example above, you were given one side and an acute angle. In the next one, you're given two sides and asked to find an angle. Finding an angle will usually involve using an inverse trigonometric function. The Greek letter theta, θ, is commonly used to represent an unknown angle. In this example, θ represents the angle of elevation.

✓ Example

Problem: A wheelchair ramp is placed over a set of stairs so that one end is 2 feet off the ground. The other end is at a point that is a horizontal distance of 28 feet away, as shown in the diagram. What is the angle of elevation to the nearest tenth of a degree?



Answer

The angle of elevation is labeled θ° in the diagram. The lengths given are the sides opposite and adjacent to this angle, so you can use the tangent function to find θ.

You want to find the measure of an angle that gives you a certain tangent value. This means that you need to find the inverse tangent. Remember that you have to use the keys 2ND and TAN on your calculator. Look at the hundredths place to round to the nearest tenth.

$$\begin{aligned} \tan\theta^\circ &= \frac{2}{28} \\ \tan\theta^\circ &= \frac{1}{14} \\ \theta^\circ &= \tan^{-1}\left(\frac{1}{14}\right) \\ \theta &= 4.0856\ldots \end{aligned}$$

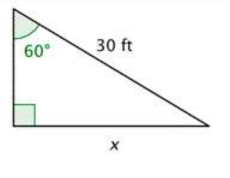
The angle of elevation is approximately 4.1°.

Remember that problems involving triangles with certain special angles can be solved without the use of a calculator.

Example

Problem: A fence is used to make a triangular enclosure with the longest side equal to 30 feet, as shown below. What is the exact length of the side opposite the 60° angle?

Answer



Call the unknown length x. Since you know the length of the hypotenuse, you can use the sine function.

This is a 30° - 60° - 90° triangle. Therefore, you can find the exact value of the trigonometric function without using a calculator.

$$\sin 60^\circ = \frac{x}{30}$$
$$\frac{\sqrt{3}}{2} = \frac{x}{30}$$

Solve the equation for x.

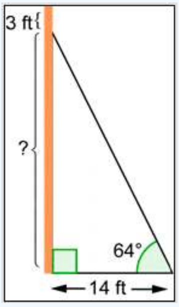
$$x = \frac{30\sqrt{3}}{2} = 15\sqrt{3}$$

The exact length of the side opposite the 60° angle is $15\sqrt{3}$ feet.

Sometimes the right triangle can be part of a bigger picture.

Exercise

A guy wire is attached to a telephone pole 3 feet below the top of the pole, as shown below. The guy wire is anchored 14 feet from the telephone pole and makes a 64° angle with the ground. How high up the pole is the guy wire attached? Round your answer to the nearest tenth of a foot.



A. $14 \cdot \sin 64^\circ \approx 12.6 \text{ feet}$
B. $14 \cdot \tan 64^\circ \approx 28.7 \text{ feet}$
C. $14 \cdot \tan 64^\circ + 3 \approx 31.7 \text{ feet}$
D. $\frac{14}{\cos 64^\circ} \approx 31.9 \text{ feet}$

Answer

A. $14 \cdot \sin 64^\circ \approx 12.6 \text{ feet}$. Incorrect. You may have been confused as to which ratio corresponds to which trigonometric function. You need to solve the equation $\tan 64^\circ = \frac{x}{14}$, where x represents the vertical distance from the base of the telephone pole up to where the guy wire is attached. The correct answer is $14 \cdot \tan 64^\circ \approx 28.7 \text{ feet}$.
B. $14 \cdot \tan 64^\circ \approx 28.7 \text{ feet}$. Correct. Let x represent the vertical distance from the base of the telephone pole up to where the guy wire is attached. Then $\tan 64^\circ = \frac{x}{14}$. Solving this equation for x gives you $x = 14 \cdot \tan 64^\circ \approx 28.7 \text{ feet}$.
C. $14 \cdot \tan 64^\circ + 3 \approx 31.7 \text{ feet}$. Incorrect. It looks like you set up and solved the correct equation to find the unknown length. However, you misread the problem. When you added the 3 you found the height of the entire pole. The correct answer is $14 \cdot \tan 64^\circ \approx 28.7 \text{ feet}$.

D. $\frac{14}{\cos 64^\circ} \approx 31.9 \text{ feet}$. Incorrect. It looks like you set up and solved an equation to find the length of the wire (the hypotenuse of the triangle). You need to solve the equation $\tan 64^\circ = \frac{x}{14}$, where x represents the vertical distance from the base of the telephone pole up to where the guy wire is attached. The correct answer is $14 \cdot \tan 64^\circ \approx 28.7 \text{ feet}$.

Summary

There are many ways to find the missing side lengths or angle measures in a right triangle. Solving a right triangle can be accomplished by using the definitions of the trigonometric functions and the Pythagorean Theorem. This process is called solving a right triangle. Being able to solve a right triangle is useful in solving a variety of real-world problems such as the construction of a wheelchair ramp.

You can find the exact values of the trigonometric functions for angles that measure 30°, 45°, and 60°. You can find exact values for the sides in 30°, 45°, and 60° triangles if you remember that $\sin 45^\circ = \frac{1}{\sqrt{2}}$ and $\sin 30^\circ = \frac{1}{2}$. For other angle measures, it is necessary to use a calculator to find approximate values of the trigonometric functions.

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19.1.3: Unit Circle Trigonometry

Learning Objectives

- Understand unit circle, reference angle, terminal side, standard position.
- Find the exact trigonometric function values for angles that measure 30°, 45°, and 60° using the unit circle.
- Find the exact trigonometric function values of any angle whose reference angle measures 30°, 45°, or 60°.
- Determine the quadrants where sine, cosine, and tangent are positive and negative.

Introduction

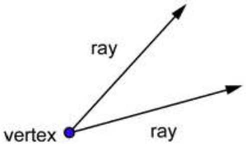
Mathematicians create definitions because they have a use in solving certain kinds of problems. For example, the six trigonometric functions were originally defined in terms of right triangles because that was useful in solving real-world problems that involved right triangles, such as finding angles of elevation. The domain, or set of input values, of these functions is the set of angles between 0° and 90°. You will now learn *new* definitions for these functions in which the domain is the set of *all* angles. The new functions will have the same values as the original functions when the input is an acute angle. In a right triangle you can only have acute angles, but you will see the definition extended to include other angles.

One use for these new functions is that they can be used to find unknown side lengths and angle measures in any kind of triangle. These new functions can be used in many situations that have nothing to do with triangles at all.

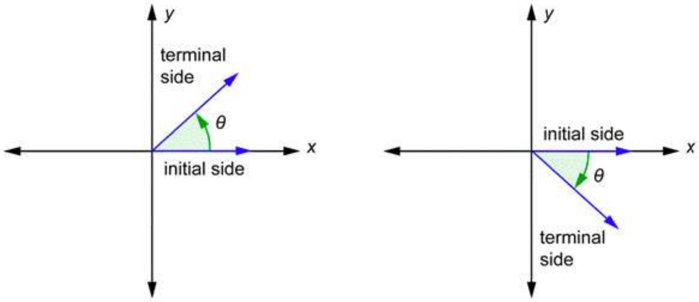
Before looking at the new definitions, you need to become familiar with the standard way that mathematicians draw and label angles.

General Angles

From geometry, you know that an angle is formed by two rays. The rays meet at a point called a vertex.



In trigonometry, angles are placed on coordinate axes. The vertex is always placed at the origin and one ray is always placed on the positive x-axis. This ray is called the **initial side** of the angle. The other ray is called the **terminal side** of the angle. This positioning of an angle is called **standard position**. The Greek letter theta (θ) is often used to represent an angle measure. Two angles in standard position are shown below.



When an angle is drawn in standard position, it has a direction. There are little curved arrows in the above drawing. The one on the left goes counterclockwise and is defined to be a positive angle. The one on the right goes clockwise and is defined to be a negative angle. If you used a protractor to measure the angles, you would get 50° in both cases. We refer to the first one as a 50° angle, and we refer to the second one as a -50° angle.

Why would you even have negative angles? As with all definitions, it is a matter of convenience. A spaceship in a circular orbit around Earth's equator could be traveling in either of two directions. So you could say that it traveled through a -50° angle to indicate that it went in the opposite direction of a spaceship that went through a 50° angle. Why is counterclockwise positive? This is just a convention, something that mathematicians have agreed on, because one way has to be positive and the other way negative.

Supplemental Interactive Activity

To understand how positive angles result from counterclockwise rotation and negative angles result from clockwise rotation, try the interactive exercise below. Either enter an angle measure in the box labeled Angle and hit enter or use the slider to move the terminal side of angle θ through the quadrants.

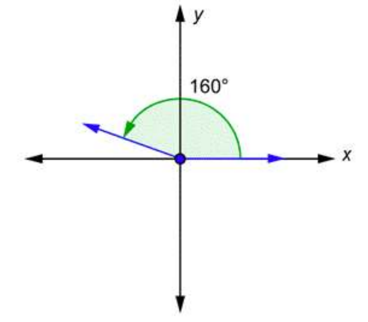
*Insert Interactive Activity

✓ Example

Problem: Draw a 160° angle in standard position.

Answer

The angle is positive, so you start at the x-axis and go 160° counterclockwise.

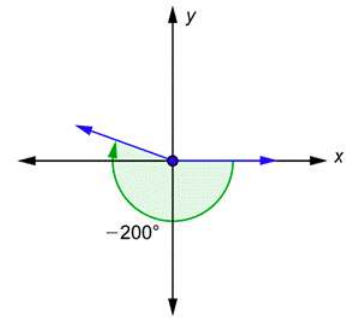


✓ Example

Problem: Draw a -200° angle in standard position.

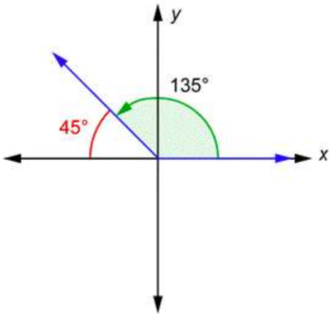
Answer

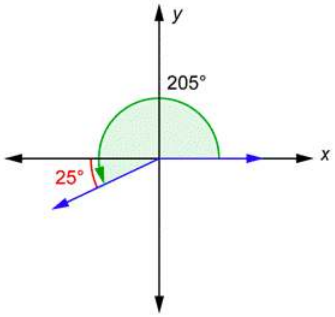
The angle is negative, so you start at the x-axis and go 200° clockwise. Remember that 180° is a straight line. That will bring you to the negative x-axis, and then you have to go 20° farther.



Notice that the terminal sides in the two examples above are the same, but they represent different angles. Such pairs of angles are said to be **coterminal angles**.

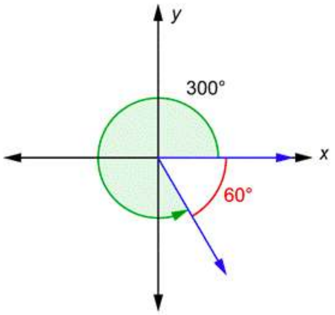
For each angle drawn in standard position, there is a related angle known as a **reference angle**. This is the angle formed by the terminal side and the x-axis. The reference angle is always considered to be positive, and has a value anywhere from 0° to 90°. Two angles are shown below in standard position.

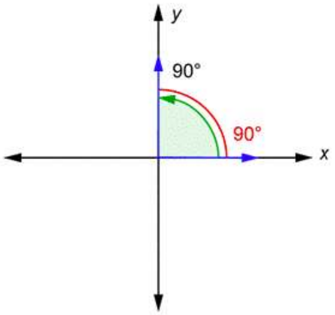




You can see that the terminal side of the 135° angle and the x-axis form a 45° angle (this is because the two angles must add up to 180°). This 45° angle, shown in red, is the reference angle for 135°. The terminal side of the 205° angle and the x-axis form a 25° angle. It is 25° because $205^\circ = 180^\circ + 25^\circ$. This 25° angle, shown in red, is the reference angle for 205°.

Here are two more angles in standard position.



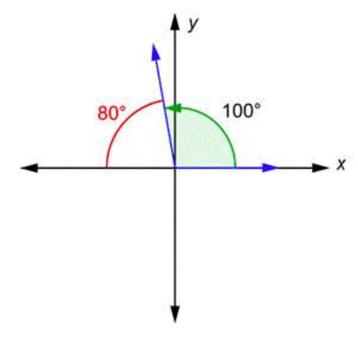


The terminal side of the 300° angle and the x-axis form a 60° angle (this is because the two angles must add up to 360°). This 60° angle, shown in red, is the reference angle for 300°. The terminal side of the 90° angle and the x-axis form a 90° angle. The reference angle is the same as the original angle in this case. In fact, any angle from 0° to 90° is the same as its reference angle.

✓ Example

Problem: What is the reference angle for 100°?

Answer



The terminal side is in Quadrant II. The original angle and the reference angle together form a straight line along the x-axis, so their sum is 180°.

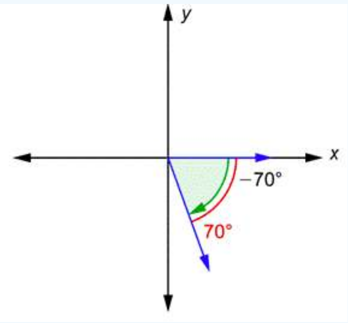
Therefore, the reference angle is 80°.

The reference angle for 180° is 80°.

✓ Example

Problem: What is the reference angle for -70°?

Answer



The terminal side and the x-axis form the “same” angle as the original. A reference angle is always a positive number, so the reference angle here is 70°, shown in red.

The reference angle for -70° is 70°.

? Exercise

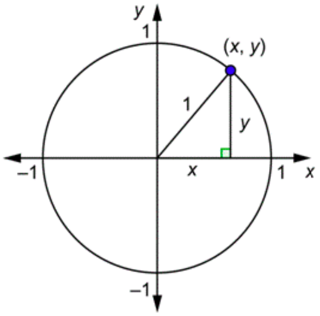
What is the reference angle for 310°?

A. 40°
B. 50°
C. -40°
D. -50°

Answer

1. 40°. Incorrect. You may have correctly drawn the angle in standard position, turning counterclockwise and landing in the fourth quadrant. However, you may have looked at the angle formed by the terminal side and the y-axis, instead of the x-axis. The correct answer is 50°.
2. 50°. Correct. In standard position, the angle will turn counterclockwise through the first, second, third, and into the fourth quadrant. The reference angle and the original angle together make one full circle, or 360°. Therefore, the reference angle is $360^{\circ}-310^{\circ}=50^{\circ}$.
3. -40°. Incorrect. You may have correctly drawn the angle in standard position, turning counterclockwise and landing in the fourth quadrant. However, you may have looked at the angle formed by the terminal side and the y-axis, instead of the x-axis. Also, the reference angle is always positive. The correct answer is 50°.
4. -50°. Incorrect. You may have correctly drawn the angle in standard position, turning counterclockwise and landing in the fourth quadrant. However, you may have confused the reference angle with a coterminal angle. Remember that the reference angle is always positive. The correct answer is 50°.

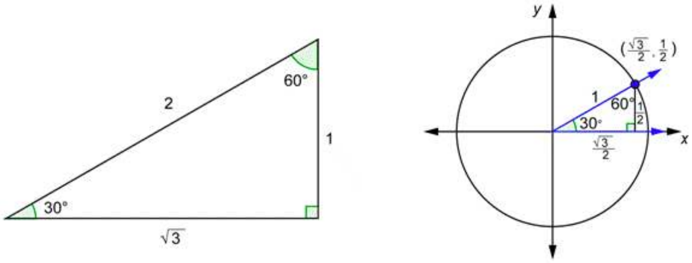
A **unit circle** is a circle that is centered at the origin and has radius 1, as shown below.



If (x, y) are the coordinates of a point on the circle, then you can see from the right triangle in the drawing and the Pythagorean Theorem that $x^2 + y^2 = 1$. This is the equation of the unit circle.

The General Definition of the Trigonometric Functions

The 30° - 60° - 90° triangle is seen below on the left. Next to that is a 30° angle drawn in standard position together with a unit circle.



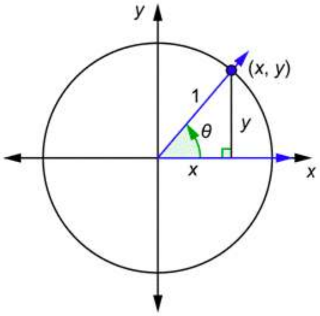
The two triangles have the same angles, so they are similar. Therefore, corresponding sides are proportional. The hypotenuse on the right has length 1 (because it is a radius). Since this is half of the hypotenuse on the left, all of the sides on the right are half of the corresponding sides on the left. For example, the side adjacent to the 30° angle on the left is $\sqrt{3}$; therefore the corresponding side on the triangle on the right has to be half that, or $\frac{\sqrt{3}}{2}$.

Look at the right triangle on the left. Using the definitions of sine and cosine:

$$\cos 30^\circ = \frac{\sqrt{3}}{2} \text{ and } \sin 30^\circ = \frac{1}{2}$$

Now look at the point where the terminal side intersects the unit circle. The x-coordinate is equal to $\cos 30^\circ$, and the y-coordinate is equal to $\sin 30^\circ$. This is not a coincidence. Let's look at a more general case.

Suppose you draw any acute angle θ in standard position together with a unit circle, as seen below.



The terminal side of the angle intersects the unit circle at the point (x, y). Let's write the definitions of the six trigonometric functions and then rewrite them by referring to the triangle above and using the variables x and y.

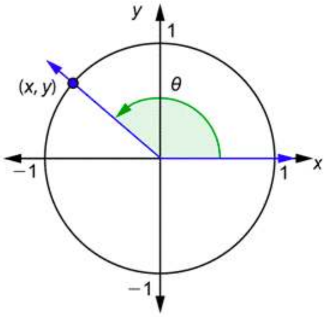
$$\begin{aligned} \sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = y & \csc\theta &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{y} \\ \cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = x & \sec\theta &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{x} \\ \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} & \cot\theta &= \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y} \end{aligned}$$

The first equation and the one below it, with the middle steps cut out, tell you:

$$\sin\theta = y \text{ and } \cos\theta = x$$

Now you can see that the y-coordinate of this point is always equal to the sine of the angle, and the x-coordinate of this point is always equal to the cosine of the angle.

Now consider *any* angle θ in standard position together with a unit circle. The terminal side will intersect the circle at some point (x, y). Depending on the angle, that point could be in the first, second, third, or fourth quadrant.



The sine of the angle θ is equal to the y-coordinate of this point and the cosine of the angle θ is equal to the x-coordinate of this point. In fact, the six trigonometric functions for any angle θ are now defined by the six equations listed above.

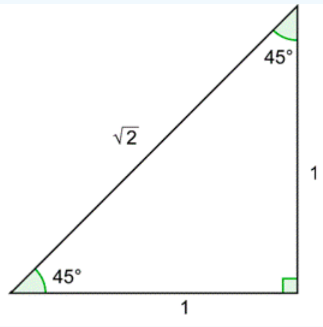
The next few examples will help you confirm that when θ is an acute angle, these new definitions give you the same results as the original definitions.

✓ Example

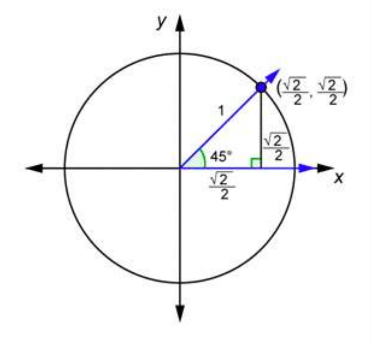
Problem: Draw a 45° angle in standard position together with a unit circle. Confirm that the x- and y-coordinates of the point of intersection of the terminal side and the circle are equal to $\cos 45^\circ$ and $\sin 45^\circ$.

Answer

Start with the 45° - 45° - 90° triangle. Then draw the 45° angle in standard position.



This unit circle triangle is similar to the 45° - 45° - 90° triangle. Each side length can be obtained by dividing the lengths of the 45° - 45° - 90° triangle by $\sqrt{2}$. So each leg on the unit circle triangle is: $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$



Look at the x- and y-coordinates of the point on the unit circle, then use the triangle to find $\cos 45^\circ$ and $\sin 45^\circ$.

From the coordinates on the unit circle: $x = \frac{\sqrt{2}}{2}$

From the triangle: $\cos 45^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

From the coordinates on the unit circle: $y = \frac{\sqrt{2}}{2}$

From the triangle: $\sin 45^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

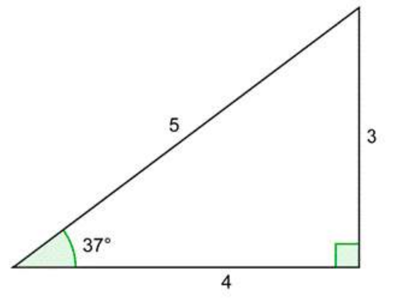
The x-coordinate is equal to $\cos 45^\circ$, and the y-coordinate is equal to $\sin 45^\circ$.

Yes, $x = \cos 45^\circ$ and $y = \sin 45^\circ$.

In the next two examples, the angle labels of 37° and 53° are actually very close approximations. The main idea of the examples (that those fractions involving x and y are equal to the various trigonometric functions) still holds true.

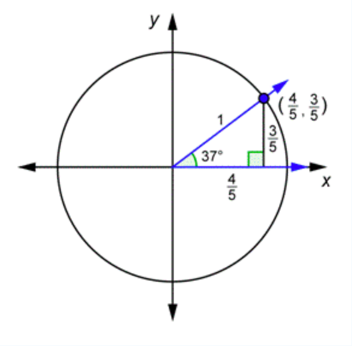
✓ Example

Problem: Draw a 37° angle in standard position together with a unit circle. Use the triangle below to find the x- and y-coordinates of the point of intersection of the terminal side and the circle. Compute $\frac{y}{x}$ and $\frac{x}{y}$. Confirm that they are equal to $\tan 37^\circ$ and $\cot 37^\circ$.



Answer

Draw the 37° angle in standard position. The unit circle triangle is similar to the 3 - 4 - 5 right triangle. Because this hypotenuse equals the original hypotenuse divided by 5, you can find the leg lengths by dividing the original leg lengths by 5.



Find the x- and y-coordinates.

$x = \frac{4}{5}, y = \frac{3}{5}$

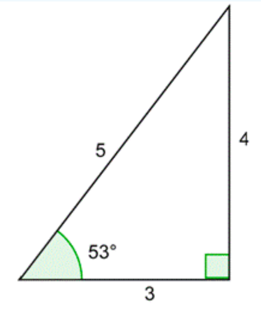
Compute the ratios. Compare the results to what you would get for $\tan 37^\circ$ and $\cot 37^\circ$ using the original triangle. They are the same!

$$\frac{y}{x} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{5} \cdot \frac{5}{4} = \frac{3}{4}$$
$$\tan 37^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$
$$\frac{x}{y} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{5} \cdot \frac{5}{3} = \frac{4}{3}$$
$$\cot 37^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}$$

So yes, $\frac{y}{x} = \tan 37^\circ$ and $\frac{x}{y} = \cot 37^\circ$

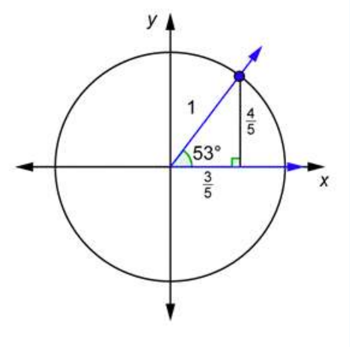
✓ Example

Problem: Draw a 53° angle in standard position together with a unit circle. Use the triangle below to find the x- and y-coordinates of the point of intersection of the terminal side and the circle. Compute $\frac{1}{x}$ and $\frac{1}{y}$. Confirm that they are equal to $\sec 53^\circ$ and $\csc 53^\circ$.



Answer

Draw the 53° angle in standard position. The unit circle triangle is similar to the 3 - 4 - 5 right triangle. Because this hypotenuse equals the original hypotenuse divided by 5, you can find the leg lengths by dividing the original leg lengths by 5.



Find the x- and y-coordinates.

$x = \frac{3}{5}, y = \frac{4}{5}$

Compute the ratios. Compare the results to what you would get for $\sec 53^\circ$ and $\csc 53^\circ$ using the original triangle. They are the same!

Yes, $\frac{1}{x} = \sec 53^\circ$ and $\frac{1}{y} = \csc 53^\circ$

The first three of our new definitions lead us to one more important identity:

$$\sin\theta = y, \cos\theta = x, \tan\theta = \frac{y}{x}$$

We can replace y by $\sin\theta$ and x by $\cos\theta$ in $\tan\theta = \frac{y}{x}$ to get the trigonometry identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

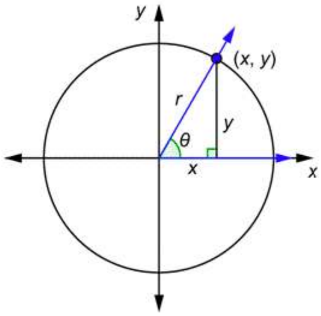
Since cotangent is the reciprocal of tangent, this gives you another trigonometric identity.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Remember, an identity is true for every possible value of the variable. So no matter what angle you are using, the values of tangent and cotangent are given by these quotients.

Although some textbooks give slightly different general definitions of the trigonometric functions, the important thing to know is that they end up giving you the same values as the definitions already given to you.

For example, start with a circle of radius r (in place of radius 1) and an angle θ in standard position. The terminal side will intersect the circle at some point (x, y). That point could be in any quadrant, but we show one in the first quadrant.



Now write down the original definitions and then rewrite them using the variables x, y, and r.

$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$ $\quad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{r}{y}$

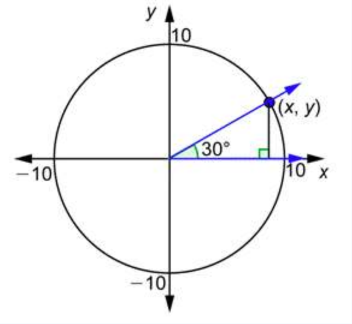
$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$ $\quad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{r}{x}$

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$ $\quad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$

These six fractions are used as the general definitions of the trigonometric functions for any angle θ, in any quadrant.

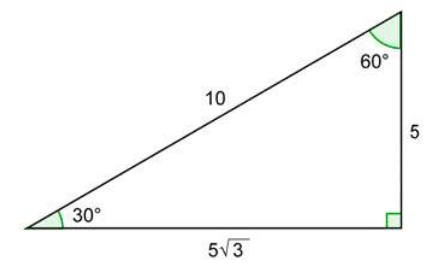
Example

Problem: Compute $\frac{y}{x}$ using the diagram below. Confirm that this is the same as the value of $\tan 30^{\circ}$.



Answer

The above diagram contains a 30° - 60° - 90° triangle.



The hypotenuse equals the radius, so it is 10. The side opposite 30° is half of 10, or 5. The adjacent side is $\sqrt{3}$ times the opposite side, or $5\sqrt{3}$.

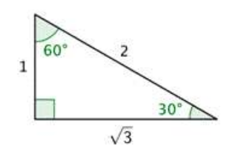
The sides of the triangle give you the values of x and y in the first diagram.

$$y=5$$
$$x=5\sqrt{3}$$

Substitute these into the definition.

$$\tan\theta=\frac{y}{x}=\frac{5}{5\sqrt{3}}=\frac{5}{5\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{5\sqrt{3}}{5\cdot 3}=\frac{\sqrt{3}}{3}$$

Here is our standard 30° - 60° - 90° triangle.



Compute $\tan 30^{\circ}$ using the right triangle definition.

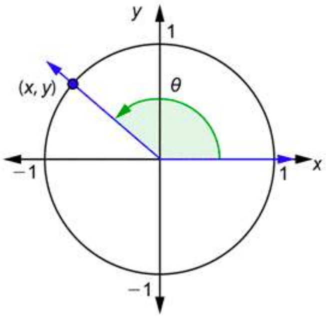
\tan 30^{\circ}=\frac{\text { opposite }}{\text { adjacent }}=\frac{1}{\sqrt{3}}=\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{3}

$$\text{So yes, }\frac{y}{x}=\frac{\sqrt{3}}{3}=\tan 30^{\circ}$$

Using the General Definitions of the Trigonometric Functions

You have been given new or “general” definitions of the six trigonometric functions and have seen that, when you compute these functions using acute angles, the result is the same as the result you would get from using the original definitions. Now you will learn how to apply these definitions to angles that are not acute and to negative angles.

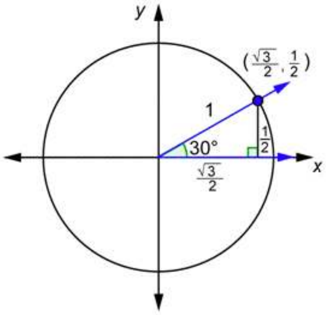
Given any angle θ, draw it in standard position together with a unit circle. The terminal side will intersect the circle at some point (x, y), as shown below.



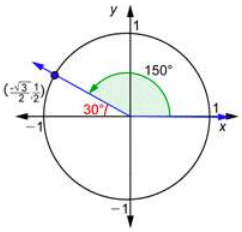
Here again are the general definitions of the six trigonometric functions using a unit circle.

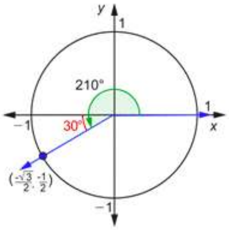
$\sin\theta = y$	$\csc\theta = \frac{1}{y}$
$\cos\theta = x$	$\sec\theta = \frac{1}{x}$
$\tan\theta = \frac{y}{x}$	$\cot\theta = \frac{x}{y}$

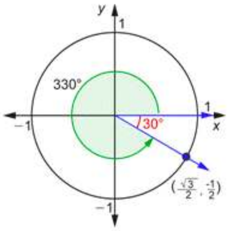
Now let's use these definitions with the angles 30°, 150°, 210°, and 330°. You have already done this for 30°. Here is that drawing:



The angles 150°, 210°, and 330° have something in common. Every one of them has a reference angle of 30°, as you can see from the drawings below. Because of this, it is easy to find the coordinates of the points where the terminal sides intersect the unit circle using the drawing above.







- For the angle 150°, this intersection point is the mirror image of $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ over the y-axis, so the coordinates for 150° are $\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$.
- For the angle 210°, this point is the mirror image of $\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$ over the x-axis, so the coordinates for 210° are $\left(\frac{-\sqrt{3}}{2}, -\frac{1}{2}\right)$.
- For the angle 330°, this point is the mirror image of $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ over the x-axis, so the coordinates for 330° are $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

You can now find the values of all six trigonometric functions for 150°, 210°, and 330°. Let's pick a few trigonometric functions and evaluate them using these angles. For example, using the leftmost diagram above and the definition of cosine:

$$\cos\theta = x$$
$$\cos 150^\circ = \frac{-\sqrt{3}}{2}$$

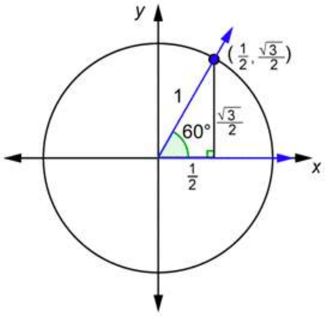
Using the middle diagram and the definition of cotangent:

$$\cot\theta = \frac{x}{y}$$
$$\cot 210^\circ = \frac{\frac{-\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{-\sqrt{3}}{2}\cdot\frac{2}{-1} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$

Using the rightmost diagram and the definition of cosecant:

$$\csc\theta = \frac{1}{y}$$
$$\csc 330^\circ = \frac{1}{-\frac{1}{2}} = -2$$

If you take the drawing above with the 30° angle in standard position, and turn the triangle so that the shorter leg is on the x-axis, you get a drawing of a 60° angle in standard position, as seen below.

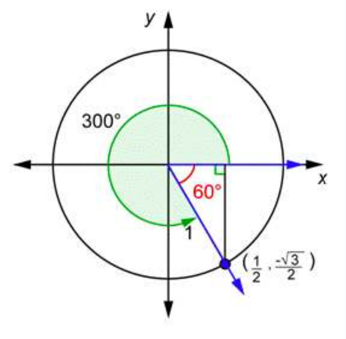


You can use the information in this diagram to find the values of the six trigonometric functions for any angle that has a reference angle of 60°.

✓ Example

Problem: Determine $\cos(300^\circ)$ and $\sec(300^\circ)$.

Answer



Draw 300° in standard position and find the reference angle. Find the x-coordinate of the point (x, y) where the terminal side intersects the unit circle. Because $\cos 60^\circ = \frac{1}{2}$, we know $x = \frac{1}{2}$.

Use the definition of cosine. Substitute the value of the x-coordinate that you found above.

$$\cos\theta = x$$
$$\cos 300^\circ = \frac{1}{2}$$

Use the definition of secant. Substitute the value of the x-coordinate that you found above. Note that, just as with acute angles, secant and cosine are reciprocals.

$$\begin{aligned}\sec\theta &= \tfrac{1}{x}\\ \sec 300^\circ &= \tfrac{1}{\frac{1}{2}} = 2\end{aligned}$$

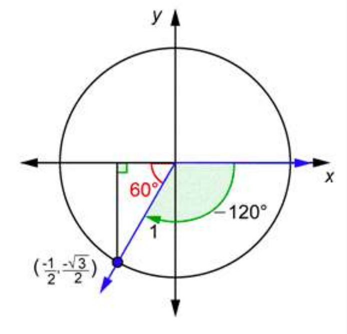
$\cos 300^\circ = \frac{1}{2}, \sec 300^\circ = 2$

The procedure is the same even if the angle is negative. Remember that a negative angle is simply one whose direction is clockwise.

Example

Problem: Find the values of $\sin(-120^\circ)$ and $\csc(-120^\circ)$.

Answer



Draw -120° in standard position and find the reference angle. Find the y-coordinate of the point (x, y) where the terminal side intersects the unit circle.

Because $\sin 60^\circ = \frac{\sqrt{3}}{2}$ and we are in the third quadrant, we know $y = -\frac{\sqrt{3}}{2}$.

Use the definition of sine. Substitute the value of the y-coordinate that you found above.

$$\begin{aligned}\sin\theta &= y\\ \sin(-120^\circ) &= -\tfrac{\sqrt{3}}{2}\end{aligned}$$

Use the definition of cosecant. Substitute the value of the y-coordinate that you found above. Rationalize the denominator. Note that, just as with acute angles, cosecant and sine are reciprocals.

$$\begin{aligned}\csc\theta &= \frac{1}{y}\\ \csc(-120^\circ) &= \frac{1}{\frac{-\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}\end{aligned}$$

$\sin(-120^\circ) = -\frac{\sqrt{3}}{2}, \csc(-120^\circ) = -\frac{2\sqrt{3}}{3}$

Look at the results from the last two examples and observe the following:

In each case, the value of the trigonometric function was either the same as the value of that function for the reference angle (60°), or the negative of the value of that function for the reference angle. Why did this happen? The computations for 60° were done using the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The computations for 300° and -120° were done using the points $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The x-coordinates have the same absolute value. The y-coordinates also have the same absolute value. When you substitute into the expressions x, $\frac{1}{x}$, y, and $\frac{1}{y}$, the result will be the same, or have a negative sign.

You will get a similar result with other angles. So the procedure for finding the value of a trigonometric function simplifies to the following:

- Determine the reference angle.
- Calculate the trigonometric function value of the reference angle.
- Determine if the value of the function is positive or negative.

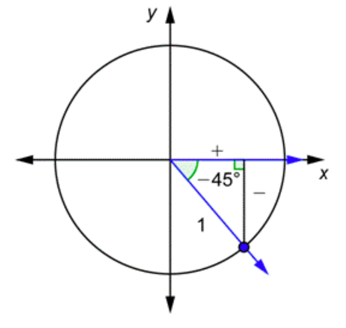
Let's try this procedure in the following example.

Example

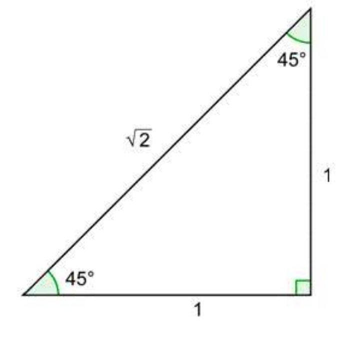
Problem: Compute $\tan(-45^\circ)$ and $\cot(-45^\circ)$.

Answer

Draw the angle in standard position. The reference angle is 45°.



Use the 45° - 45° - 90° triangle. The value of $\tan 45^\circ$ is 1.



In the first diagram, we put a + sign to indicate that x is positive, and a - sign to indicate that y is negative. Use this to determine the sign of $\tan(-45^\circ)$.

$$\tan\left(-45^\circ\right)=\frac{y}{x}=\frac{(-)}{(+)}=(-)$$

Since the result was negative, the value of $\tan(-45^\circ)$ is negative.

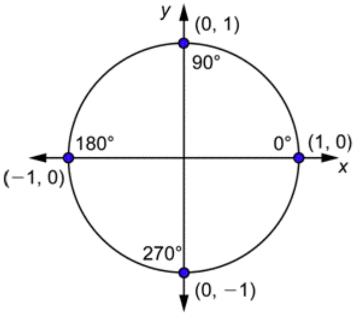
$$\tan\left(-45^\circ\right)=-\tan 45^\circ=-1$$

You can go through a similar procedure with cotangent or use the fact that it is the reciprocal of tangent.

$$\cot\left(-45^\circ\right)=\frac{1}{\tan(-45^\circ)}=\frac{1}{-1}=-1$$

$$\tan\left(-45^\circ\right)=-1$$
$$\cot\left(-45^\circ\right)=-1$$

The angles whose measures are a multiple of 90° have terminal sides on the axes. This can be confusing, because the terminal side is not in one quadrant, but rather on a border between quadrants. So let's look at these angles separately. The drawing below shows the points of intersection of the terminal sides of 0°, 90°, 180°, and 270° with the unit circle.



You can use this drawing and the definitions to find the trigonometric functions for (0^{\circ}\), 90°, 180°, and 270°. For example:

$$\begin{gathered}\sin\theta = y \\ \sin 0^\circ = 0 \\ \sin 90^\circ = 1 \\ \sin 180^\circ = 0 \\ \sin 270^\circ = -1\end{gathered}$$

For all six functions, you substitute the values of x and y as you did earlier. However, what happens if you try to compute $\tan 90^\circ$ using the definition $\tan\theta = \frac{y}{x}$?

$$\begin{gathered}\tan\theta = \frac{y}{x} \\ \tan 90^\circ = ?\frac{1}{0}\end{gathered}$$

You cannot divide by 0, so $\tan 90^\circ$ is simply undefined. Similarly, $\csc 180^\circ$ is undefined, because if you try to apply the definition, you will end up dividing by 0. The same is true any time one of the definitions leads to division by 0: the trigonometric function is undefined for that angle.

? Exercise

What are the values of $\cos 90^\circ$ and $\cot 180^\circ$?

A. $\cos 90^\circ = 0 \text{ and } \cot 180^\circ = 0$
B. $\cos 90^\circ = 0 \text{ and } \cot 180^\circ = \text{ undefined}$
C. $\cos 90^\circ = 1 \text{ and } \cot 180^\circ = 0$
D. $\cos 90^\circ = 1 \text{ and } \cot 180^\circ = \text{ undefined}$

Answer

A. $\cos 90^\circ = 0 \text{ and } \cot 180^\circ = 0$. Incorrect. The value for cosine is correct. However, when evaluating cotangent, you may have switched the x- and y-coordinates. The correct answer is B.
B. $\cos 90^\circ = 0 \text{ and } \cot 180^\circ = \text{ undefined }$. Correct. Since $\cos\theta = x$ and 90° corresponds to the point (0,1), $\cos 90 = 0$. Now $\cot\theta = \frac{x}{y}$ and 180° corresponds to the point (-1,0). If you tried to apply the definition you would be dividing by 0, because $y = 0$. This means that $\cot 180^\circ$ is undefined.
C. $\cos 90^\circ = 1 \text{ and } \cot 180^\circ = 0$. Incorrect. When evaluating both of these functions, you may have switched the x- and y-coordinates. The correct answer is B.
D. $\cos 90^\circ = 1 \text{ and } \cot 180^\circ = \text{ undefined }$. Incorrect. The value for cotangent is correct. However, when evaluating cosine, you may have switched the x- and y-coordinates. The correct answer is B.

You can use the following charts to help you remember the values of the trigonometric functions for the reference angles 0°, 30°, 45°, 60°, and 90° for sine and cosine. Once you have these, you can get the value of tangent from the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$, and the values of the other three trigonometric functions using reciprocals.

Make a table as follows:

	0°	30°	45°	60°	90°
sine					

cosine					

As an initial step, put the numbers 0, 1, 2, 3, and 4 in the “sine” row and 4, 3, 2, 1, and 0 in the “cosine” row. Do this in pencil. You are going to replace these numbers!

	0°	30°	45°	60°	90°
sine	0	1	2	3	4
cosine	4	3	2	1	0

Now replace the numbers 0 through 4 by taking their square roots and dividing by 2. The rows now contain the correct, but unsimplified, values for sine and cosine.

	0°	30°	45°	60°	90°
sine	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
cosine	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$

You can simplify $\sqrt{0}$ to 0, $\sqrt{1}$ to 1, and $\sqrt{4}$ to 2, and then divide by 2. This will give you the final table with the correct values of sine and cosine at these angles.

	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Determining the Sign of the Trigonometric Functions

First you learned the definitions for the trigonometric functions of an acute angle. Then you learned the general definitions of these functions, which can be used for any angle, and the method for applying them. Finally, you learned a simpler procedure for finding the values of trigonometric functions:

- Determine the reference angle.
- Calculate the trigonometric function value of the reference angle.
- Determine if the value of the function is positive or negative.

Now you’ll learn an easy way to remember where the trigonometric functions are positive and where they are negative.

The sine function: since $\sin\theta = y$, sine is positive when $y > 0$. This occurs in Quadrants I and II.

The cosine function: since $\cos\theta = x$, cosine is positive when $x > 0$. This occurs in Quadrants I and IV.

The tangent function: since $\tan\theta = \frac{y}{x}$, tangent is positive when x and y are both positive or both negative. This occurs in Quadrants I and III.

We can summarize this information by quadrant:

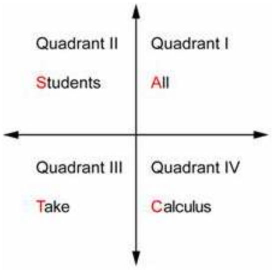
Quadrant I: sine, cosine, and tangent are positive.

Quadrant II: sine is positive (cosine and tangent are negative).

Quadrant III: tangent is positive (sine and cosine are negative).

Quadrant IV: cosine is positive (sine and tangent are negative).

Let A stand for all (three functions, sine, cosine, and tangent), S stand for sine, T stand for tangent, and C stand for cosine. Now you can use these single letters to remember in which quadrant sine, cosine, and tangent are positive. The final step is to replace each letter by a word to give you a phrase that’s easy to remember: “All Students Take Calculus.” Going counterclockwise, place these words in the four quadrants.



Now if you look in Quadrant II, for example, you see the word Students. The S tells you that sine is positive (while cosine and tangent are negative).

✓ Example

Problem: What signs are $\tan 200^\circ$ and $\cos 200^\circ$?

Answer

Since $200^\circ = 180^\circ + 20^\circ$, is in Quadrant III.

The word "Take" represents the fact that tangent is positive, so $\tan 200^\circ$ is positive.

Sine and cosine are negative in Quadrant III, so $\cos 200^\circ$ is negative.

$\tan 200^\circ$ is positive and
$\cos 200^\circ$ is negative.

✓ Example

Problem: In which quadrant must an angle lie if its sine is positive and its tangent is negative?

Answer

The words "All" and "Students" tell us that sine is positive in Quadrants I and II.

Tangent is positive in Quadrant I, but negative in Quadrant II.

Quadrant II

This device applies to the functions sine, cosine, and tangent. The other three trigonometric functions are reciprocals of these three. Recall the basic fact that the reciprocal of a positive number is positive, and the reciprocal of a negative number is negative. This implies that sine and cosecant have the same sign, cosine and secant have the same sign, and tangent and cotangent have the same sign. So if you want to know the sign of cosecant, secant, or cotangent, find the sign of sine, cosine, or tangent, respectively.

? Exercise

What signs are $\cos(-25^\circ)$ and $\sec(-25^\circ)$?

A. They are both positive.
B. They are both negative.
C. $\cos(-25^\circ) > 0$ and $\sec(-25^\circ) < 0$
D. $\cos(-25^\circ) < 0$ and $\sec(-25^\circ) > 0$

Answer

A. They are both positive. Correct. Angle -25° is in Quadrant IV. The word "Calculus" tells you that cosine is positive. Since secant is the reciprocal of cosine, it is also positive.
B. They are both negative. Incorrect. Perhaps you thought that because the angle is negative, the values of these functions would be negative. The values depend on the quadrant in which the angle lies. Angle -25° lies in Quadrant IV. The correct answer is A.

C. $\cos(-25^{\circ}) > 0$ and $\sec(-25^{\circ}) < 0$. Incorrect. The sign of cosine is correct. Perhaps you thought that when you take a reciprocal, the sign changes, but the sign doesn't change. Therefore, secant and cosine have the same sign. The correct answer is A.

D. $\cos(-25^{\circ}) < 0$ and $\sec(-25^{\circ}) > 0$. Incorrect. The sign of secant is correct. Perhaps you thought that because the angle is negative, the value of cosine would be negative. The values depend on the quadrant in which the angle lies. Angle -25° lies in Quadrant IV. The correct answer is A.

Summary

The trigonometric functions were originally defined for acute angles. There are general definitions of these functions, which apply to angles of any size, including negative angles. Values of trigonometric functions are computed by finding the reference angle, determining the value of the trigonometric function of the reference angle, and then determining if the value of the function is positive or negative. A useful way to remember this last step is "All Students Take Calculus."

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19.2: Graphing Trigonometric Functions

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19.2.1: Degree and Radian Measure

Learning Objectives

- Understand radian measure.
- Convert from degree to radian measure.
- Convert from radian to degree measure.

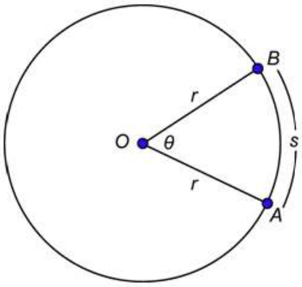
Introduction

You know that there can be different units of measurement to measure the same thing. For example, length can be measured in feet and meters and temperature can be measured in degrees Celsius and degrees Fahrenheit. We often use formulas to convert between different units of measurement.

There are also two ways to measure angles. You know how to measure them in degrees. Now you will learn how to measure them in radians and how to convert between these two measurements. While degree measure is used in everyday activities such as building construction and surveying land, **radian measure** is used for many calculations, such as the speed and distance traveled by satellites above the Earth’s surface, including the International Space Station! It is important to be able to measure angles in radians as well as in degrees and to be able to convert between the two systems.

Radian Measure

In order for us to be able to define radians, it is necessary to introduce the concept of a central angle. A **central angle** is an angle whose vertex is at the center of a circle. In the circle below, the center is point O, the length of the radius is r, and $\angle AOB$ is a central angle.



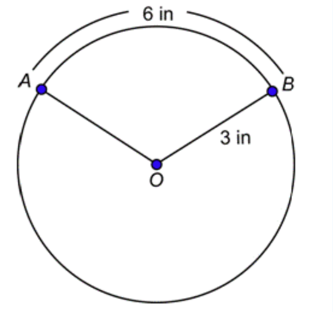
Notice that $\angle AOB$ cuts off or determines an arc $\overset{\frown}{AB}$ that has length s. The radian measure of a central angle, often denoted by the Greek letter theta (θ), is defined to be the ratio of the arc length to the length of the radius. So the radian measure of $\angle AOB$ is given by:

$$\theta=\frac{\text{arc length}}{\text{radius length}}=\frac{s}{r}$$

Arc length, s, and radius, r, must be in the same units.

Example

Problem: What is the radian measure of central angle AOB?



Answer

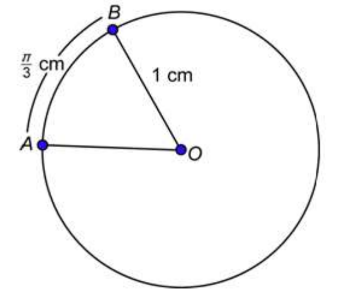
In this circle, $r=3$ inches. The arc determined by $\angle AOB$ has length $s=6$ inches. Substitute these into the formula.

$\theta=\frac{s}{r}=\frac{6\text{ inches}}{3\text{ inches}}=2$

The measure of central angle AOB is 2 radians.

Example

Problem: Find the measure of central angle AOB in radians.



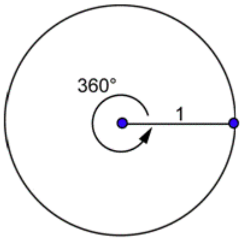
Answer

In this circle, $r=1$ cm. The arc determined by $\angle AOB$ has length $s=\frac{\pi}{3}$ cm. Substitute these into the formula.

$\theta=\frac{s}{r}=\frac{\frac{\pi}{3}\text{ cm}}{1\text{ cm}}=\frac{\pi}{3}$

The measure of central angle AOB is $\frac{\pi}{3}$ radians.

What is the connection between degrees and radians? Let’s see what happens when the arc is the entire circle:



The circumference of this circle is given by $C=2\pi r=2\pi\cdot 1=2\pi$. This is the arc length of the entire circle, or the arc length corresponding to a central angle of 360°. What is the radian measure of this angle?

$$\theta=\frac{\text{arc length}}{\text{radius length}}=\frac{2\pi}{1}=2\pi$$

A central angle of 360° has a radian measure of 2π. That is:

$$360^{\circ}=2\pi\text{ radians}$$

If you divide both sides by 2, you will obtain:

$$180^{\circ}=\pi\text{ radians}$$

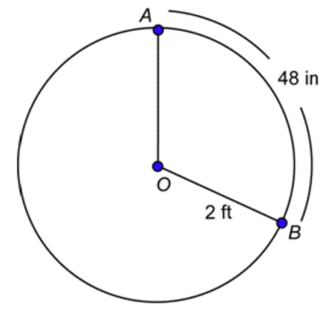
We’ll use this to convert from degrees to radians and vice versa.

While degrees are always written with a degree symbol (°), radians are usually written without any symbol or unit attached. So, for example, $\tan 3^{\circ}$ means the tangent of an angle that measures 3°, while $\tan 3$ means the tangent of an angle that measures 3 radians.

Make sure that you use the same units of length for the length of the radius r and the arc length s. In the first example, they were both in inches; in the second example, they were both in centimeters. When you divide, those units cancel, and you are left with a unitless number.

✓ Example

Problem: Find the measure of central angle AOB in radians.



Answer

You want the arc length and the radius to have the same units, but they are given to you in inches and feet. You could convert either one. The arc determined by $\angle AOB$ has length $s = 48$ inches.

$$s = 48 \text{ inches}$$
$$= 48 \text{ inches } \cdot \frac{1 \text{ foot}}{12 \text{ inches}}$$

$$; = 4 \text{ feet}$$

In this circle, $r = 2$ feet. Substitute this and the value above for s into the formula.

$\theta = \frac{s}{t} = \frac{4 \text{ feet}}{2 \text{ feet}} = 2$

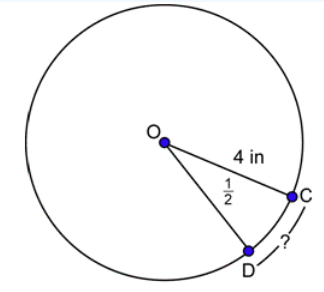
The measure of central angle AOB is 2 radians.

If you take the definition of radian measure, $\theta = \frac{s}{r}$, and multiply both sides by r, you get the formula for arc length:

$$s = r\theta$$

✓ Example

Problem: The central angle shown has a measure of $\frac{1}{2}$ radian. What is the length of $\overset{\frown}{CD}$?



Answer

In this circle, $r = 4$ inches. You know that $\theta = \frac{1}{2}$. Substitute these numbers into the arc length formula.

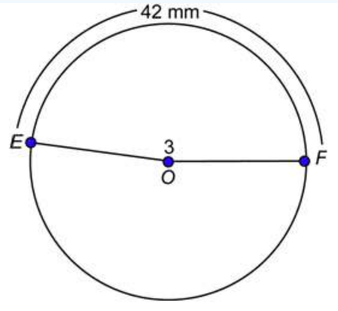
$s = r\theta = (4 \text{ inches }) \cdot \frac{1}{2} = 2 \text{ inches}$

The length of $\overset{\frown}{CD}$ is 2 inches.

We have used $\theta = \frac{s}{t}$ and $s = r\theta$ to find the radian measure of the central angle as well as the arc length. But you can also use the arc length formula to find the length of the radius, as seen in the following example.

Example

Problem: If the length of $\overset{\frown}{EF}$ is 42 millimeters, and the measure of $\angle EOF$ is 3 radians, what is the length of the radius of the circle?



Answer

You are given that $s = 42$ mm, and $\theta = 3$. Substitute these values into the arc length formula.

$$\begin{aligned} s &= r\theta \\ 42 \text{ mm} &= r \cdot 3 \end{aligned}$$

Solve the equation for r.

$$\frac{42 \text{ mm}}{3} = r$$

Rewrite as a mixed number.

$$r = 14 \text{ mm}$$

The radius of the circle is 14 millimeters.

Exercise

A circle has a radius of 8 inches and central angle AOB makes an arc AB of length 2 feet.

What is the radian measure of angle AOB?

A. $\frac{1}{4}$
B. $\frac{1}{3}$
C. 3
D. 4

Answer

A. $\frac{1}{4}$. Incorrect. You probably divided 2 by 8. You need to convert the feet to inches so that you have the same units. The correct answer is 3.
B. $\frac{1}{3}$. Incorrect. You probably correctly converted both measurements to inches, but then divided the radius by the arc length. You need to divide the arc length by the radius. The correct answer is 3.
C. 3. Correct. The arc length is 2 feet or 24 inches. Divide this by the radius, 8 inches, to get 3.
D. 4. Incorrect. You probably divided 8 by 2. You need to convert the feet to inches so that you have the same units. You also need to divide the arc length by the radius. The correct answer is 3.

Converting from Degrees to Radians

Earlier we saw that $180^\circ = \pi$ radians. Divide both sides of this equation by 180. You get:

$$1^\circ = \frac{\pi}{180} \text{ radians}$$

You can use this to convert the measure of an angle from degrees to radians. For example, if you wanted to convert 30° to radians, you could multiply both sides by 30:

$$30^\circ = 30 \cdot \tfrac{\pi}{180} \text{ radians } = \tfrac{30\pi}{180} \text{ radians } = \tfrac{\pi}{6} \text{ radians}$$

To say this another way, you multiply 30° by $\frac{\pi}{180^\circ}$ to get $\frac{\pi}{6}$ radians.

To convert 60° to radians, you multiply 60° by $\frac{\pi}{180^\circ}$ to get $\frac{\pi}{3}$ radians.

In a similar fashion, multiply any degree measure by $\frac{\pi}{180^\circ}$ to convert it to radians.

✓ Example

Problem: Convert 0°, 45°, 90° to radians.

Answer

In each case, just multiply by $\frac{\pi}{180^\circ}$ and simplify.

$$0^\circ = 0^\circ \cdot \tfrac{\pi}{180^\circ} \text{ radians } = 0 \text{ radians}$$

$$\begin{aligned} 45^\circ &= 45^\circ \cdot \frac{\pi}{180^\circ} \text{ radians} \\ &= \frac{45\pi}{180} \text{ radians} \\ &= \frac{\pi}{4} \text{ radians} \end{aligned}$$

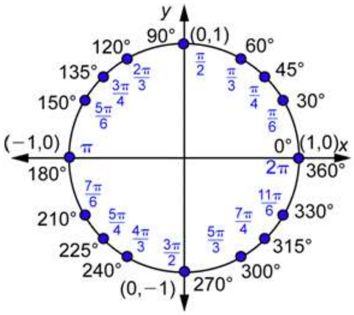
$$\begin{aligned} 90^\circ &= 90^\circ \cdot \frac{\pi}{180^\circ} \text{ radians} \\ &= \frac{90\pi}{180} \text{ radians} \\ &= \frac{\pi}{2} \text{ radians} \end{aligned}$$

$$0^\circ = 0 \text{ radians, } 45^\circ = \tfrac{\pi}{4} \text{ radians, } 90^\circ = \tfrac{\pi}{2} \text{ radians}$$

Here is a chart with the angles that we have converted so far.

Degree Measure	**Radian Measure**
0°	0
30°	$\frac{\pi}{6}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
180°	π
360°	2π

Here is the same information, with some additional angles, in a picture format:



The same procedure works just as well for negative angles. Remember that a negative angle is just an angle that is oriented clockwise when drawn in standard position. A negative number of degrees becomes a negative number of radians.

Exercise

How many radians are there in -120°?

A. $\frac{\pi}{60}$
B. $-\frac{2}{3}$
C. $-\frac{3\pi}{2}$
D. $-\frac{2\pi}{3}$

Answer

A. $\frac{\pi}{60}$. Incorrect. You may have correctly found $-\frac{120\pi}{180}$, but then reduced incorrectly. The correct answer is D.
B. $-\frac{2}{3}$. Incorrect. It looks like you divided by 180. You need to multiply by $\frac{\pi}{180^\circ}$. The correct answer is D.
C. $-\frac{3\pi}{2}$. Incorrect. You may have correctly found $-\frac{120\pi}{180}$, but made a mistake while reducing. The correct answer is D.
D. $-\frac{2\pi}{3}$. Correct. Multiply by $\frac{\pi}{180^\circ}$ to get $-\frac{120\pi}{180}=-\frac{2\pi}{3}$.

Converting from Radians to Degrees

Suppose you had angles measured in radians and you wanted to know their degree measure. The procedure is similar to the one going from degrees to radians.

Again, you know that π radians $=180^\circ$. Divide both sides of this equation by π. You get:

$$\text{radian} \ = \frac{180^\circ}{\pi}$$

If you wanted to convert $\frac{\pi}{6}$ radians to degrees, you could multiply both sides of the above equation by $\frac{\pi}{6}$:

$$\frac{\pi}{6}\cdot 1 \text{ radian} \ = \frac{\pi}{6}\cdot\frac{180^\circ}{\pi}=\frac{180^\circ}{6}=30^\circ$$

To state this in another way, we multiplied $\frac{\pi}{6}$ by $\frac{180^\circ}{\pi}$ to get 30°. In a similar fashion, multiply any radian measure by $\frac{180^\circ}{\pi}$ to convert it to degrees.

Example

Problem: Convert $\frac{2\pi}{3}$ and $\frac{3\pi}{4}$ to degrees.

Answer

In each case, just multiply by $\frac{180^\circ}{\pi}$ and simplify.

$$\begin{aligned}\frac{2\pi}{3} &= \frac{2\pi}{3}\cdot\frac{180^\circ}{\pi}\\ &= \frac{360^\circ}{3}\\ &= 120^\circ\\ \frac{3\pi}{4} &= \frac{3\pi}{4}\cdot\frac{180^\circ}{\pi}\\ &= \frac{540^\circ}{4}\\ &= 135^\circ\end{aligned}$$

$\frac{2\pi}{3}=120^\circ$, $\frac{3\pi}{4}=135^\circ$

Make sure that your answer has a degree symbol. An angle measure written without a degree symbol is always assumed to be in radians.

Here is a chart showing equivalent radian and degree measures for some common angles.

Radian Measure	**Degree Measure**
0	0°
\frac{\pi}{6}	30°
\frac{\pi}{4}	45°

\frac{\pi}{3}	60°
\frac{\pi}{2}	90°
π	180°
2π	360°

✓ Example

Problem: Convert 1 radian to degrees and give your answer to the nearest tenth of a degree.

Answer

Actually, you already know the value of 1 radian in degrees. However, you need to use a calculator to do the division and obtain a decimal.

$$\begin{aligned} 1 \text{ radian } &= \frac{180^\circ}{\pi} \\ &= 57.29577951^\circ \ldots \\ &= \approx 57.3^\circ \\ 1 \text{ radian } &\approx 57.3^\circ \end{aligned}$$

So 1 radian is a little less than 60°, or a little less than one-sixth of a circle.

You can use the same procedure for negative angles.

? Exercise

What is the equivalent degree measure of $-\frac{5\pi}{6}$ radians written in simplest terms?

A. 150°
B. -150°
C. $\frac{-900^\circ}{6}$
D. $\frac{-\pi^2}{216}$

Answer

A. 150°. Incorrect. You probably correctly multiplied by $\frac{180^\circ}{\pi}$, but you left out the negative sign. The correct answer is -150°.
B. -150°. Correct. Multiply by $\frac{180^\circ}{\pi}$ to get $\frac{-900\pi^\circ}{6\pi}$, and then simplify the fraction.
C. $\frac{-900^\circ}{6}$. Incorrect. You probably correctly multiplied by $\frac{180^\circ}{\pi}$, but you forgot to simplify. The correct answer is -150°.
D. $\frac{-\pi^2}{216}$. Incorrect. You multiplied by $\frac{\pi}{180^\circ}$ instead of $\frac{180^\circ}{\pi}$. Also, when you write an angle in degrees, you must use a degree symbol, and this answer choice does not have one. The correct answer is -150°.

Summary

Angles can be measured in two ways: with degrees and radians. Radian measure of a central angle is defined as the ratio of the arc length to the length of the radius. This definition leads to the arc length formula: $s = r\theta$.

Radians and degrees are connected by the relationship $360^\circ = 2\pi \text{ radians}$. If you wish to convert from degrees to radians, multiply the degree measure by $\frac{\pi}{180^\circ}$. If you wish to convert from radians to degrees, multiply the radian measure by $\frac{180^\circ}{\pi}$.

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19.2.2: Graphing the Sine and Cosine Functions

Learning Objectives

- Determine the coordinates of points on the unit circle.
- Graph the sine function.
- Graph the cosine function.
- Compare the graphs of the sine and cosine functions.

Introduction

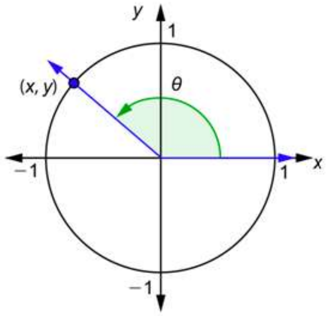
You know how to graph many types of functions. Graphs are useful because they can take complicated information and display it in a simple, easy-to-read manner. You'll now learn how to graph the sine and cosine functions, and see that the graphs of the sine function and cosine function are very similar.

Values of the Sine and Cosine Functions

We have seen a point (x, y) on a graph of a function. The first coordinate is the input or value of the variable, and the second coordinate is the output or value of the function.

Each point on the graph of the sine function will have the form $(\theta, \sin\theta)$, and each point on the graph of the cosine function will have the form $(\theta, \cos\theta)$. It is customary to use the Greek letter theta, θ, as the symbol for the angle. Graphing points in the form $(\theta, \sin\theta)$ is just like graphing points in the form (x, y). Along the x-axis, we will be plotting θ, and along the y-axis, we will be plotting the value of $\sin\theta$. The graphs that we'll draw will use values of θ in radians. Before drawing the graphs, it will be useful to find some values of $\sin\theta$ and $\cos\theta$, and then gather them together in a table.

Let's review the general definitions of these functions. Given any angle θ, draw it in standard position together with a unit circle. The **terminal side** will intersect the circle at some point (x, y), as shown below.



The value of $\cos\theta$ has been defined to be the x-coordinate of this point, and the value of $\sin\theta$ has been defined to be the y-coordinate of this point.

Example

Problem: Find the values of $\sin\theta$ and $\cos\theta$ for $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$.

Answer

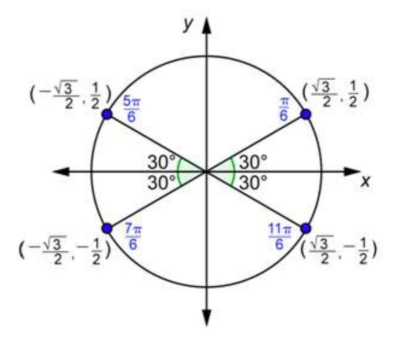
You might find it useful to convert these angles to degrees. All four angles have a reference angle of 30° or $\frac{\pi}{6}$ radians.

$$\frac{\pi}{6} = 30^\circ$$
$$\frac{5\pi}{6} = 150^\circ$$
$$\frac{7\pi}{6} = 210^\circ$$
$$\frac{11\pi}{6} = 330^\circ$$

Use the right triangle definition to find the $\sin\theta$ and $\cos\theta$ for $\theta = \frac{\pi}{6}$.

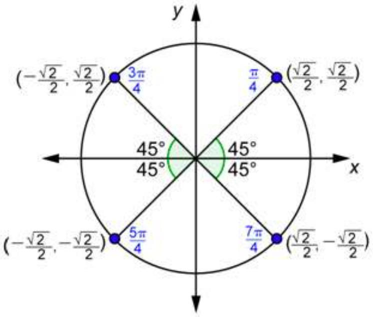
$\frac{\cos \pi}{6} = \cos 30^\circ = \frac{\sqrt{3}}{2}$
$\frac{\sin \pi}{6} = \sin 30^\circ = \frac{1}{2}$

Graph the four angles in standard position. The coordinates of the point in the first quadrant were found above. The x-coordinate is the value of $\cos\theta$, and the y-coordinate is the value of $\sin\theta$. The other points are reflections of the first point over the x-axis, the y-axis, or both.


y
x
$(-\frac{\sqrt{3}}{2}, \frac{1}{2})$
$\frac{5\pi}{6}$
$\frac{\pi}{6}$
$(\frac{\sqrt{3}}{2}, \frac{1}{2})$
30°
30°
30°
30°
$(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$
$\frac{7\pi}{6}$
$\frac{11\pi}{6}$
$(\frac{\sqrt{3}}{2}, -\frac{1}{2})$

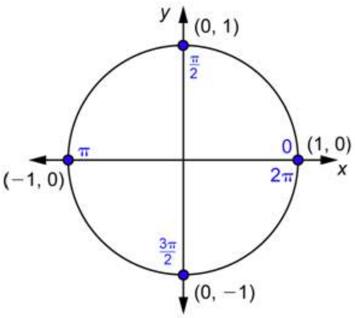
$\cos\left(\frac{\pi}{6}\right) = \cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}$
$\cos\left(\frac{5\pi}{6}\right) = \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$
$\sin\left(\frac{\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}, \sin\left(\frac{7\pi}{6}\right) = \sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$

You can go through a similar procedure to find the values of $\cos\theta$ and $\sin\theta$ for $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$. All four angles have a reference angle of $\frac{\pi}{4}$ radians or 45°.


y
x
$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
$\frac{3\pi}{4}$
$\frac{\pi}{4}$
$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
45°
45°
45°
45°
$(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$
$\frac{5\pi}{4}$
$\frac{7\pi}{4}$
$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$

Using the fact that $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \sin\left(\frac{\pi}{4}\right)$ gives you the coordinates of the point in the first quadrant. Since the other points are reflections of this one, the coordinates have the same or the opposite values.

The diagram below can be used to find the values of $cos\theta$ and $sin\theta$ for $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2},$ and 2π. Note that because $2\pi = 360^\circ$, when you draw the angle 2π in standard position, you end up back at the x-axis. 2π radians or 360° corresponds to the same point as 0 radians does, namely (1,0).


y
(0, 1)
$\frac{\pi}{2}$
π
(−1, 0)
0
(1, 0)
2π
x
$\frac{3\pi}{2}$
(0, −1)

Using the coordinates of the four points gives you:

$$\begin{array}{cc} \cos 0 = \cos 2\pi = 1 & \sin\left(\frac{\pi}{2}\right) = 1 \\ \cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0 & \sin 0 = \sin\pi = \sin 2\pi = 0 \\ \cos\pi = -1 & \sin\left(\frac{3\pi}{2}\right) = -1 \end{array}$$

Supplemental Interactive Activity

To become more familiar with the coordinates of points on the unit circle, try the following interactive exercise.

*Insert Interactive Activity Module

The Graph of the Sine Function

Our goal right now is to graph the function $y = \sin\theta$. Each point on this function's graph will have the form $(\theta, \sin\theta)$ with the values of θ in radians. The first step is to gather together in a table all the values of $\sin\theta$ that you know. To start, we will use values of θ between 0° and 180° (or $0 \le \theta \le \pi$).

θ (in degrees)	θ (in radians)	$\sin\theta$	$(\theta, \sin\theta)$
0°	0	0	(0,0)
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$(\frac{\pi}{6}, \frac{1}{2})$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$
90°	$\frac{\pi}{2}$	1	$(\frac{\pi}{2}, 1)$
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$(\frac{2\pi}{3}, \frac{\sqrt{3}}{2})$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$(\frac{3\pi}{4}, \frac{\sqrt{2}}{2})$
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$(\frac{5\pi}{6}, \frac{1}{2})$
180°	π	0	$(\pi, 0)$

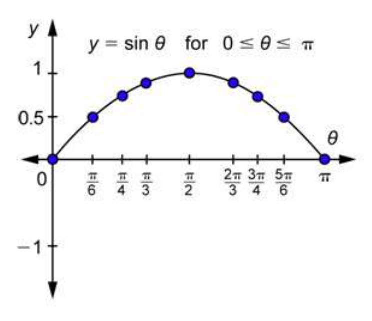
When we graph functions, we often say to graph the function on an interval. We use interval notation to describe the interval. Interval notation has the form $[a, b]$, which means the interval begins at a and ends at b. In the example, the notation $[0, \pi]$ has the same meaning as $0 \le \theta \le \pi$.

✓ Example

Problem: Graph the sine function on the interval $[0, \pi]$. Describe the values of the function as θ goes from 0 to π.

Answer

Plot all the points from the last column of the table above. Note that $\frac{\sqrt{2}}{2} \approx 0.7$ and that $\frac{\sqrt{3}}{2} \approx 0.9$. Connect the points with a smooth curve.

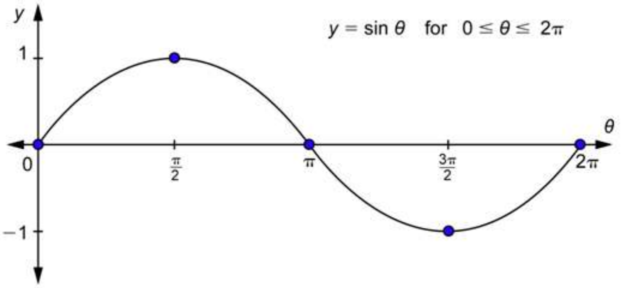


The values increase from 0 to 1 and then decrease from 1 to 0.

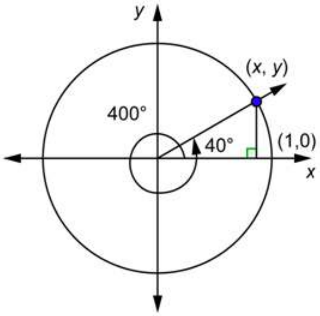
Note that our input is θ, the measure of the angle in radians, and that the horizontal axis is labeled θ, not x. Next we will gather together all the values of $\sin\theta$ that you know for $\pi \le \theta \le 2\pi$ in one table.

θ (in degrees)	θ (in radians)	$\sin\theta$	$(\theta, \sin\theta)$
180°	π	0	(π,0)
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$(\frac{7\pi}{6},-\frac{1}{2})$
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2})$
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$(\frac{4\pi}{3}, -\frac{\sqrt{3}}{2})$
270°	$\frac{3\pi}{2}$	-1	($3\frac{\pi}{2}$, -1)
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$(\frac{5\pi}{3}, -\frac{\sqrt{3}}{2})$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2})$
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$(\frac{11\pi}{6}, -\frac{1}{2})$
360°	2π	0	(2π, 0)

You could simply plot all the points from the last column and continue the graph in the last example. But notice the following: the values in the third column (or y-coordinates of the points) have the opposite values of the points that we just graphed. This means that instead of plotting points above the θ-axis, you will be plotting points below the θ-axis. Also, the inputs and outputs are spaced out in the same fashion for this part of the graph as they were for the first part of the graph. So instead of having a "hill" that goes from 0 up to 1 and down to 0, you will have a "valley" that goes down from 0 to -1 and then up to 0.



We have used values of θ from 0 to 2π to draw the graph of the sine function. What does the graph look like if we continue to the right of 2π for $2\pi \le \theta \le 4\pi$, which is one more time around the unit circle? In terms of degrees, these are angles between 360° and 720°. Let's go back and look at one of these angles in standard position with the unit circle. A 400° angle is shown below.



Because $400^\circ = 360^\circ + 40^\circ$, the angle travels a full rotation plus another 40°, as shown by the curved arrow in the diagram. Imagine you are on (1,0) and you walk around the circle one complete time and then you walk a little more to end up at (x, y). This point where the terminal side intersects the unit circle is the same point that you would get for a 40° angle. All of the trigonometric functions for these two angles are computed using the coordinates of this point. This means that $\sin 400^\circ = \sin 40^\circ$, $\cos 400^\circ = \cos 40^\circ$, $\tan 400^\circ = \tan 40^\circ$, and the same is true for the three reciprocal functions.

There is nothing special about 400°. You could draw other angles that are greater than 360° and find similar results. The results above for sine and cosine can be rewritten as $\sin\left(40^\circ+360^\circ\right)=\sin 40^\circ$ and $\cos\left(40^\circ+360^\circ\right)=\cos 40^\circ$. In general, it is true that $\sin\left(\theta^\circ+360^\circ\right)=\sin\theta^\circ$ and $\cos\left(\theta^\circ+360^\circ\right)=\cos\theta^\circ$, or, using radians:

$$\sin(\theta+2\pi)=\sin\theta \quad \text{and} \quad \cos(\theta+2\pi)=\cos\theta$$

These two equations tell us that when we go around the circle a second time, we are going to get the same values for $\sin(\theta+2\pi)$ as we did for $\sin\theta$ and the same values for $\cos(\theta+2\pi)$ as we did for $\cos\theta$. In other words, as we go around the same circle a second time, at the same locations on the circle we will get the same values for the y-coordinate and the x-coordinate that we did the first time around the circle.

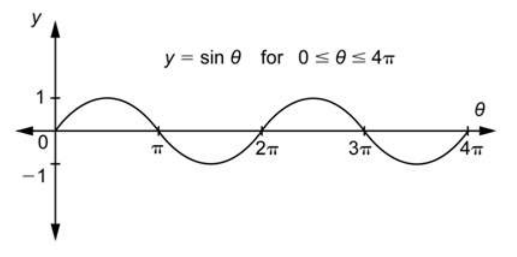
✓ Example

Problem: Sketch the graph of the sine function on the interval $[0,4\pi]$ and find the range.

Answer

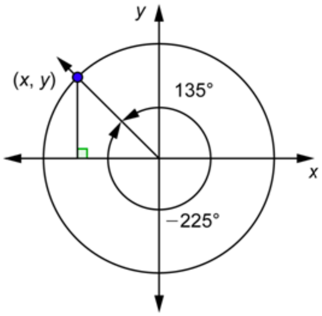
Because the values of the sine function between 2π and 4π are the same as the values between 0 and 2π, the shape of the graph between 2π and 4π is the same as the shape of the graph between 0 and 2π.

The range is the set of *all* y-values that the function can have, so the range of $y=\sin\theta$ is $-1\leq y\leq 1$.



The range of $y=\sin\theta$ is $-1\leq y\leq 1$.

The same reasoning that we used above works for negative angles. For example, the angles -225° and 135° are drawn in standard position with the unit circle below.



Because they are **coterminal angles**, they intersect the unit circle at the same point and therefore have the same coordinates. Therefore, $\sin\left(-225^\circ\right)=\sin 135^\circ$, $\cos\left(-225^\circ\right)=\cos 135^\circ$, and so on for the other trigonometric functions. Notice that we could rewrite the first equation as $\sin\left(-225^\circ\right)=\sin\left(-225^\circ+360^\circ\right)$. Likewise, $\sin\theta=\sin\left(\theta+360^\circ\right)$, or $\sin\theta=\sin(\theta+2\pi)$, is true for any angle θ including negative angles. The equation $\sin\theta=\sin(\theta+2\pi)$ tells us that each time we go one additional full revolution around the circle, we get the same values for the sine and the cosine as we did the first time around the circle.

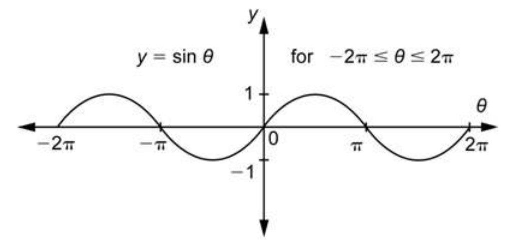
✓ Example

Problem: Sketch the graph of the sine function on the interval $[-2\pi,2\pi]$.

Answer

Because $\sin\theta = \sin(\theta + 2\pi)$ is true for negative angles as well as positive angles, the values of the sine function between -2π and 0 are the same as the values of the sine function between 0 and 2π.

Therefore, the shape of the graph between -2π and 0 is the same as the shape of the graph between 0 and 2π.



Since the equation $\sin(\theta + 2\pi) = \sin\theta$ is true for any angle, it is an identity. We can use this identity to continue the graph of the sine function in either direction. This "hill and valley" pattern over an interval of length 2π will continue in both directions forever.

Each time we add 2π to an angle, say $\frac{\pi}{6}$ or 30°, we will get the same function value.

$$\sin\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6} + 2\pi\right) = \sin\left(\frac{\pi}{6} + 4\pi\right) = \sin\left(\frac{\pi}{6} + 6\pi\right) = \ldots = \frac{1}{2}$$

You can also use the identity above to simplify calculations of the sine function by repeatedly subtracting 2π from the angle. For example:

$$\sin\left(\frac{17\pi}{4}\right) = \sin\left(4\pi + \frac{\pi}{4}\right) = \sin\left(2\pi + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

? Exercise

What is the value of $\sin\left(\frac{25\pi}{6}\right)$?

A. $\frac{1}{2}$
B. $-\frac{1}{2}$
C. $\frac{\sqrt{3}}{2}$
D. $-\frac{\sqrt{3}}{2}$

Answer

A. $\frac{1}{2}$. Correct. Rewrite $\frac{25\pi}{6}$ as $4\pi + \frac{\pi}{6}$. The 4π represents having gone around the circle twice. Using the identity $\sin(\theta + 2\pi) = \sin\theta$ to remove one revolution at a time, we get $\sin\left(4\pi + \frac{\pi}{6}\right) = \sin\left(2\pi + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \sin 30^\circ = \frac{1}{2}$.
B. $-\frac{1}{2}$. Incorrect. Perhaps you simplified incorrectly and thought this was equal to $\sin\left(\frac{7\pi}{6}\right)$. Use the identity $\sin(\theta + 2\pi) = \sin\theta$ to simplify $\sin\left(4\pi + \frac{\pi}{6}\right)$. The correct answer is $\frac{1}{2}$.
C. $\frac{\sqrt{3}}{2}$. Incorrect. You may have incorrectly converted from degrees to radians or confused sine and cosine. Use the identity $\sin(\theta + 2\pi) = \sin\theta$ to simplify $\sin\left(4\pi + \frac{\pi}{6}\right)$. Remember that $\frac{\pi}{6}\text{ radians } = 30^\circ$. The correct answer is $\frac{1}{2}$.
D. $-\frac{\sqrt{3}}{2}$. Incorrect. You may have simplified incorrectly and thought this was equal to $\sin\left(\frac{4\pi}{3}\right)$. Use the identity $\sin(\theta + 2\pi) = \sin\theta$ to simplify $\sin\left(4\pi + \frac{\pi}{6}\right)$. The correct answer is $\frac{1}{2}$.

The Graph of the Cosine Function

Now our goal is to graph $y = \cos\theta$. We will go through the same procedure as we did for the sine function, and the result will be similar.

Each point on the function's graph will have the form $(\theta, \cos\theta)$ with the values of θ in radians. The first step is to gather together in a table all the values of $\cos\theta$ that you know. To start, we will use values of θ between 0° and 180° (or $0 \le \theta \le \pi$).

θ (in degrees)	θ (in radians)	$\sin\theta$	$(\theta, \sin\theta)$
0°	0	1	(0,1)
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$

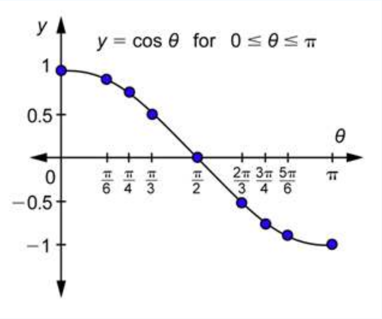
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$(\frac{\pi}{3}, \frac{1}{2})$
90°	$\frac{\pi}{2}$	0	$(\frac{\pi}{2}, 0)$
120°	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$(\frac{2\pi}{3}, -\frac{1}{2})$
135°	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$(\frac{3\pi}{4}, -\frac{\sqrt{2}}{2})$
150°	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$(\frac{5\pi}{6}, -\frac{\sqrt{3}}{2})$
180°	π	-1	(π, -1)

✓ Example

Problem: Graph the cosine function on the interval $[0, \pi]$. Describe the values of the function as θ goes from 0 to π.

Answer

Plot all the points from the last column of the table above. Note that $\frac{\sqrt{2}}{2} \approx 0.7$ and that $\frac{\sqrt{3}}{2} \approx 0.9$. Connect them with a smooth curve.

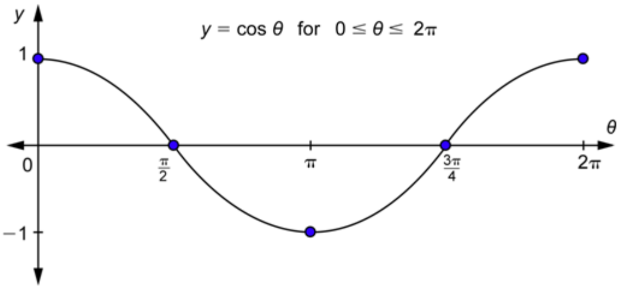


The values decrease from 1 to 0 and also continue to decrease from 0 to -1.

Once again, our input is θ, the measure of the angle in radians, and the horizontal axis is labeled θ, not x. Next, we will gather together all the values of $\cos\theta$ that you know for $\pi \leq \theta \leq 2\pi$ in one table.

θ (in degrees)	θ (in radians)	$\sin\theta$	$(\theta, \sin\theta)$
180°	π	-1	(0,-1)
210°	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$(\frac{\pi}{6}, -\frac{\sqrt{3}}{2})$
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$(\frac{\pi}{4}, -\frac{\sqrt{2}}{2})$
240°	$\frac{4\pi}{3}$	$-\frac{1}{2}$	$(\frac{\pi}{3}, -\frac{1}{2})$
270°	$\frac{3\pi}{2}$	0	$(\frac{\pi}{2}, 0)$
300°	$\frac{5\pi}{3}$	$\frac{1}{2}$	$(\frac{2\pi}{3}, \frac{1}{2})$
315°	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$(\frac{3\pi}{4}, \frac{\sqrt{2}}{2})$
330°	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$(\frac{5\pi}{6}, \frac{\sqrt{3}}{2})$
360°	2π	1	(π, 1)

Again, you could simply plot all the points from the last column and continue the graph. Instead, compare the values in the third columns of the two tables: they are the same numbers, but in the reverse order. These are the y1-coordinates of the points. This means that while the first part that we graphed decreased from 1 down to -1, this second part that we are graphing will increase from -1 to 1 and have the "same shape" (turned around the other way). Here it is:



The next step is to continue the graph for input values $2\pi \le \theta \le 4\pi$. When we were in the process of graphing the sine function, we established the following identity:

$$\cos(\theta + 2\pi) = \cos\theta$$

This equation tells us that when we go around the circle a second time, we are going to get the same values for $\cos(\theta + 2\pi)$ as we did for $\cos\theta$.

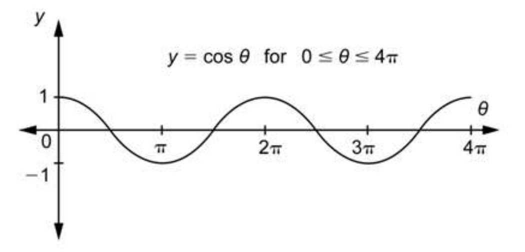
In other words, as we go around the same circle a second time, at the same locations on the circle we will get the same values for the x-coordinate that we did the first time around the circle.

Example

Problem: Sketch the graph of the cosine function on the interval $[0, 4\pi]$.

Answer

Because the outputs between 2π and 4π are the same as the outputs between 0 and 2π, the shape of the graph between 2π and 4π is the same as the shape of the graph between 0 and 2π.



Just as the identity $\sin\theta = \sin(\theta + 2\pi)$ is true for negative angles, the identity $\cos(\theta + 2\pi) = \cos\theta$ is also true for any negative angle θ.

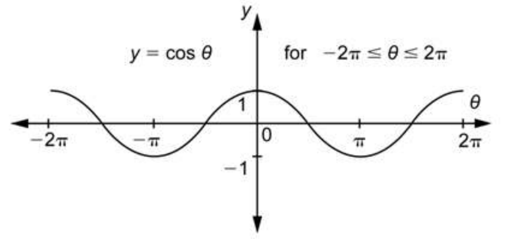
Example

Problem: Sketch the graph of the cosine function on the interval $[-2\pi, 2\pi]$.

Answer

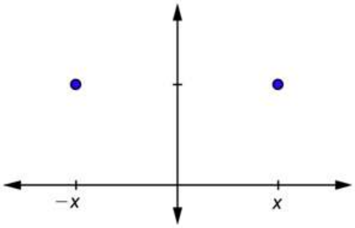
Because $\cos\theta = \cos(\theta + 2\pi)$ is true for negative angles as well as positive angles, the values of the cosine function between -2π and 0 are the same as the values of the cosine function between 0 and 2π.

Therefore, the shape of the graph between -2π and 0 is the same as the shape of the graph between 0 and 2π.



The identity $\cos(\theta + 2\pi) = \cos\theta$ was used above to extend the graph of the cosine function to the right and to the left. You can use this to continue to extend the graph in both directions. You will get another “hill and valley” pattern that repeats after intervals of length 2π in both directions.

Another important feature of the graph of $y = \cos\theta$ is that the left and right halves are mirror images of each other over the y-axis. The graph of $y = x^2$ has this same property. Another way to describe this is to say that if you substitute a number and its opposite into the function, you will get the same value as the previous equation. For example, $\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$, $\cos(-\pi) = \cos\pi$, or in general, $\cos(-\theta) = \cos\theta$. We say that the graph is **symmetric about the y-axis**. The diagram below shows two points taken from a symmetric graph.



The height of the points at opposite inputs is the same. The height is the value of the function. A function whose graph is symmetric about the y-axis has $f(-x) = f(x)$.

Exercise

What is the range of the cosine function?

A. all values in the interval [0,1]
B. all values in the interval [-1,1]
C. all values in the interval [0, 2π]
D. all real numbers

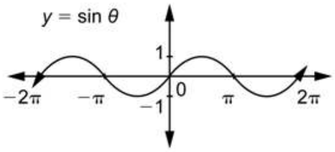
Answer

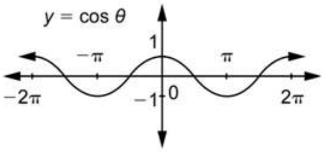
A. all values in the interval [0,1]. Incorrect. You were probably looking at the y-values, which is the correct thing to do. However, you chose only part of the range. The range is the set of *all* y-values that the function can have; in this case that would be $-1 \leq y \leq 1$. The correct answer is B.
B. all values in the interval [-1,1]. Correct. The graph of the function extends forever in both directions, so its domain is all real numbers. It consists of a repeating hill and valley pattern. The valley goes down to a y-value of -1, and the hill goes up to a y-value of 1. All y-values between these two y-values are also outputs of the function. So the set of outputs, or range, is all numbers from -1 to 1.
C. all values in the interval [0, 2π]. Incorrect. This interval, as a set of inputs, will give you one complete pattern. It is the set of outputs that you are looking for. The correct answer is B.
D. all real numbers. Incorrect. Perhaps you were thinking of the domain of the function, which is all real numbers. The range is the set of all outputs or y-values. The correct answer is B.

A Comparison of the Graphs of Sine and Cosine

The graphs of sine and cosine both have hills and valleys in a repeating pattern.

Since this repeating pattern can be extended indefinitely to the left and right, the domain for both functions is the real numbers. The range for both of them is the interval [-1,1].



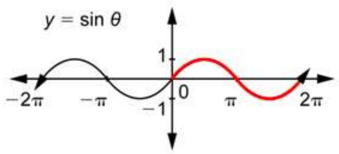


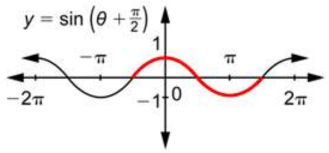
Let’s now compare these graphs in some other ways.

First, we want to see what happens to the graph of a function when you change the input by adding a constant to it. Compare $y = \sin\theta$ and $y = \sin\left(\theta + \frac{\pi}{2}\right)$. Here is a table with some values for these two functions.

θ **(in radians)**	$\sin\theta$	$\theta + \frac{\pi}{2}$ **(in radians)**	$\sin\left(\theta + \frac{\pi}{2}\right)$
0	0	$\frac{\pi}{2}$	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{5\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	π	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
π	0	$\frac{3\pi}{2}$	-1

Now we'll graph these two functions. As a reminder, the input is θ for both of these functions. To graph $y = \sin\theta$, you use the numbers in the first and second columns. To graph $y = \sin\left(\theta + \frac{\pi}{2}\right)$, you use the numbers in the first and fourth columns. (The third column was just written as a convenient intermediate step. You do not use that for the graph.)





First observe, as shown by the pieces in red, that the effect of adding $\frac{\pi}{2}$ to the input is to shift the graph to the left by $\frac{\pi}{2}$ units. You may recall seeing this effect when you graphed radical functions such as $y = \sqrt{x}$ and $y = \sqrt{x+1}$ (adding the 1 to the input shifted the graph of $y = \sqrt{x}$ to the left 1 unit). In general, if you add a positive constant c to the input of a function, that will have the effect of shifting the original graph to the left by c units. If you *subtract* a positive constant from the input of a function, that will have the effect of shifting the original graph to the *right* by c units.

Next, observe that the graph on the right is already familiar to you. It is the graph of $y = \cos\theta$! So you can say that the graph of $y = \sin\left(\theta + \frac{\pi}{2}\right)$ is the same as the graph of $y = \cos\theta$, or you can say that the graph of $y = \sin\theta$ shifted to the left by $\frac{\pi}{2}$ units is the graph of $y = \cos\theta$.

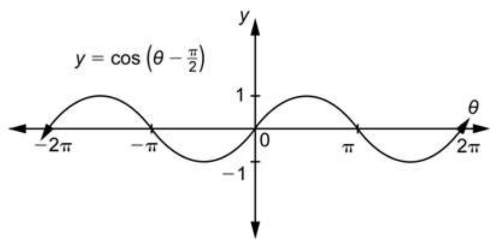
The next example looks at a shift in the other direction.

✓ Example

Problem: Sketch the graph of $y = \cos\left(\theta - \frac{\pi}{2}\right)$ on the interval $[-2\pi, 2\pi]$. How does the graph compare to the graph of $y = \sin\theta$?

Answer

The graph of $y = \cos\left(\theta - \frac{\pi}{2}\right)$ is the same as the graph of $y = \cos\theta$ shifted $\frac{\pi}{2}$ units to the right.



The graph of $y = \cos\left(\theta - \frac{\pi}{2}\right)$ is the same as the graph of $y = \sin\theta$.

Because the patterns repeat, you could start with the graph of either sine or cosine and shift it by different amounts to the right or left to get the graph of the other function.

? Exercise

Which comparison of the graphs of $y=\sin\theta$ and $y=\cos\theta$ is true?

A. They are the same.
B. The graph of $y=\sin\theta$ shifted $\frac{\pi}{2}$ units to the right is the graph of $y=\cos\theta$.
C. The graph of $y=\sin\theta$ shifted π units to the right is the graph of $y=\cos\theta$.
D. The graph of $y=\sin\theta$ shifted $\frac{\pi}{2}$ units to the left is the graph of $y=\cos\theta$.

Answer

A. They are the same. Incorrect. The two graphs have the same repeating pattern or the same general shape, but they are not identical. The correct answer is D.
B. The graph of $y=\sin\theta$ shifted $\frac{\pi}{2}$ units to the right is the graph of $y=\cos\theta$. Incorrect. If you switch sine and cosine in this option you get a correct statement. The two graphs described in this option have the same shape, but the hills and valleys do not match up. The correct answer is D.
C. The graph of $y=\sin\theta$ shifted π units to the right is the graph of $y=\cos\theta$. Incorrect. If you shift the graph of the sine function by π units to the right, you get a graph that "begins" (at $\theta=0$) with a valley. This is not the graph of the cosine function. The correct answer is D.
D. The graph of $y=\sin\theta$ shifted $\frac{\pi}{2}$ units to the left is the graph of $y=\cos\theta$. Correct. If you shift the graph of $y=\sin\theta$ by $\frac{\pi}{2}$ units to the left, you will get a graph that "begins" (at $\theta=0$) at the top of a hill. This is the graph of $y=\cos\theta$.

Supplemental Interactive Activity

For practice graphing sine and cosine, try the following interactive exercise.

*Insert Interactive Activity Module

Summary

The graphs of sine and cosine have the same shape: a repeating "hill and valley" pattern over an interval on the horizontal axis that has a length of 2π. The sine and cosine functions have the same domain, the real numbers, and the same range, the interval of values [-1,1].

The graphs of the two functions, though similar, are not identical. One way to describe their relationship is to say that the graph of $y=\cos\theta$ is identical to the graph of $y=\sin\theta$ shifted $\frac{\pi}{2}$ units to the left. Another way to describe their relationship is to say that the graph of $y=\sin\theta$ is identical to the graph of $y=\cos\theta$ shifted $\frac{\pi}{2}$ units to the right.

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19.2.3: Amplitude and Period

Learning Objectives

- Understand amplitude and period.
- Graph the sine function with changes in amplitude and period.
- Graph the cosine function with changes in amplitude and period.
- Match a sine or cosine function to its graph and vice versa.

Introduction

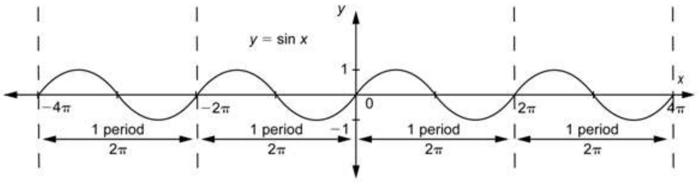
You know how to graph the functions $y = \sin x$ and $y = \cos x$. Now you'll learn how to graph a whole "family" of sine and cosine functions. These functions have the form $y = a \sin bx$ or $y = a \cos bx$, where a and b are constants.

Periodic Functions

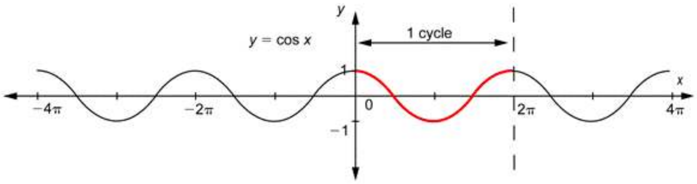
We used the variable θ previously to show an angle in standard position, and we also referred to the sine and cosine functions as $y = \sin \theta$ and $y = \cos \theta$. Often the sine and cosine functions are used in applications that have nothing to do with triangles or angles, and the letter x is used instead of θ for the input (as well as to label the horizontal axis). So from this point forward, we'll refer to these same functions as $y = \sin x$ and $y = \cos x$. This change does not affect the graphs; they remain the same.

You know that the graphs of the sine and cosine functions have a pattern of hills and valleys that repeat. The length of this repeating pattern is 2π. That is, the graph of $y = \sin x$ (or $y = \cos x$) on the interval $[0,2\pi]$ looks like the graph on the interval $[2\pi, 4\pi]$ or $[4\pi, 6\pi]$ or $[-2\pi, 0]$. This pattern continues in both directions forever.

The graph below shows four repetitions of a pattern of length 2π. Each one contains exactly one complete copy of the "hill and valley" pattern.



If a function has a repeating pattern like sine or cosine, it is called a **periodic function**. The **period** is the length of the smallest interval that contains exactly one copy of the repeating pattern. So the period of $y = \sin x$ or $y = \cos x$ is 2π. Any part of the graph that shows this pattern over one period is called a **cycle**. For example, the graph of $y = \cos x$ on the interval $[0,2\pi]$ is one cycle.



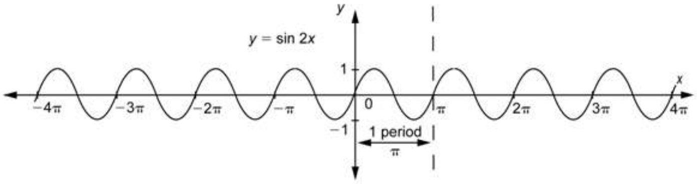
You know from graphing quadratic functions of the form $y = ax^2$ that as you changed the value of a, you changed the "width" of the graph. Now we'll look at functions of the form $y = \sin bx$ and see how changes to b will affect the graph. For example, is $y = \sin 2x$ periodic, and if so, what is the period? Here is a table with some inputs and outputs for this function.

for this function.

x **(in radians)**	$2x$ **(in radians)**	$\sin 2x$
0	0	0
$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\pi}{2}$	1

$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	π	0
$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	-1
$\frac{5\pi}{6}$	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
π	2π	0

As the values of x go from 0 to π, the values of $2x$ go from 0 to 2π. We can see from the graph that the function $y=\sin 2x$ is a periodic function, and goes through one full cycle on the interval [0, π], so its period is π. If you substituted values of x from π to 2π, the values of $2x$ would go from2π to 4π, and $\sin 2x$ would go through another complete cycle of the sine function.



Notice that $y=\sin 2x$ has two cycles on the interval [0, 2π] which is the interval $y=\sin x$ needs to complete one full cycle.

Exercise

What is the smallest positive value for x where $y=\sin 2x$ is at its minimum?

A. $-\frac{\pi}{4}$
B. $\frac{\pi}{4}$
C. $\frac{3\pi}{4}$
D. $\frac{\pi}{2}$

Answer

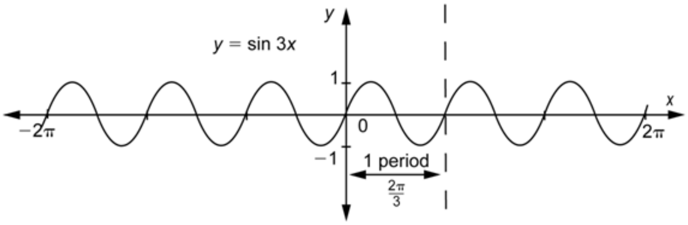
A. $-\frac{\pi}{4}$. Incorrect. The function does attain its minimum value at this point, but $-\frac{\pi}{4}$ is not a positive value. The correct answer is $\frac{3\pi}{4}$.
B. $\frac{\pi}{4}$. Incorrect. Perhaps you confused minimum and maximum. The correct answer is $\frac{3\pi}{4}$.
C. $\frac{3\pi}{4}$Correct. The minimum value for the sine function is -1. Look at the graph of $y=\sin 2x$. It attains this minimum at the bottom of every valley. The bottom of the first valley where x is positive is at $x=\frac{3\pi}{4}$.
D. $\frac{\pi}{2}$. Incorrect. You may have thought of 0 as the minimum value, but the sine function takes on negative values. The correct answer is $\frac{3\pi}{4}$.

What is the period of the function $y=\sin 3x$? Here is a table with some inputs and outputs for this function.

x **(in radians)**	$3x$ **(in radians)**	$\sin 3x$
0	0	0
$\frac{\pi}{9}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{6}$	$\frac{\pi}{2}$	1
$\frac{2\pi}{9}$	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{3}$	π	0
$\frac{4\pi}{9}$	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	$\frac{3\pi}{2}$	-1

$\frac{5\pi}{9}$	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{2\pi}{3}$	2π	0

As the values of x go from 0 to $\frac{2\pi}{3}$, the values of $3x$ go from 0 to 2π. We can see from the graph that $y=\sin 3x$ goes through one full cycle on the interval [0, $\frac{2\pi}{3}$], so its period is $\frac{2\pi}{3}$.



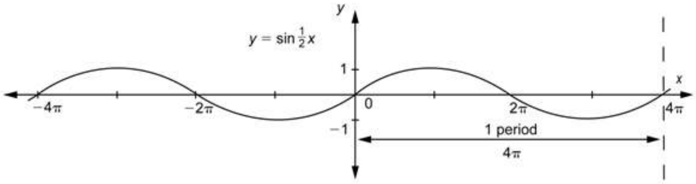
Notice that $y=\sin 3x$ has three cycles on the interval [0, 2π], which is the interval $y=\sin x$ needs to complete one full cycle.

What is the period of the function $y=\sin\left(\frac{1}{2}\right)x$? Here is a table with some inputs and outputs for this function.

x **(in radians)**	$\frac{1}{2}x$ (in radians)	$\sin\left(\frac{1}{2}\right)$
0	0	0
$\frac{2\pi}{3}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
π	$\frac{\pi}{2}$	1
$\frac{4\pi}{3}$	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$
2π	π	0
$\frac{8\pi}{3}$	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
3π	$\frac{3\pi}{2}$	-1
$\frac{10\pi}{3}$	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
4π	2π	0

As the values of x go from 0 to 4π, the values of $\frac{1}{2}x$ go from 0 to 2π.

We can see from the graph that $y=\sin\left(\frac{1}{2}x\right)$ goes through one full cycle on the interval $[0,4\pi]$, so its period is 4π.



Notice that $y=\sin\left(\frac{1}{2}x\right)$ has half of one full cycle on the interval $[0,2^{\pi}]$, which is the interval $y=\sin x$

Let’s put these results into a table. For the first three functions, we have rewritten their periods with the numerator 2π so that the pattern becomes clear. Can you see a relationship between the function and the denominator in the periods? needs to complete one full cycle.

Function	**Period**
$y=\sin\left(\frac{1}{2}x\right)$	$4\pi=\frac{2\pi}{\frac{1}{2}}$
$y=\sin 1x$	$2\pi=\frac{2\pi}{1}$

Function	Period
$y=\sin 2x$	$\pi=\frac{2\pi}{2}$
$y=\sin(3x)$	$\frac{2\pi}{3}=\frac{2\pi}{3}$

In each case, the period could be found by dividing 2π by the coefficient of x. In general, the period of $y=\sin bx$ is $\frac{2\pi}{|b|}$, and the period of $y=\cos bx$ is $\frac{2\pi}{|b|}$. Since the period is the length of an interval, it must always be a positive number. Since it is possible for b to be a negative number, we must use $|b|$ in the formula to be sure the period, $\frac{2\pi}{|b|}$, is always a positive number.

> The period of $y=\sin bx$ or $y=\cos bx$ is $\frac{2\pi}{|b|}$.

You can think of the different values of b as having an "accordion" (or a spring) effect on the graphs of sine and cosine. A large value of b squeezes them in and a small value of b stretches them out.

There is another way to describe this effect. In the interval [0, 2π], $y=\sin x$ goes through one cycle while $y=\sin 2x$ goes through two cycles. If you go back and check all of the examples above, you will see that $y=\sin bx$ has $|b|$ cycles in the interval $[0,2\pi]$. Likewise, $y=\cos bx$ has $|b|$ cycles in the interval [0, 2π].

✓ Example

Problem: What are the periods of $y=\sin 4x$ and $y=\cos\left(-\frac{1}{2}x\right)$?

Answer

For $y=\sin 4x$, $b=4$. Substitute this value into the formula.

$$\text{period }=\frac{2\pi}{|b|}=\frac{2\pi}{4}=\frac{\pi}{2}$$

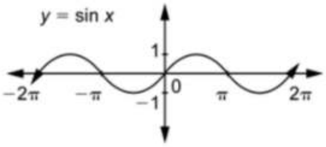
For $y=\cos\left(-\frac{1}{2}x\right)$, $b=-\frac{1}{2}$.

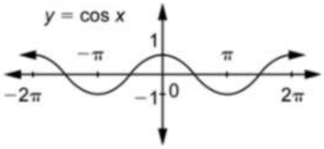
$$\begin{aligned}\text{period }&=\frac{2\pi}{|b|}=\frac{2\pi}{\left|-\frac{1}{2}\right|}=\frac{2\pi}{\frac{1}{2}}\\&=\frac{2\pi}{1}\cdot\frac{2}{1}=4\pi\end{aligned}$$

The period of $y=\sin 4x$ is $\frac{\pi}{2}$, and the period of $y=\cos\left(-\frac{1}{2}x\right)$ is 4π.

Amplitude

As you have seen, the graphs of all of these sine and cosine functions alternate between hills and valleys. The height of one hill (which equals the depth of one valley) is called the **amplitude**. You can see that for all the graphs we have looked at so far, the amplitude equals 1.





The formal way to say this for any periodic function is:

$$\text{amplitude }=\frac{\text{maximum}-\text{minimum}}{2}$$

You know that the maximum value of $y=\sin x$ or $y=\cos x$ is 1 and the minimum value of either is -1. So if you applied the above definition, you would get:

$$\text{amplitude }=\frac{1-(-1)}{2}=\frac{1+1}{2}=\frac{2}{2}=1$$

This result agrees with what was observed from the graph.

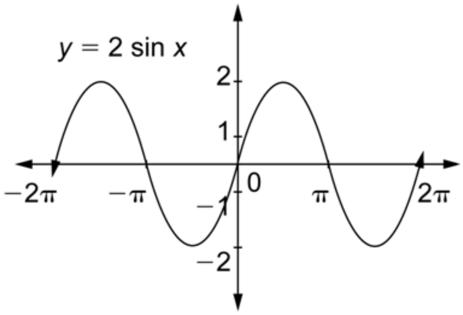
You have seen that changing the value of b in $y=\sin bx$ or $y=\cos bx$ either stretches or squeezes the graph like an accordion or a spring, but it does not change the maximum or minimum values. For all of these functions, the maximum is 1 and the minimum is

-1. So if you applied the definition of amplitude, you would be doing the exact same calculation as we just did above. The amplitude of any of these functions is 1.

Let's look at a different kind of change to a function by graphing the function $y=2\sin x$. Here's a table with some values of this function.

x **(in radians)**	$\sin x$	$2\sin x$
0	0	0
$\frac{\pi}{6}$	$\frac{1}{2}$	1
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	2
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	1
π	0	0

We'll take the first and third columns to make part of the graph and then extend that pattern to the left and to the right.



Now you can use this graph in the following example.

Example

Problem: What is the amplitude of $y=2\sin x$?

Answer

You can find the maximum and minimum values of the function from the graph. For example, at $x=\frac{\pi}{2}$ the value is 2, and at $x=-\frac{\pi}{2}$ the value is -2.

$$\begin{aligned}\text{maximum } &= 2\\ \text{minimum } &= -2\end{aligned}$$

Use the definition of amplitude.

$$\text{amplitude } =\frac{2-(-2)}{2}=\frac{2+2}{2}=\frac{4}{2}=2$$

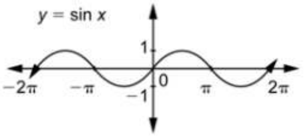
Notice that the height of each hill is 2, and the depth of each valley is 2. This is equal to the amplitude, as we mentioned at the start.

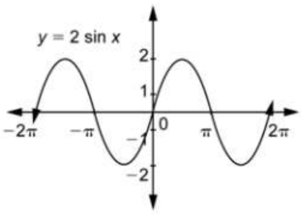
Notice also that the amplitude is equal to the coefficient of the function: $y=2\sin x$

This is not a coincidence.

The amplitude is 2.

Let’s compare the graph of this function to the graph of the sine function.

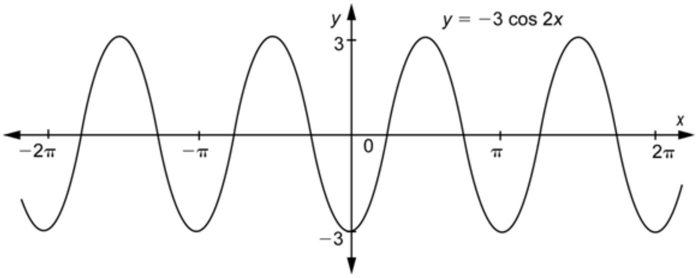




The effect of multiplying $\sin x$ by 2 is to stretch the graph vertically by a factor of 2. Because it has been stretched vertically by this factor, the amplitude is twice as much, or 2. If we had looked at $y = 3\sin x$, the graph would have been stretched vertically by a factor of 3, and the amplitude of this function is 3. Again, this is equal to the coefficient of the function. In general, we have the following rule.

> The amplitude of $y = a\sin bx$ or $y = a\cos bx$ is $|a|$.

As the last example, $y = 2\sin x$, shows, multiplying by a constant on the outside affects the amplitude. If you multiply by a constant on the outside and on the inside, as in $y = -3\sin 2x$, you will affect both the amplitude and the period. Here is the graph of $y = -3\sin 2x$:



✓ Example

Problem: Determine the amplitude and period of $y = -3\cos 2x$.

Answer

Use the formula for amplitude, with $a = -3$.

$$\text{amplitude } = |a| = |-3| = 3$$

Use the formula for period, with $b = 2$.

$$\text{period } = \frac{2\pi}{|b|} = \frac{2\pi}{2} = \pi$$

The amplitude is 3 and the period is π.

In this example, you could have found the period by looking at the graph above. One complete cycle is shown, for example, on the interval [0, π], so the period is π.

In the functions $y = a\sin bx$ and $y = a\cos bx$, multiplying by the constant a only affects the amplitude, not the period. As we said earlier, changing the value of b only affects the period, not the amplitude. The general result is as follows.

> The amplitude of $y = a\sin bx$ or $y = a\cos bx$ is $|a|$.
>
> The period of $y = a\sin bx$ or $y = a\cos bx$ is $\frac{2\pi}{|b|}$.

Supplemental Interactive Activity

To help you understand changes in amplitude and period for both the sine function and cosine function, try the following interactive exercise.

*Insert Interactive Activity

Exercise

What are the amplitude and period of $y=\frac{1}{2}\sin 4x$?

A. The amplitude is $\frac{1}{2}$, and the period is $\frac{\pi}{2}$.
B. The amplitude is $\frac{1}{2}$, and the period is 8π.
C. The amplitude is 1, and the period is $\frac{\pi}{2}$.
D. The amplitude is 1, and the period is 8π.

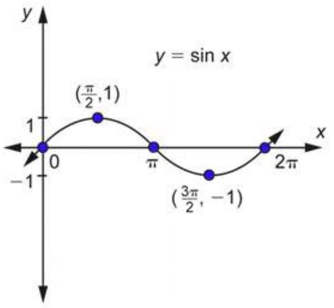
Answer

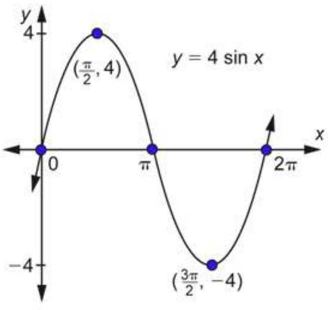
A. Correct. In this function, $a=\frac{1}{2}$, so this is the amplitude. The period equals $\frac{2\pi}{|b|}=\frac{2\pi}{4}=\frac{\pi}{2}$.
B. Incorrect. The amplitude is correct, but the period is not. You probably multiplied 2π by 4 instead of dividing. The correct answer is A.
C. Incorrect. The period is correct, but the amplitude is not. You may have thought the amplitude is the maximum minus the minimum, but it is half of this. The correct answer is A.
D. Incorrect. You may have thought the amplitude is the maximum minus the minimum, but it is half of this. You probably multiplied 2π by 4, instead of dividing, to find the period. The correct answer is A.

Graphs of Sine Functions

You know the function $y=a\sin bx$ has amplitude $|a|$ and period $\frac{2\pi}{|b|}$. You can use these facts to draw the graph of any function in the form $y=a\sin bx$ by starting with the graph of $y=\sin x$ and modifying it.

For example, suppose you wanted the graph of $y=4\sin x$. Since $b=1$, this function has the same period as $y=\sin x$. Since $a=4$, the amplitude is 4. Therefore, you would take the graph of $y=\sin x$ and simply stretch it vertically by a factor of 4. Here is one cycle for these two functions.





Note that the points that were on the x-axis “remain” on the x-axis. At these points (where $x=0,\pi,2\pi$), the value of $\sin x$ is 0. If you multiply 0 by 4 (or anything else), you will still have a value of 0. So the points will still be on the x-axis. On the other hand, the highest and lowest points have moved away from the x-axis. They had y-values of 1 and -1 for $\sin x$, and they have y-values of 4 and -4 for $4\sin x$.

Example

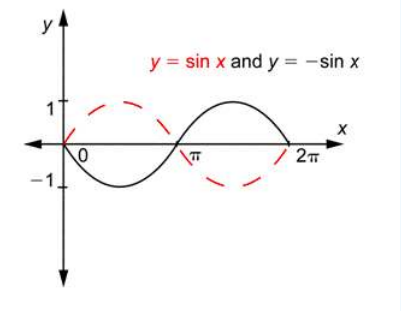
Problem: Graph $y=-\sin x$ on the interval [0, 2π].

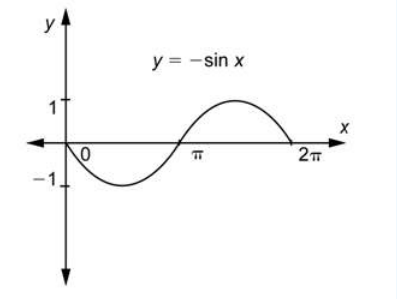
Answer

The value of b is 1, so the graph has a period of 2π, as does $y=\sin x$.

The value of a is -1, so the graph has an amplitude of 1, as does $y=\sin x$.

Though the amplitude and the period are the same as the function $y = \sin x$, the graph is not exactly the same. The effect of multiplying by -1 is to replace y-values by their opposites. So the graph of $y = \sin x$ gets reflected over the x-axis.





If you want to check these graphs with a graphing calculator, make sure that the graphing window has the correct settings. For this last example, you would use $0 \leq x \leq 2\pi$ and $-1 \leq y \leq 1$. In general, you would probably want to adjust the x-values to show one full cycle and the y-values using the amplitude.

If the values of a and b are both different from 1, then you need to combine the effects of the two changes.

✓ Example

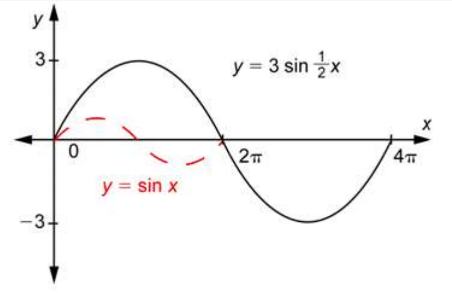
Problem: Graph $y = 3\sin\left(\frac{1}{2}x\right)$ on the interval [0, 4π].

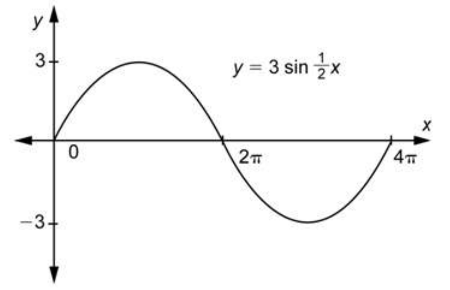
Answer

The value of a is 3, so the graph has an amplitude of 3. This has the effect of taking the graph of $y = \sin x$ and stretching it vertically by a factor of 3.

The value of b is $\frac{1}{2}$, so the graph has a period of $\frac{2\pi}{\frac{1}{2}} = \frac{2\pi}{1} \cdot \frac{2}{1} = 4\pi$. This is twice the period of $y = \sin x$. This has the effect of taking the graph of $y = \sin x$ and stretching it horizontally by a factor of 2. (The alternative way to say this is that $y = 3\sin\left(\frac{1}{2}x\right)$ has $\frac{1}{2}$ of a cycle on the interval [0, 2π].)

To make the graph of $y = 3\sin\left(\frac{1}{2}x\right)$, you must combine the two effects described above.





Sometimes you need to stretch the graph of the sine function, and sometimes you need to shrink it.

Example

Problem: Graph two cycles of a sine function whose amplitude is $\frac{1}{2}$ and whose period is $\frac{2\pi}{3}$.

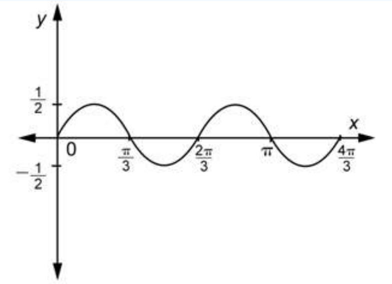
Answer

There are different functions of the form $y = a\sin bx$ that fit this description because a and b could be positive or negative. We will draw the graph assuming these are positive.

Because the amplitude is $\frac{1}{2}$, this has the effect of taking the graph of $y = \sin x$ and shrinking it vertically by a factor of 2.

The period is $\frac{2\pi}{3} = \frac{1}{3} \cdot 2\pi$, which is $\frac{1}{3}$ the period of $y = \sin x$. This has the effect of taking the graph of $y = \sin x$ and shrinking it horizontally by a factor of 3.

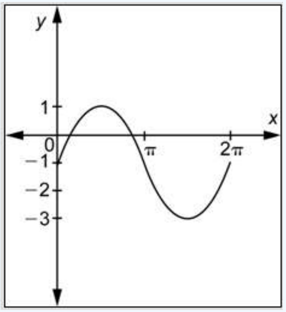
To make the graph, you must combine the two effects described above.

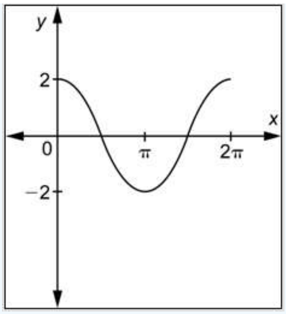


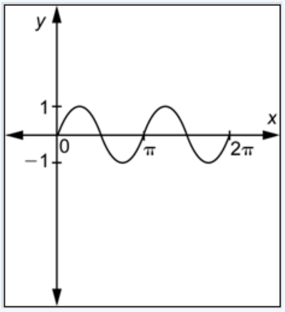
Even without knowing the specific value of a constant, you can sometimes still narrow down the possibilities for the shape of a graph.

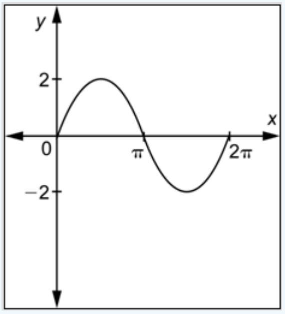
Exercise

The graph of a function $y = a\sin x$, where a is a constant, is drawn on the interval $0 \le x \le 2\pi$. Which of the following options could be this graph?

A. 

B. 

C. 

D. 

Answer

A. Incorrect. This has the correct shape and period, but it is in the wrong position. Regardless of the value of a, the graph must pass through the x-axis at $x=0, \pi, 2\pi$, which it does not. The correct answer is D.
B. Incorrect. This is the graph of a cosine function. Regardless of the value of a, the graph must pass through the x-axis at $x=0, \pi, 2\pi$. The correct answer is D.
C. Incorrect. Because the coefficient of x is 1, the graph should have a period of 2π, but this graph has a period of π. The correct answer is D.
D. Correct. Because the coefficient of x is 1, the graph has a period of 2π, which this option has. The factor a could stretch or shrink the graph, but it must still pass through the x-axis at the points $x=0, \pi, 2\pi$, which it does.

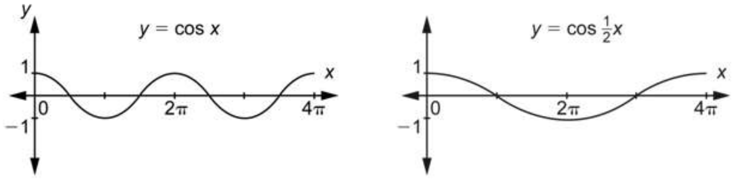
Graphs of Cosine Functions

You know the function $y=a\cos bx$ has amplitude $|a|$ and period $\frac{2\pi}{|b|}$. Just as you did with sine functions, you can use these facts to draw the graph of any function in the form $y=a\cos bx$ by starting with the graph of $y=\cos x$ and modifying it.

For example, suppose you wanted the graph of $y=\cos\left(\frac{1}{2}x\right)$. Since $a=1$, it has the same amplitude as $y=\cos x$. And, because $b=\frac{1}{2}$, the period is given by:

$$\frac{2\pi}{|b|}=\frac{2\pi}{\frac{1}{2}}=\frac{2\pi}{1}\cdot\frac{2}{1}=4\pi$$

Since this is twice the period of $y=\cos x$, you would take the graph of $y=\cos x$ and stretch it horizontally by a factor of 2. Here is a side-by-side comparison of these two graphs.



Note that in the interval [0, 2π], the graph of $y=\cos x$ has one full cycle. Since $b=\frac{1}{2}$, the graph of $y=\cos\left(\frac{1}{2}x\right)$ has $\frac{1}{2}$ of a cycle in that interval.

If you are using a graphing calculator, you need to adjust the settings for each graph to get a graphing window that shows all the features of the graph. For the last example, you would use $0\leq x\leq 4\pi$ and $-1\leq y\leq 1$. In general, you would probably want to adjust the x-values to show one full cycle and the y-values to show the highest and lowest points. In the next example, you would use $-\pi\leq x\leq\pi$ and $-2\leq y\leq 2$ for the graphing window because you are specifically asked to graph it over the domain $[-\pi,\pi]$ and the graph will have an amplitude of 2, going as low as -2 and as high as +2.

✓ Example

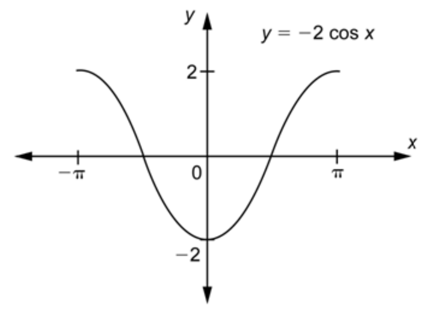
Problem: Graph $y=-2\cos x$ on the interval $[-\pi,\pi]$.

Answer

The amplitude equals $|a|=|-2|=2$. This has the effect of taking the graph of $y=\cos x$ and stretching it vertically by a factor of 2. The negative sign on the “outside” has an additional effect: the y-values are replaced by their opposites, so the graph is also flipped over the x-axis.

The value of b is 1, so the graph has a period of 2π, as does $y=\cos x$.

When the only change is a vertical stretch, compression, or flip, the x-intercepts remain the same. So the graph will pass through the x-axis at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.



Again, if the values of a and b are both different from 1, you need to combine the effects of the two changes. In the next example, you will see a variation that you have not seen before. Up to this point, all of the values of b have been rational numbers, but here we are using the irrational number π. This situation does not really change the procedure, but you will see that it changes the scale on the x-axis in a new way.

✓ Example

Problem: Graph $y=4\cos\pi x$ on the interval [0, 4].

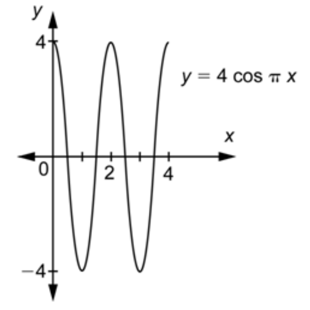
Answer

The value of a is 4, so the graph has an amplitude of 4. This has the effect of taking the graph of $y=\cos x$ and stretching it vertically by a factor of 4.

The value of b is π, so the graph has a period of $\frac{2\pi}{\pi}=2$. This has the effect of shrinking the graph of $y=\cos x$ horizontally by a factor of π, causing it to complete one complete cycle on the interval [0, 2].

Note the effect on the x-values of the intercepts, the high points, and the low points. Because the period is 2, the first cycle of the graph will have high points at $x = 0$ and 2. The low point will still be midway between these, so it is at $x = 1$. The x-intercepts are still midway between the high and the low points, so they will be at $x = \frac{1}{2}$ and $\frac{3}{2}$. The second cycle of the graph has all of these points shifted to the right 2 units.

To make the graph of $y = 4\cos \pi x$, you must combine the effects described above.

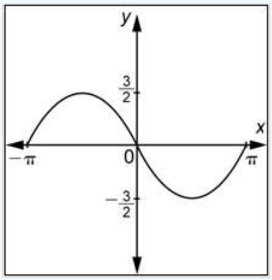


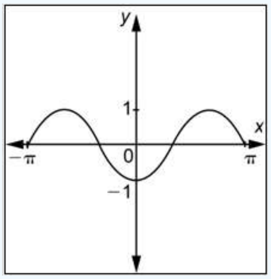
In two previous examples ($y = -\sin x$ and $y = -\cos x$), you saw that a negative sign on the outside (a negative value of a) has the effect of flipping the graph around the x-axis. In the next example, you will see the effect of a negative sign on the “inside” (a negative value of b).

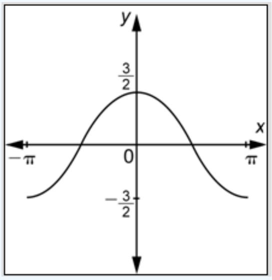
One last hint: besides trying to figure out the overall effect of the value of a or b on the graph, you might want to check specific points. For example, just substitute $x = 0$ into the function and see where that point will end up.

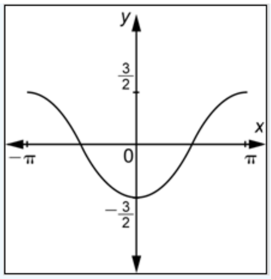
? Exercise

Which of the following options is the graph of $y = \frac{3}{2}\cos(-x)$ on the interval $-\pi \leq x \leq \pi$?

A.


B.


C.


D. 

Answer

A. Incorrect. This is the graph of $y=\frac{3}{2}\sin(-x)$. Remember to check specific points like $=0$. At that point, $y=\frac{3}{2}\cos(0)=\frac{3}{2}\cdot 1=\frac{3}{2}$. So the point $\left(0,\frac{3}{2}\right)$ should be on the graph. The correct answer is C.

B. Incorrect. You confused the effects of a and b. This is the graph of $y=-\cos\left(\frac{3}{2}x\right)$. The correct answer is C.

C. Correct. The value of a is $\frac{3}{2}$, which will stretch the graph vertically by a factor of $\frac{3}{2}$. The period of the graph is $\frac{2\pi}{|-1|}=2\pi$, as is the period of $y=\cos x$. The effect of the negative sign on the inside is to replace x-values by their opposites. This will flip the graph around the y-axis. However, because the graph of cosine is symmetric about the y-axis, this has no effect at all. So the only change to the graph of $y=\cos x$ is the vertical stretch.

D. Incorrect. This graph has the correct period and amplitude. However, you have confused the effect of a minus sign on the inside with a minus sign on the outside. The correct answer is C.

Matching Graphs and Functions

Given any function of the form $y=a\sin bx$ or $y=a\cos bx$, you know how to find the amplitude and period and how to use this information to graph the functions.

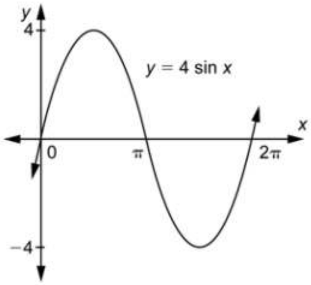
Given a graph of a sine or cosine function, you also can determine the amplitude and period of the function. From this information, you can find values of a and b, and then a function that matches the graph.

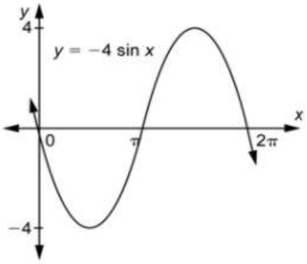
Remember that along with finding the amplitude and period, it’s a good idea to look at what is happening at $x=0$. If a and b are any nonzero constants, the functions $y=a\sin bx$ and $y=a\cos bx$ will have the following values at $x=0$:

$$y=a\sin(b\cdot 0)=a\sin 0=a\cdot 0=0$$
$$y=a\cos(b\cdot 0)=a\cos 0=a\cdot 1=a\neq 0$$

This tells you that the graph of $y=a\sin bx$ passes through (0, 0) regardless of the values of a and b, and the graph of $y=a\cos bx$ never passes through (0, 0) regardless of the values of a and b. So, for example, if you are given a graph passing through the origin and are asked to determine which function it represents, you know right away that it is not in the form $y=a\cos bx$.

You need to be careful about the sign of a. You might determine that a function has an amplitude of 4, for example. Though it is possible that $a=4$, it is also possible that $a=-4$. Here is an example of each of these two possibilities.



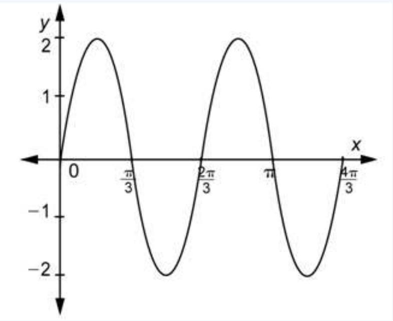


You will need to compare the graph to that of $y=\sin x$ or $y=\cos x$ to see if, in addition to any stretching or shrinking, there has been a **reflection** over the x-axis. The graph above on the right can be thought of as the result of stretching and reflecting the graph of $y=\sin x$ across the x-axis. If there has been a reflection, then the value of a will be negative. Once you have determined if a is positive or negative, you can always choose a positive value of b.

Here are some examples of this process.

Example

Problem: Determine a function of the form $y = a\sin bx$ or $y = a\cos bx$ whose graph is shown below.



Answer

First, observe that the graph passes through the origin, so you are looking for a function of the form $y = a\sin bx$.

Next, observe that the maximum value of the function is 2 and the minimum is -2, so the amplitude is 2. The graph has the same “orientation” as $y = \sin x$. (It starts with a hill to the right of the y-axis.) This implies that a is positive, and in particular, a $= 2$.

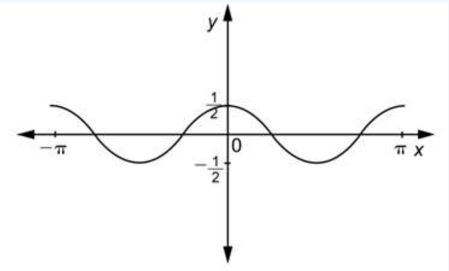
Finally, observe that the graph shows two cycles and that one entire cycle is contained in the interval $\left[0, \frac{2\pi}{3}\right]$. Therefore, the period is $\frac{2\pi}{3}$. Since $\frac{2\pi}{3} = \frac{2\pi}{|b|}$, $b = \pm 3$. According to our process, once you have determined if a is positive or negative, you can always choose a positive value of b. So $b = 3$.

Combine these three pieces of information.

$y = 2\sin 3x$

Example

Problem: Determine a function of the form $y = a\sin bx$ or $y = a\cos bx$ whose graph is shown below.



Answer

First, observe that the graph does *not* pass through the origin, but rather crests, reaching a maximum when $x = 0$, so you are looking for a function of the form $y = a\cos bx$.

Next, observe that the maximum value of the function is $\frac{1}{2}$ and the minimum is $-\frac{1}{2}$, so the amplitude is $\frac{1}{2}$. The graph has the same “orientation” as $y = \cos x$. (It has a hill with the y-axis running through the middle.) This implies that a is positive, and in particular, $a = \frac{1}{2}$.

Finally, observe that the graph shows two cycles and that one entire cycle is contained in the interval [0, π]. Therefore, the period is π. Because $\pi = \frac{2\pi}{|b|}$, $b = \pm 2$. According to our process, once you have determined if a is positive or negative, you can always choose a positive value of b. So $b = 2$.

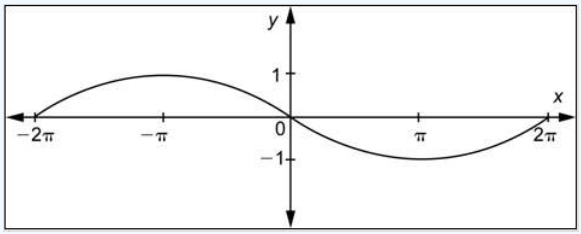
Combine these three pieces of information.

$y = \frac{1}{2}\cos 2x$

Make sure that you recognize where a cycle starts and ends. The period is the length of the interval over which one cycle runs.

Exercise

Problem: Which of the following functions is represented by the graph below?



A. $y = -\sin x$
B. $y = \sin\left(\frac{1}{2}x\right)$
C. $y = -\sin 2x$
D. $y = -\sin\left(\frac{1}{2}x\right)$

Answer

A. $y = -\sin x$. Incorrect. You correctly recognized the graph as a reflected sine function, but the period is incorrect. Perhaps you saw the 2π on the right and used that as the length of one cycle. However, the entire graph is one cycle, and the period equals $2\pi - (-2\pi) = 4\pi$. The correct answer is D.

B. $y = \sin\left(\frac{1}{2}x\right)$. Incorrect. You correctly found the amplitude and period of this sine function. However, you also need to check the orientation of the graph. Notice that to the right of the y-axis you have a valley instead of a hill. The correct answer is D.

C. $y = -\sin 2x$. Incorrect. You correctly found the amplitude and the orientation of this sine function. However, the period is incorrect. Perhaps you recognized that the period of the graph is twice the period of $y = \sin x$, and thought that the value of b would be 2. However, to find the value of b, you must set the period equal to $\frac{2\pi}{|b|}$. The correct answer is D.

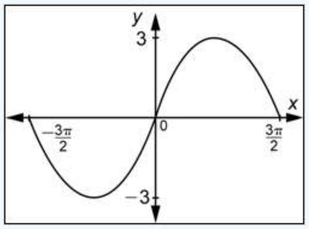
D. $y = -\sin\left(\frac{1}{2}x\right)$. Correct. The graph passes through the origin, so the function could have the form $y = a\sin bx$, but not $y = a\cos bx$. The amplitude is 1. The graph has a valley on the right, which could be the result of a reflection of $y = \sin x$ over the x-axis. Therefore, $a = -1$. The graph shows one cycle, so the period is 4π. Since $4\pi = \frac{2\pi}{|b|}$, the value of b could be $\frac{1}{2}$. So this could be the graph of $y = -\sin\left(\frac{1}{2}x\right)$.

Remember that when writing a function you can use the notation $f(x)$ in place of the variable y.

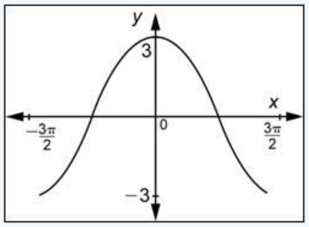
Exercise

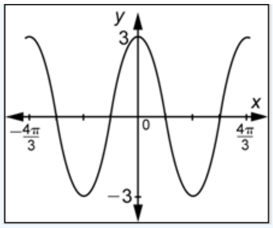
Which of the following graphs represents $f(x) = 3\cos\left(\frac{2}{3}x\right)$?

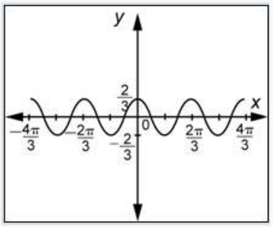
A.



B.



C. 

D. 

Answer

A. Incorrect. This is the graph of a function of the form $f(x) = a\sin bx$. Remember to check the value of the function at $x = 0$. Since $f(0) = 3$, the function $f(x) = 3\cos\left(\frac{2}{3}x\right)$ passes through (0, 3), not the origin as shown in this graph. The correct answer is B.

B. Correct. First, this graph has the shape of a cosine function. Second, because $a = 3$ in the equation, the amplitude is 3. Finally, because $b = \frac{2}{3}$,, the period of this function is $\frac{2\pi}{\frac{2}{3}} = \frac{2\pi}{1} \cdot \frac{3}{2} = 3\pi$. The graph in this answer completes one full cycle between $x = -\frac{3}{2}\pi$ and $x = \frac{3}{2}\pi$ so its period is $\frac{3}{2}\pi - \left(-\frac{3}{2}\pi\right) = 3\pi$ as needed.

C. Incorrect. This graph does have the shape of a cosine function, and the amplitude is 3, which is correct. However, the period is incorrect. You probably multiplied 2π by $\frac{2}{3}$ instead of dividing. The correct answer is B.

D. Incorrect. This graph does have the shape of a cosine function. However, in determining the graph, it appears that you switched the values of a and b. The correct answer is B.

Summary

The functions $y = a\sin bx$ and $y = a\cos bx$ are periodic functions: their graphs have a repeating pattern of hills and valleys that continues in both directions forever. The height of the hill or the depth of the valley is called the amplitude, and is equal to $|a|$. Any one full pattern in the graph is called a cycle, and the length of an interval over which a cycle occurs is called the period. The period is equal to the value $\frac{2\pi}{|b|}$.

You can use this information to graph any of these functions by starting with the basic graph of $y = \sin x$ or $y = \cos x$ and then doing a combination of stretching or shrinking the graph vertically based on the value of a, stretching or shrinking the graph horizontally based on the value of b, or reflecting it based on the signs of a and b.

You can also start with a graph, determine the values of a and b, and then determine a function that it represents.

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Glossary

absolute value | The absolute value of a number is its distance from 0 on a number line.

actor pair | A pair of numbers whose product is a given number. For example, 2 and 15 are a factor pair of of 30 because 2·15=30. Both 2 and 15 are factors of 30.

acute angles | An angle measuring less than 90^o.

Acute Triangle | A triangle with three angles that each measure between 0^o and 90^o.

addends | A number added to one or more numbers to form a sum.

addends | A number added to one or more other numbers to form a sum.

addition property of 0 | The sum of any number and 0 is equal to that number. The number 0 is often called the additive identity.

addition property of equality | For all real numbers a, b, and c, if $a = b$ then $a + c = b + c$. If two expressions are equal to each other and you add the same value to both sides of the equation, the equation will remain equal.

additive identity | The number 0 is called the additive identity because when you add it to a number, the result you get is the same number. For example, 4+0=4.

additive inverses | Any two numbers whose sum is zero, such as 3 and -3, because 3+(-3)=0.

adjacent side | For a given acute angle in a right triangle, the adjacent side to that angle is the side that, along with the hypotenuse, forms that acute angle.

amount | In a percent problem, the portion of the whole corresponding to the percent.

amplitude | The distance between the highest point and the rest position (zero position) in a wave.

angle | A figure formed by the joining of two rays with a common endpoint.

area | The amount of space inside a two-dimensional shape, measured in square units.

arithmetic operations | The operations of addition, subtraction, multiplication and division.

associative law of addition | For three or more numbers, the sum is the same regardless of how you group the numbers. For example, (6+2)+1=6+(2+1).

associative law of multiplication | For three or more numbers, the product is the same regardless how you group the numbers. For example, (3·5)·7=3·(5·7)

associative property of addition | For three or more real numbers, the sum is the same regardless of how you group the numbers. For example, $(6 + 2) + 1 = 6 + (2 + 1)$.

associative property of multiplication | For three or more real numbers, the product is the same regardless of how you group the numbers. For example, $(3 \cdot 5) \cdot 7 = 3 \cdot (5 \cdot 7)$.

axis | One of two perpendicular lines of a coordinate place that intersect at the origin. The plural form of axis is axes.

Bar graphs | A graph that uses horizontal or vertical bars to represent data.

bar graphs | A graph that uses horizontal or vertical bars to represent data.

base | In a percent problem, the base represents how much should be considered 100% (the whole); in exponents, the base is the value that is raised to a power when a number is written in exponential notation. In the example of 5^3, 5 is the base.

base | The expression that is being raised to a power when using exponential notation. In 5^3, 5 is the base, which is the number that is repeatedly multiplied. $5^3 = 5 \cdot 5 \cdot 5$ In a^b, a is the base.

base | The expression that is being raised to a power when using exponential notation. In 5^3, 5 is the base (which is the number that is repeatedly multiplied). $5^3 = 5 \cdot 5 \cdot 5$ And in a^b, the base is a.

binomial | A polynomial with exactly two terms, such as $5y^2 - 4^x$ and $x^5 + 6$.

binomials | A polynomial with exactly two terms, such as $5y^2 - 4x$ and $x^5 + 6$.

boundary line | A line that divides the coordinate plane into two regions. If points along the boundary line are included in the solution set, then a solid line is used; if points along the boundary line are not included then a dotted line is used.

box-and-whisker plot | A graph that uses a number line to show the distribution of a set of data.

Capacity | The amount of liquid (or other pourable substance) that an object can hold when it's full.

categorical data | Data that details non-numerical features of an object. Examples of categorical data include eye color, blood type, and types of computers.

Celsius | A measure of temperature commonly used in countries that use the metric system. On the Celsius scale, water freezes at 0^o Celsius and boils at 100^o Celsius.

central angle | An angle whose vertex is at the center of a circle.

circle graph | Also called a pie chart, a type of graph where categorical data is represented as sections of a whole circle.

circle graphs | Also called a pie chart, a type of graph where categorical data is represented as sections of a whole circle.

coefficient | A number that multiplies a variable.

coefficient | A number that multiplies a variable.

cofunctions | Two trigonometric functions, such as sine and cosine, for which the value of the first function at an acute angle equals the value of the second function at the complement of that angle.

common denominator | A number that is a multiple of all of the denominators in a group of fractions.

common log | A logarithm using 10 as the base ($\log_{10}$).

common logarithm | A logarithm using 10 as the base, such as $\log_{10}$.

commutative law of addition | Two numbers can be added in any order without changing the sum. For example: 6+4=4+6

commutative law of multiplication | Two numbers can be multiplied in any order without changing the product. For example, 8·9=9·8

commutative property of addition | Two real numbers can be added in any order without changing the sum. For example, 6+4=4+6.

commutative property of multiplication | Two real numbers can be multiplied in any order without changing the product. For example, $8 \cdot 9 = 9 \cdot 8$.

complementary angles | Two angles whose measurements add up to 90^o.

completing the square | A method for solving quadratic equations by rewriting one side of the equation as a squared binomial.

complex fraction | A quotient of two fractions.

complex number | A number in the form $a + bi$, where a and b are real numbers and i is the square root of -1.

complex numbers | A number in the form $a + bi$, where a and b are real numbers and i is the square root of -1.

complex rational expression | A quotient of two rational expressions.

composite number | A natural number that has at least one factor other than 1 and itself.

compound event | An event with more than one outcome.

compound inequality | A statement including two inequality statements joined either by the word “or” or “and.” For example, $2x - 3 < 5$ and $x + 14 > 11$.

cone | A solid figure with a single circular base and a round, smooth face that diminishes to a single point.

congruent | Having the same size and shape.

conjugate | One binomial in a conjugate pair. Given the binomial $a + b$, the conjugate is $a - b$; given $a - b$ the conjugate is $a + b$.

conjugate pair | A pair of binomials that, when multiplied, follow the pattern: $(a + b)(a - b) = a^2 - b^2$. The product of a pair of binomials that are conjugates is the difference of two squares.

consistent system of linear equations | A system of linear equations that has at least one solution.

constant | A symbol that represents a quantity that cannot change. It can be a number, letter or a symbol.

constant of variation | Represented by the variable k in variation problems, the constant of variation is a number that relates the input and the output.

constants | A symbol that represents a quantity that cannot change. It can be a number, letter or a symbol.

coordinate plane | A plane formed by the intersection of a horizontal number line called the x-axis and a vertical number line called the y-axis.

corresponding angles | Angles of separate figures that are in the same position within each figure.

corresponding sides | Sides of separate figures that are opposite corresponding angles.

cosine | If A is an acute angle of a right triangle, then the cosine of angle A is the ratio of the length of the side adjacent to angle A over the length of the hypotenuse.

coterminal angles | The description of two angles drawn in standard position that share their terminal side.

counting numbers | Also called natural numbers, the numbers 1, 2, 3, 4, ...

cube | A six-sided polyhedron that has congruent squares as faces.

cube root | The number which, when multiplied together three times yields the original number. For example, the cube root of 64 is 4 because $4\cdot 4\cdot 4=64$.

cubing |

Raising a number to a power of 3. 2^3 is read “2 to the third power” or “2 cubed,” and means use 2 as a factor three times in the multiplication.

2^3= 2·2·2=8.

cup | A unit of capacity equal to 8 fluid ounces.

cycle | Any part of a graph of a periodic function that is one period long.

cylinder | A solid figure with a pair of circular, parallel bases and a round, smooth face between them.

data | Mathematical term for information such as values or measurements.

decimal fractions | A fraction written as a decimal point and digits to the right of the decimal point.

Decimal numbers | Decimal numbers are numbers whose place value is based on 10s, including whole numbers and decimal fractions, which have decimal points and digits to the right of the decimal point. The numbers 18, 4.12 and 0.008 are all decimal numbers.

degree | The value of an exponent.

degree of a monomial | The degree of a monomial is the power to which the variable is raised. For example, the monomial $5y^2$ has a degree of 2. If the monomial contains several variables then the degree of the monomial is the sum of the degree of all the variables. For example, the monomial $7x^2y^3$ has a degree of 5.

degree of a polynomial | The highest exponent or sum of exponents of a term in a polynomial. For example, $7x^2y^3+3x^2y-8$ is a 5th degree polynomial because the highest sum of exponents in a term is $2+3=5$.

denominator | The bottom number of a fraction that tells how many equal parts are in the whole.

dependent linear equations | Equations that graph as the same straight line.

diameter | The length across a circle, passing through the center of the circle. A diameter is equal to the length of two radii.

difference | The quantity that results from subtracting one number from another, or from subtracting the subtrahend from the minuend.

digit | One of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9.

direct variation | A type of variation where the output varies directly with the input. Direct variation is represented by the formula $y=kx$.

discriminant | In the Quadratic Formula, the expression underneath the radical symbol: b^2-4ac. The discriminant can be used to determine the number and type of solutions the formula will reveal.

distributing | To rewrite the product of the number and a sum or difference using the distributive property.

distributive property of multiplication | The product of a sum (or a difference) and a number is the same as the sum (or difference) of the product of each addend (or each number being subtracted) and the number. For example, $3(4+2)=3(4)+3(2)$, and $3(4-2)=3(4)-3(2)$.

distributive property of multiplication over addition | The product of a number and a sum is the same as the sum of the product of the number and each of the addends making up the sum. For example, 3(4+2)=3(4)+3(2).

distributive property of multiplication over subtraction | The product of a number and a difference is the same as the difference of the product of the number and each of the numbers being subtracted. For example, 8(10-2)=8(10)-8(2).

dividend | The number to be divided up in a division problem. In the problem 8÷2=4, 8 is the dividend.

Divisibility tests | A rule that tells quickly whether dividing a number by another number can be done without leaving a remainder.

divisible | Can be divided by a number without leaving a remainder. For example, 20 is divisible by 4 because $20\div 4=5$ (no remainder).

divisor | The number that is being divided into the dividend in a division problem. In the problem 8÷2=4, 2 is the divisor.

divisor | The number that is being divided into the dividend in a division problem. In the problem $8\div 2=4$, 2 is the divisor.

domain | The set of all possible input values for the variable in a function.

domain of the function | The set of all input values or x-coordinates of the function.

e | An irrational number, approximately 2.718281828459; sometimes called Euler’s number.

elimination method | A method of solving a system of equations. Given a system, the elimination method allows you to add the two equations in order to eliminate a common variable.

equally likely | Having the same likelihood of occurring, such that in a large number of trials, two equally likely outcomes would happen roughly the same number of times.

equation | A mathematical statement that two expressions are equal.

equations | A mathematical statement that two expressions are equal.

Equilateral Triangle | A triangle with 3 equal sides.

equivalent fractions | Two or more fractions that name the same part of the whole.

estimate | An answer to a problem that is close to the exact number, but not necessarily exact.

evaluate | To find the value of an expression.

even number | A whole number that is divisible by 2.

event | A collection of possible outcomes, often describable using a common characteristic, such as rolling an even number with a die or picking a card from a specific suit.

event space | The set of possible outcomes in an event: for example, the event “rolling an even number” on a die has the event space of 2, 4, and 6.

excluded values | A value for the variable that is not included in the domain because it would cause the function to be undefined.

expanded form | A way to write a number as a sum of the value of its digits. For example, thirty-two is written in expanded form as 30+2, or 3 tens +2 ones, or (3·10)+(2·1).

exponent | The number that indicates how many times the base is used as a factor. In the example of 5^3, 3 is the exponent and means that 5 is used three times as a factor: 5·5·5

exponent | When a number is expressed in the form a^b, b is the exponent. The exponent indicates how many times the base is used as a factor. Power and exponent mean the same thing.

exponent | When a number is expressed in the form a^b, b is the exponent. The exponent indicates how many times the base is used as a factor. Power and exponent mean the same thing.

exponent | When a number is expressed in the form a^b, b is the exponent. The exponent indicates how many times the base is used as a factor. Power and exponent mean the same thing.

exponential decay | An exponential function of the form $f(x)=b^x$, where $0<b<1$. The function decreases as x increases.

exponential functions | A function of the form $f(x)=b^x$ where $b>0$ and $b\neq 1$.

exponential growth | An exponential function of the form $f(x)=b^x$, where $b>1$ and $b\neq 0$. The function increases as x increases.

Exponential notation | A notation that represents repeated multiplication using a base and an exponent. For example, 2^2 is notation that means 2·2·2·2. This notation tells you that 2 is used as a factor 4 times. 2^4=16. (Also called exponential form.)

exponential notation | A shorter way to write repeated multiplication. For example, 2^4 means $2\cdot 2\cdot 2\cdot 2$. Two is used as a factor 4 times.

Exponential notation | A shorter way to write repeated multiplication. For example, 2^4 means $2\cdot 2\cdot 2\cdot 2$. Two is used as a factor 4 times.

exponents | The number that indicates how many times the base is used as a factor. In the example of 5^3, 3 is the exponent and means that 5 is used three times as a factor: $5\cdot 5\cdot 5$

exponents | When a number is expressed in the form a^b, b is the exponent. The exponent indicates how many times the base is used as a factor. Power and exponent mean the same thing.

expression | A mathematical phrase. For example, $8\cdot 2+3$ is an expression. It represents the quantity 19.

expression | A mathematical phrase that can contain a combination of numbers, variables, or operations.

Expressions | A mathematical phrase that can contain a combination of numbers, variables, or operations.

extraneous solutions | A solution of the simplified form of an equation that does not satisfy the original equation and must be discarded.

face | The flat surface of a solid figure.

factor | A number that is multiplied by another number or numbers to get a product. For example, in the equation 4·5=20, 4 and 5 are factors.

factor label method | One method of converting a measurement from one unit of measurement to another unit of measurement. In this method, you multiply the original measurement by unit fractions containing different units of measurement to obtain the new unit of measurement.

factor tree | A diagram showing how a number can be written as factors, and those factors written as a product of factors, and so on until only prime numbers are used.

Factoring | The process of breaking a number down into its multiplicative factors.

factors | A number that is multiplied by another number or numbers to get a product. For example, in the equation $4 \cdot 5=20$, 4 and 5 are factors.

Fahrenheit | A measure of temperature commonly used in the United States. On the Fahrenheit scale, water freezes at 32^{o} Fahrenheit and boils at 212^{o} Fahrenheit.

fluid ounce | A unit of capacity equal to $\frac{1}{8}$ of a cup. One fluid ounce of water at 62^{o}F weighs about one ounce.

foot | A unit for measuring length in the U.S. customary measurement system. 1 foot = 12 inches

formulas | An equation or an expression that states a rule for a relationship among quantities. For example, the formula for finding the area of a rectangle can be represented as $A = l \cdot w$, or simply $l \cdot w$

Fractions | An expression used to refer to a part of a whole.

function | A relation that assigns to each x-value exactly one y-value.

function notation | An equation that takes the form $f(x) =$, and is read " f of x is..." For example, $f(x) = 3x + 7$.

Fundamental Counting Principle | If one event has p possible outcomes, and another event has m possible outcomes, then there are a total of $p \cdot m$ possible outcomes for the two events.

gallon | A unit equal to 4 quarts, or 128 fluid ounces.

gram | The base unit of mass in the Metric system.

greatest common factor | The largest number (or expression) that is a factor of a set of two or more numbers (or expressions).

greatest common factor (GCF) | The product of the prime factors that two or more terms have in common. The greatest common factor of xyz and $3xy$ is xy.

Grouping symbols | Symbols such as parentheses, braces, brackets, and fraction bars that indicate the numbers to be grouped together.

half-life | The amount of time it takes a substance to decrease to half its original amount.

histogram | A graph using bars to show continuous quantitative data over a series of similar-sized intervals. The height of the bar shows the frequency of the data, and the width of the bar represents the interval for the data.

histograms | A graph using bars to show continuous quantitative data over a series of similar-sized intervals. The height of the bar shows the frequency of the data, and the width of the bar represents the interval for the data.

hypotenuse | The side opposite the right angle in any right triangle. The hypotenuse is the longest side of any right triangle.

identity | An equation that is true for any possible value of the variable.

identity property of 0 | When you add 0 to any number, the sum is the same as the original number. For example, 55+0=55.

identity property of 1 | When you multiply any number by 1, the product is the same as the original number. For example, 9(1)=9.

imaginary number | A number in the form bi, where b is a real number and i is the square root of -1.

imaginary part | The imaginary term, bi, in a complex number $a + bi$.

improper fraction | A fraction in which the numerator is equal to or greater than the denominator.

inch | A unit for measuring length in the U.S. customary measurement system. 1 foot = 12 inches

inconsistent system of linear equations | A system of linear equations that has no solutions.

independent linear equations | Equations that graph as different straight lines.

index | The small positive integer just outside and above the radical symbol that denotes the root. For example, $\sqrt[3]{}$ denotes the cube root.

inequalities | A mathematical statement that shows the relationship between two expressions where one expression can be greater than or less than the other expression. An inequality is written by using an inequality sign (>, <, ≤, ≥, ≠).

inequality | A mathematical sentence that compares two numbers that are not equal.

inequality | A mathematical statement that shows the relationship between two expressions where one expression can be greater than or less than the other expression. An inequality is written by using an inequality sign (>, <, ≤, ≥, ≠).

initial side | The stationary ray that forms an angle in standard position and lies on the positive x-axis.

Integers | The numbers …, -3, -2, -1, 0, 1, 2, 3...

integers | The numbers ..., -3, -2, -1, 0, 1, 2, 3 ...

inverse function | If you take a function and reverse its inputs and outputs, then you get its inverse function.

inverse operation | A mathematical operation that can reverse or “undo” another operation. Addition and subtraction are inverse operations. Multiplication and division are inverse operations.

inverse operation | A mathematical operation that can reverse or “undo” another operation. Addition and subtraction are inverse operations. Multiplication and division are inverse operations.

inverse operations | A mathematical operation that can reverse or “undo” another operation. Addition and subtraction are inverse operations. Multiplication and division are inverse operations.

inverse variation | A type of variation where the output varies inversely with the input. Inverse variation is represented by the formula $y = \frac{k}{x}$.

Irrational numbers | Numbers that cannot be written as the ratio of two integers—the decimal representation of an irrational number is nonrepeating and nonterminating.

isolate the variable | A method for solving an equation that involves rewriting an equivalent equation in which the variable is on one side of the equation and everything else is on the other side of the equation.

isosceles trapezoid | A trapezoid with one pair of parallel sides and another pair of opposite sides that are congruent.

Isosceles Triangle | A triangle with 2 equal sides.

joint variation | A type of variation where the output varies jointly with multiple inputs. Joint variation is represented by the formula $y = kxz$.

least common denominator | (LCD) The least, or smallest, number that is a multiple of all the denominators in a group of fractions.

least common denominator | The smallest number (or expression) that is a multiple of all the denominators in a group of fractions (or rational expressions).

least common multiple | (LCM) The least, or smallest, number that is a multiple of two or more numbers.

least common multiple | The smallest number (or expression) that is a multiple of a set of two or more numbers (or expressions).

legs | In a right triangle, one of the two sides creating a right angle.

Length | The distance from one end to the other or the distance from one point to another.

like denominators | Denominators that are the same.

like terms | Terms that contain the same variables raised to the same powers. For example, $3x$ and $-8x$ are like terms, as are $8xy^2$ and $0.5xy^2$.

like terms | Terms that contain the same variables raised to the same powers. For example, $3x$ and $-8x$ are like terms, as are $8xy^2$ and $0.5xy^2$

like terms | Terms that contain the same variables raised to the same powers. For example, $3x$ and $-8x$ are like terms, as are $8xy^2$ and $0.5xy^2$.

line | A line is a one-dimensional figure, which extends without end in two directions.

line graphs | Used to show continuous data, a graph where individual data points are connected with line segments. Line graphs are typically used for data sets that track a quantity over time.

line of reflection | The line that cuts a parabola into two halves (which are mirror images of each other).

line segment | A finite section of a line between any two points that lie on the line.

linear equation | An equation in two variables whose ordered pairs graph as a straight line.

linear inequality | A mathematical statement in two variables using the inequality symbols <, >, ≤, or ≥ to show the relationship between two expressions. When the inequality symbol is replaced by an equal sign, the resulting related equation will graph as a straight line.

linear relationships | A linear relationship exists between two variables if, when you plot their values on a coordinate system, you get a straight line.

liter | The base unit of volume in the Metric system.

logarithm | A calculation in which the exponent y in $x = b^y$ is found when given x and b; the corresponding notation is $\log_b x = y$.

logarithmic functions | A function using a logarithm, of the form $f(x) = \log_b x$, $b > 0$ and $b \neq 1$. A calculation in which the exponent y in $x = b^y$ is found when given x and b; the corresponding notation is $\log_b x = y$.

logarithms | A calculation in which the exponent y in $x = b^y$ is found when given x and b; the corresponding notation is $\log_b x = y$.

lowest terms | A fraction is in lowest terms if the numerator and denominator have no common factors other than 1.

Mean | The sum of all the data values in a data set divided by the number of items in the data set; also called the average.

Measurement | The use of standard units to find out the size or quantity of items such as length, width, height, mass, weight, volume, temperature or time.

median | The middle number or the mean of the two middle numbers of a set of ordered data.

meter | The base unit of length in the Metric system.

metric system | A widely used system of measurement that is based on the decimal system and multiples of 10.

midrange | The mean of the greatest and least values of a data set.

mile | A unit for measuring length in the U.S. customary measurement system. 1 mile = 5,280 feet or 1,760 yards.

minuend | The number from which another number is subtracted.

mixed number | An expression in which a whole number is combined with a proper fraction. For example $5\frac{2}{3}$ is a mixed number.

mixed numbers | An expression in which a whole number is combined with a proper fraction. For example $5\frac{2}{3}$ is a mixed number.

mode | The number that appears most often in a data set.

monomial | A polynomial with exactly one term. $4x, \ -5y^2$, and 6 are all examples of monomials.

monomial | A polynomial with exactly one term. $4x, \ -5y^2$, and 6 are all examples of monomials.

monomials | A polynomial with exactly one term. $4x, \ -5y^2$, and 6 are all examples of monomials.

multi-step equation | An equation that requires more than one step to solve.

multiple | Any number that has a given number as a factor. For example, 4, 8, 16, and 200 are multiples of 4, because 4 is a factor of each of these numbers.

multiplication property of 1 | The product of any number and 1 is equal to that number. The number 1 is often called the multiplicative identity.

multiplication property of equality | For all real numbers a, b, and c, $c \neq 0$: If $a = b$, then $ac = bc$. If two expressions are equal to each other and you multiply both sides of the equation by the same non-zero number, the equation will remain equal.

multiplicative inverses | Two numbers are multiplicative inverses if their product is 1. For example, $\frac{3}{1} \cdot \frac{1}{3} = \frac{3}{3} = 1$.

natural log | A logarithm using e as the base, written as $\log_e$.

Natural logarithms | A logarithm that uses e as the base ($\log_e$).

Natural numbers | The numbers 1, 2, 3, 4 and so on. Also called counting numbers.

natural numbers | Also called counting numbers, the numbers 1, 2, 3, 4, ...

negative numbers | Numbers less than 0.

nonrepeating decimal | Numbers whose decimal parts continue without repeating—these are irrational numbers.

nonterminating decimals | Numbers whose decimal parts continue forever (without ending in an infinite sequence of zeros)—these decimals can be rational (if they repeat) or irrational (if they are nonrepeating).

numerator | The top number of a fraction that tells how many parts of a whole are being represented.

obtuse angles | An angle measuring more than 90^o and less than 180^o.

Obtuse Triangle | A triangle with one angle that measures between 90^o and 180^o.

one-step equations | An equation that requires only one step to solve.

operation | A mathematical process; the four basic operations are addition, subtraction, multiplication, and division.

opposite | An opposite of a number is the number with the opposite sign, but same absolute value. For example, the opposite of 72 is -72. A number plus its opposite is always 0.

opposite side | For a given acute angle in a right triangle, the opposite side to that angle is the side that is not one of the two sides that form that acute angle.

order of operations | The rules that determine the sequence of calculations in an expression with more than one type of computation.

order of operations | The rules that determine the sequence of calculations in an expression with more than one type of computation.

ordered pairs | A pair of numbers that indicates a point on a coordinate plane.

ordered pairs | A pair of numbers that indicates a point on a coordinate plane.

origin | The point where the *x*-axis and the *y*-axis intersect on the coordinate plane (0, 0).

ounce | A unit for measuring weight in the U.S. customary measurement system. 16 ounces = 1 pound.

outcome | A result of a trial.

parabola | A u-shaped graph which is produced by a quadratic function.

Parallel lines | Two or more lines that lie in the same plane but which never intersect.

parallel lines | Two or more lines that lie in the same plane but which never intersect.

percent | A ratio that compares a number to 100. “Per cent” means “per 100,” or “how many out of 100.”

perfect cube | A number whose cube root is an integer.

perfect square | A whole number that can be expressed as a whole number raised to a power of 2. For example, 25 is a perfect square because $25=5\cdot5=5^2$

perfect square trinomial | A trinomial that is the product of a binomial times itself, such as $a^2 + 2ab + b^2$ (from $(a+b)^2$), and $a^2 - 2ab + b^2$ (from $(a-b)^2$).

Perfect squares | A square of a whole number. Since $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, etc., 1, 4, and 9 are perfect squares.

perimeter | The distance around a two-dimensional shape.

period | The length of the smallest interval that contains exactly one copy of the repeating pattern of a periodic function.

periodic function | A function whose graph has a pattern that repeats forever in both directions.

periods | Each group of three digits in a number separated by a comma.

Perpendicular lines | Two lines that lie in the same plane and intersect at a angle.

perpendicular lines | Two lines that lie in the same plane and intersect at a 90^o angle.

pi | The ratio of a circle’s circumference to its diameter. Pi is denoted by the Greek letter π. It is often approximated as 3.14 or $\frac{22}{7}$.

pictograph | A graph that uses small icons or pictures to represent data.

pint | A unit of capacity equal to 16 fluid ounces, or 2 cups.

place value | The value of a digit based on its position within a number.

place-value chart | A chart that shows the value of each digit in a number.

plane | In geometry, a two-dimensional surface that continues infinitely. Any three individual points that don't lie on the same line will lie on exactly one plane.

point | A zero-dimensional object that defines a specific location on a plane. It is represented by a small dot.

polygon | A closed plane figure bounded by three or more line segments.

polygon | A closed plane figure with three or more straight sides.

polygon | A closed plane figure with three or more straight sides.

polygons | A closed plane figure with three or more straight sides.

Polyhedrons | A solid whose faces are polygons.

polynomial | A monomial or the sum or difference of two or more monomials.

polynomials | A monomial or the sum or difference of two or more monomials.

positive numbers | Numbers greater than 0.

pound | A unit for measuring weight in the U.S. customary measurement system. 16 ounces = 1 pound.

power | In an exponent a^b, the power is represented by b

Power Rule for Exponents | To raise a power to a power, multiply the exponents. \(\ (x^a)^b=x^{a\cdot b}

prefix | A short set of letters that denote the size of measurement units in the Metric System. Metric prefixes include centi-, milli-, kilo-, and hecto-.

prime factor | A factor that only has itself and as factors.

prime factorization | A number written as the product of its prime factors.

prime factorization | A number written as the product of its prime factors.

prime factorization | A number written as the product of its prime factors.

prime factorization | The process of breaking down a number (or expression) into its prime multiplicative factors. For example, the prime factorization of $12xy$ is $2 \cdot 2 \cdot 3 \cdot x \cdot y$.

prime factorization | The process of breaking down a number (or expression) into its prime multiplicative factors. For example, the prime factorization of $12xy$ is $2 \cdot 2 \cdot 3 \cdot x \cdot y$.

prime number | A natural number with exactly two factors: 1 and the number itself.

prime number | A prime number is a natural number with exactly two distinct factors, 1 and itself. The number 1 is not a prime number because it does not have two distinct factors.

principal | In finance, the amount of money on which interest is calculated.

principal root | The positive square root of a number, as in $\sqrt{16} = 4$. By definition, the radical symbol always means to find the principal root. Note that zero has only one square root, itself (since $0 \cdot 0 = 0$).

principal root | The positive square root of a number, as in $\sqrt{16} = 4$. By definition, the radical symbol always means to find the principal root. Note that zero has only one square root, itself (since $0 \cdot 0 = 0$).

Principle of Zero Products | If $ab = 0$, then either $a = 0$ or $b = 0$, or both a and b are 0.

Probability | A measure of how likely it is that something will occur.

product | The result when two numbers are multiplied. For example, the product of 4·5 is 20.

Product Raised to a Power | The product of two or more non-zero numbers raised to a power equals the product of each number raised to the same power: $(ab)^x = a^x \cdot b^x$

Product Raised to a Power Rule | The product of two or more non-zero numbers raised to a power equals the product of each number raised to the same power: $(ab)^x = a^x \cdot b^x$

Product Rule for Exponents | To multiply two exponential terms with the same base, add their exponents. $(x^a)(x^b) = x^{a+b}$

proportion | An equation that states that two ratios are equal.

pyramids | A polyhedron with a polygonal base and a collection of triangular faces that meet at a point.

Pythagoras | A Greek philosopher and mathematician who lived in the 6th Century B.C.

Pythagorean Theorem | The formula that relates the lengths of the sides of any right triangle: $a^2+b^2=c^2$, where c is the hypotenuse, and a and b are the legs of the right triangle.

quadrants | The x- and y-axes divide the coordinate plane into four regions. These regions are called quadrants.

quadratic equation | An equation that can be written in the form $ax^2 + bx + c = 0$, where x is a variable, and a, b, and c are constants with $a \neq 0$.

Quadrilaterals | A four-sided polygon.

quart | A unit of capacity equal to 32 fluid ounces, or 4 cups.

quartiles | The name of quarter sections of an ordered set of data.

quotient | The result of a division problem. In the problem 8÷2=4, 4 is the quotient.

Quotient Raised to a Power Rule | For any real numbers a and b $(b \neq 0)$ and any positive integer x: $(\frac{a}{b})^{\frac{1}{x}} = \frac{a^{\frac{1}{x}}}{b^{\frac{1}{x}}}$. For any real numbers a and b $(b \neq 0)$ and any positive integer x: $\sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$.

Quotient Rule for Exponents | For any non-zero number x and any integers a and b: $\frac{x^a}{x^b} = x^{a-b}$

radian measure | A measure of a central angle given by the ratio of the arc length to the radius.

radical equation | An equation that contains a radical expression.

radical expression | An expression that contains a radical.

Radical expressions | An expression that contains a radical.

radical sign | The symbol used for square root and other roots. It looks like $\sqrt{}$ and the number is written under it. For example, the square root of nine is written with the radical sign: $\sqrt{9}$

radical symbol | The symbol, $\sqrt{}$, used to denote the process of taking a root of a quantity.

radicand | The number or value under the radical symbol.

radius | The distance from the center of a circle to any point on the circle.

raised to the power | When a base has an exponent, it can be said that the base is "raised to the power" of the exponent. For example, 3^5 is read as "3 raised to the fifth power."

range | The set of all possible outputs in a function. Also the difference between the greatest value of a data set and the least value.

range of the function | The set of all output values or y-coordinates of the function.

Rates | A ratio that compares quantities measured in different units. For example, a speed compares the distance traveled to a length of time.

rational equations | An equation that contains one or more rational expressions.

rational exponent | An exponent that is a fraction.

Rational expressions | A fraction that contains a polynomial as the numerator, denominator, or both.

Rational formulas | A formula expressed as a rational equation.

rational numbers | Numbers that can be written as the ratio of two integers, where the denominator is not zero.

rational numbers | Numbers that can be written as the ratio of two integers, where the denominator is not zero.

rationalizing a denominator | The process by which a fraction containing radicals in the denominator is rewritten to have only rational numbers in the denominator.

Ratios | A comparison of two numbers by division. For example, the ratio of 15 boys in a class to 14 girls in the same class is 15:14.

ray | A half-line that begins at one point and goes on forever in one direction.

real numbers | All rational or irrational numbers.

real numbers | All rational or irrational numbers.

real part | The real term, a, in a complex number $a + bi$.

reciprocal | A number that when multiplied by a given number gives a product of 1. For example, $\frac{2}{7}$ and $\frac{7}{2}$ are reciprocals of each other.

reciprocals | A number that when multiplied by a given number gives a product of 1. For example, $\frac{2}{7}$ and $\frac{7}{2}$ are reciprocals of each other.

rectangle | A quadrilateral with two pairs of parallel sides and four right angles.

rectangular prism | A polyhedron that has three pairs of congruent, rectangular, parallel faces.

reference angle | The angle formed by the terminal side of an angle in standard position and the x-axis, whose measure is between 0° and 90°.

regroup | Rewriting a number so you can subtract a greater digit from a lesser digit.

relation | A correspondence between sets of values or information.

repeating decimals | Numbers whose decimal parts repeat a pattern of one or more digits—these are all rational numbers.

rhombus | A quadrilateral with four congruent sides.

right angle | An angle measuring exactly 90°.

Right Triangle | A triangle containing a right angle.

right triangles | A triangle containing a right angle.

rise | The vertical change between two points on a line.

rounding | Finding a number that's close to a given number, but is easier to think about.

run | The horizontal change between two points on a line.

sample space | The set of all possible outcomes.

Scalene Triangle | A triangle in which all three sides are a different length.

scientific notation | A positive number is written in scientific notation if it is written as $a \times 10^n$ where the coefficient a has a value such that $1 \leq a \leq 10$ and n is an integer.

set | A collection or group of things such as numbers.

sets | A collection or group of things such as numbers.

similar | Having the same shape but not necessarily the same size.

simple event | An event with only one outcome.

simplest form | A fraction is in simplest form if the numerator and denominator have no common factors other than 1.

sine | If A is an acute angle of a right triangle, then the sine of angle A is the ratio of the length of the side opposite angle A over the length of the hypotenuse.

slope | The ratio of the vertical change to the horizontal change of two points on a line. $\text{Slope} = \frac{\text{rise}}{\text{run}}$

slope-intercept form | A linear equation written in the form $y = mx + b$, where m represents the slope of the line, and b represents the y-value of the y-intercept, $(0, b)$.

sphere | A solid, round figure where every point on the surface is the same distance from the center.

square | A quadrilateral whose sides are all congruent and which has four right angles.

square root | A value that can be multiplied by itself to give the original number. For example if the original number is 9, then 3 is its square root because 3 multiplied by itself (3^2, pronounced "3 squared") equals 9. The symbol used for a square root is called a radical sign and goes on top of the number. The square root of 9 is written as: $\sqrt{9}$

square root | A number that when multiplied by itself gives the original nonnegative number. For example, $6 \cdot 6 = 36$ and $-6 \cdot -6 = 36$ so 6 is the positive square of 36 and -6 is the negative square root of 36.

Square Root Property | If $x^2 = a^2$, then $x = a$ or $x = -a$.

Square Root Property | If $x^2 = a^2$, then $x = a$ or $x = -a$.

square roots |

A value that can be multiplied by itself to give the original number. For example if the original number is 9, then 3 multiplied by itself (3^2 pronounced "3 squared") equals 9. The symbol used for a square root is called a radical sign and goes on top of the number. The square root of 9 is written as:

$\sqrt{9}$

squaring | Multiplying a number by itself, or raising the number to a power of 2. 8^2 can be read as "8 to the second power," “8 to a power of 2,” or “8 squared.”

standard form | A way to write a number using digits. For example, thirty-two is written in standard form as 32.

standard position | The placement of an angle upon a set of coordinate axes with its vertex at the origin, its initial side placed along the positive x-axis, and a directional arrow pointing to the angle’s terminal side.

stem-and-leaf plot | A type of graph used to visualize quantitative data. In a stem-and-leaf plot the digits of each number are organized separately to display a set of data.

straight angle | An angle measuring exactly 180º.

substitute | The replacement of a variable with a number.

substitution method | A method of solving a system of equations. Given a system, the substitution method allows you to create a simpler, one-variable equation by substituting one quantity in for an equivalent quantity.

subtrahend | The number that is subtracted from another number.

sum | The result when two or more numbers are added; the quantity that results from addition.

supplementary angles | Two angles whose measurements add up to 180º.

symmetric about the y-axis | The left and right halves of the graph are mirror images of each other over the y-axis.

system of linear equations | Two or more linear equations with the same variables.

system of linear inequalities | Two or more linear inequalities with the same variables.

tangent | If A is an acute angle of a right triangle, then the tangent of angle A is the ratio of the length of the side opposite angle A over the length of the side adjacent to .

term | A number or product of a number and variables raised to powers. $4x$, $-5y^2$, 6, and x^3y^4 are all examples of terms.

terminal side | The ray that has been rotated around the origin to form an angle with the stationary ray that is the initial side of the angle.

terminating decimals | Numbers whose decimal parts do not continue indefinitely but end eventually—these are all rational numbers.

terms | A number or product of a number and variables raised to powers. $4x, -5y^2, 6$, and x^3y^4 are all examples of terms.

ton | A unit for measuring the weight of heavier items in the U.S. customary measurement system. 1 ton = 2,000 pounds

trailing zeros | A placeholder 0 that occurs after the final non-0 digit in a decimal number. In the number 22.0900, the 0s in the thousandths and ten-thousandths places are trailing zeros.

trapezoid | A quadrilateral with one pair of parallel sides.

tree diagram | A diagram that shows the choices or random outcomes from multiple trials, using branches for each new outcomes.

trial | A random action or series of actions.

triangle | A polygon with three sides.

trigonometric functions | A function of an angle expressed as the ratio of two of the sides of a right triangle that contains that angle; the sine, cosine, tangent, cotangent, secant, cosecant.

trinomial | A polynomial with exactly three terms, such as $5y^2-4y+4$ and $x^2+2xy+y^2$.

U.S. customary measurement system | The most common system of measurement used in the United States. It is based on English measurement systems of the 18th century.

unit circle | A circle centered at the origin that has radius 1.

unit equivalents | Statements of equivalence between measurement units within a system or in comparison to another system of units. For example 1 foot = 12 inches or 1 inch = 2.54 centimeters are both examples of unit equivalents.

unit fractions | A fraction where the numerator and denominator are equal amounts. For example: $\frac{1 \text{ kg}}{1000 \text{ g}}$ or $\frac{12 \text{ inches}}{1 \text{ foot}}$. Unit fractions serve to help with conversions in the Factor Label method.

unit price | A rate in which the quantity is expressed as one unit. If 12 candy bars cost 4, the unit price is per 1 candy bar.

unit rate | A rate in which the second quantity is one unit. If a bird flaps its wings 240 times in 3 minutes, the unit rate of wing flapping is 80 flaps per 1 minute.

units of measurement | A standard amount or quantity. For example, an inch is a unit of measurement.

unlike denominators | Denominators that are different from each other. For example the fractions $\frac{1}{4}$ and $\frac{1}{8}$ have different denominators, one denominator being 4 and the other denominator being 8.

variable | A letter or symbol used to represent a quantity that can change.

variables | A letter or symbol used to represent a quantity that can change.

vertex | A turning point in a graph. Also the endpoint of the two rays that form an angle.

volume | A measurement of how much it takes to fill up a three-dimensional figure. Volume is measured in cubic units.

weight | A mathematical description of how heavy an object is.

whole numbers | Any of the numbers 0, 1, 2, 3, and so on.

whole numbers | The numbers 0, 1, 2, 3, ..., or all natural numbers plus 0.

whole numbers | The numbers 0, 1, 2, 3, ..., or all natural numbers plus 0.

x-axis | The horizontal axis of a coordinate plane. Also the horizontal axis of a bar graph or histogram.

x-axis | The horizontal axis of a coordinate plane. Also the horizontal axis of a bar graph or histogram.

x-coordinate | The first number in an ordered pair, which tells the distance to the right or left from the origin when graphing in a coordinate plane.

x-intercept | The point where the graph of a linear equation intersects the x-axis (x, 0).

y-axis | The vertical axis of a coordinate plane. Also the vertical axis of a bar graph or histogram.

y-axis | The vertical axis of a coordinate plane. Also the vertical axis of a bar graph or histogram.

y-coordinate | The second number in an ordered pair, which tells the distance to move up or down from the origin when graphing in a coordinate plane.

y-intercept | The point where the graph of a linear equation intersects the y-axis (0, y).

yard | A unit for measuring length in the U.S. customary measurement system. 1 = 3 feet or 36 inches

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